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ABELIAN VARIETIES OF WEIL TYPE AND KUGA-SATAKE VARIETIES

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Abstract. We analyze the relationship between abelian fourfolds of Weil type and Hodge structures of type K3, and we extend some of these correspondences to the case of arbitrary dimension.

Introduction. Abelian varieties of Weil type are examples of abelian varieties for which the Hodge conjecture is open in general. We study the structure of abelian fourfolds of Weil type, giving an explicit description of the Hodge structures of the cohomology groups, in particular the sub-Hodge structures of the second cohomology group. Starting from the observations of Paranjape [P], we show that for certain abelian varieties of Weil type (those with discriminant one) there exists a polarized sub-Hodge structure of the second cohomology group of dimension 6 with $h^{2,0} = 1$. We show that the map which associates this polarized Hodge structure to an abelian fourfold of Weil type with discriminant one admits an "inverse". Indeed, we can construct the Kuga-Satake variety associated to this Hodge structure and prove that the Kuga-Satake variety is an abelian variety of dimension 16 which is isogeneous to the product of four copies of the abelian fourfold of Weil type we started with. In the last section we generalize some of these results to higher dimensions; starting from a polarized weight two Hodge structure of type $(1, n-2, 1)$ with $n \equiv 2 \pmod 4$, the Poincaré decomposition of the Kuga-Satake variety gives (a number of copies of) an abelian variety of Weil type with discriminant one.

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1. Abelian varieties of Weil type.

1.1. Weil type. Let (X, E) be a polarized abelian variety of dimension $2n$ and $K \hookrightarrow \mathrm{End}(X) \otimes \boldsymbol{Q}$ be an imaginary quadratic field. The polarization $E \subset H^2(X, \boldsymbol{Q})$ defines by duality a bilinear antisymmetric map, which is for convenience denoted by the same letter E, from $H_1(X, \boldsymbol{Q}) \times H_1(X, \boldsymbol{Q})$ to $\boldsymbol{Q}$. X is said to be of Weil type if the action of K on the tangent space T_0X can be diagonalized as

$$\mathrm{diag}(\sigma(k), \dots, \sigma(k), \overline{\sigma(k)}, \dots, \overline{\sigma(k)}) \quad (k \in K)$$

with n entries $\sigma(k)$ and n entries $\overline{\sigma(k)}$ (where $\sigma : K \hookrightarrow \boldsymbol{C}$ is an embedding) and $E(kx, ky) = \sigma(k)\overline{\sigma(k)}E(x, y)$ for $x, y \in T_0X$.

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1.2. Discriminant. Let (X, K, E) be an abelian variety of Weil type and let $K = \boldsymbol{Q}(\varphi)$. The map

$$\begin{array}{rccc} H: & H_1(X,\boldsymbol{Q}) \times H_1(X,\boldsymbol{Q}) & \longrightarrow & K \\ & (x\ ,\ y) & \longmapsto & E(\varphi x, y) + \varphi E(x, y) \end{array}$$

is a Hermitian form on the K-vector space $H_1(X, \boldsymbol{Q})$. There exists a K-basis in which H is represented by a diagonal matrix $\mathrm{diag}(a, 1, \dots, 1, -1, \dots, -1)$, where a is a rational positive number called the discriminant of the variety ($a = \mathrm{discr}(X, K, E) = (-1)^n \det(H) \in \boldsymbol{Q}^*/\mathrm{Nm}_{\boldsymbol{Q}/K}(K^*)$).

2. Hodge structures.

2.1. Let V be a $\boldsymbol{Q}$-vector space and $h : S(\boldsymbol{R}) \to GL(V_{\boldsymbol{R}})$ be a rational representation of the group

$$S(\boldsymbol{R}) = \left\{ s(a, b) := \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in GL(2, \boldsymbol{R}) \right\} \cong \boldsymbol{C}^*$$

on $V_{\boldsymbol{R}} = V \otimes_{\boldsymbol{Q}} \boldsymbol{R}$ such that $h(s(a, 0)) = a^n \cdot \boldsymbol{1}$. The couple (V, h) is said to be a (rational) Hodge structure of weight n.

2.2. By the action of h, we have a decomposition of $V_{\boldsymbol{C}} = V \otimes_{\boldsymbol{Q}} \boldsymbol{C}$ into weight spaces:

$$V_{\boldsymbol{C}} = \bigoplus_{p+q=n} V^{p,q},$$

where $V^{p,q} = \{v \in V_{\boldsymbol{C}};\, h(z)v = z^p \bar{z}^q v\}$ and $\overline{V^{q,p}} = V^{p,q}$. This decomposition of V is a Hodge decomposition in the usual sense.

2.3. Polarization. A polarization of the Hodge structure (V, h) is a $\boldsymbol{Q}$-bilinear map $\psi : V \times V \to \boldsymbol{Q}$ such that

(i) $\psi(h(z)v, h(z)w) = (z\bar{z})^n \psi(v, w)$ for any $v, w \in V_{\boldsymbol{R}}$, $z \in \boldsymbol{C}^*$ (n =weight),

(ii) $\psi(v, h(i)w) = \psi(w, h(i)v)$ for any $v, w \in V_{\boldsymbol{R}}$,

(iii) $\psi(v, h(i)v) > 0$ for any $v \in V_{\boldsymbol{R}} - \{0\}$.

It is easy to show the following by direct computation (see [vG1]):

2.4. LEMMA. *A polarization ψ is symmetric if the weight of the Hodge structure is even, alternating if the weight is odd, and the quadratic form $Q(v) = -\psi(v, v)$ associated to the polarization is positive definite on $(V^{2,0} \oplus V^{0,2}) \cap V_{\boldsymbol{R}}$ and negative on $V^{1,1} \cap V_{\boldsymbol{R}}$.*

2.5. Weight 2 Hodge structures. Let (V, h, ψ) be a polarized Hodge structure of weight 2. The triple $(\dim V^{2,0}, \dim V^{1,1}, \dim V^{0,2})$ is said to be the type of the Hodge structure.

3. Sub-Hodge structures of the cohomology groups. Let $K := \boldsymbol{Q}(\varphi)$ $(\varphi^2 = -d)$ and V be a K-vector space. We can extend the action of K^* to $\bigwedge_{\boldsymbol{Q}}^2 V$ in a natural way: $k_2(v \wedge w) = kv \wedge kw$ $(k \in K^*)$. In order to examine the sub-Hodge structures, the following proposition concerning K-vector spaces is very useful (see also [M], [M-Z] and [vG2] in which this proposition is implicitly contained):

3.1. PROPOSITION. *Let V be a K-vector space. Then the $\boldsymbol{Q}$-linear map*

$$i : \bigwedge_K^2 V \hookrightarrow \bigwedge_{\boldsymbol{Q}}^2 V, \quad a \wedge_K b \mapsto \frac{1}{2} a \wedge b - \frac{1}{2d} \varphi a \wedge \varphi b$$

is injective and $\mathrm{Im}(i) = \{w \in \bigwedge_{\boldsymbol{Q}}^2 V; \varphi_2 w = -dw\}$ *(where φ_2 is the action of $\varphi \in K^*$ on $\bigwedge_{\boldsymbol{Q}}^2 V$ already defined).*

PROOF. Let $W := \mathrm{Hom}_K(V, K)$, $W^* := \mathrm{Hom}_K(W, K)$ and $(W^*)^{\boldsymbol{Q}} := \mathrm{Hom}_{\boldsymbol{Q}}(W, \boldsymbol{Q})$. The trace map

$$\begin{aligned} \omega : W^* &\xrightarrow{\cong} W^{*\boldsymbol{Q}}, \quad \text{where } \mathrm{Tr} : K \ni z \longmapsto z + \bar{z} \in \boldsymbol{Q} \\ f &\longmapsto \mathrm{Tr} \circ f \end{aligned}$$

is a linear isomorphism of $\boldsymbol{Q}$-vector spaces. We also have an isomorphism of K-vector spaces

$$\begin{aligned} \alpha : \bigwedge_K^2 V &\longrightarrow \mathrm{Hom}_K\Big(\bigwedge_K^2 W, K\Big) \\ a \wedge_K b &\longmapsto [\alpha \wedge_K \beta \mapsto \alpha(a)\beta(b) - \beta(a)\alpha(b)] . \end{aligned}$$

Then, since K-linear homomorphisms are also $\boldsymbol{Q}$-linear, we have a chain of maps

$$\bigwedge_K^2 V \xrightarrow{\alpha} \mathrm{Hom}_K\Big(\bigwedge_K^2 W, K\Big) \xrightarrow{\mathrm{id}} \mathrm{Hom}_{\boldsymbol{Q}}\Big(\bigwedge_{\boldsymbol{Q}}^2 W, K\Big) \xrightarrow{\mathrm{Tr}}$$

$$\xrightarrow{\mathrm{Tr}} \mathrm{Hom}_{\boldsymbol{Q}}\Big(\bigwedge_{\boldsymbol{Q}}^2 W, \boldsymbol{Q}\Big) \xrightarrow{\cong} \bigwedge_{\boldsymbol{Q}}^2 (W^*)^{\boldsymbol{Q}} \xrightarrow{(\omega\wedge\omega)^{-1}} \bigwedge_{\boldsymbol{Q}}^2 W^* \xrightarrow{\cong} \bigwedge_{\boldsymbol{Q}}^2 V .$$

Writing these maps in term of $\boldsymbol{Q}$ and K-basis, we have that $i(a \wedge_K b) = (1/2) a \wedge b - (1/2d) \varphi a \wedge \varphi b$. The map i is a composite of injective maps, hence is injective.

Obviously, $\mathrm{Im}(i) \subset \{w \in \bigwedge_{\boldsymbol{Q}}^2 V; \varphi_2 w = -dw\}$. Indeed, we have

$$\varphi_2\Big(\frac{1}{2} a \wedge b - \frac{1}{2d} \varphi a \wedge \varphi b\Big) = \frac{1}{2} \varphi(a) \wedge \varphi(b) - \frac{1}{2d} \varphi^2(a) \wedge \varphi^2(b) = (-d)\Big(\frac{1}{2} a \wedge b - \frac{1}{2d} \varphi a \wedge \varphi b\Big) .$$

Let now $V_K = V_+ \oplus V_-$ be the decomposition of $V_K = V \otimes_{\boldsymbol{Q}} K$ into the subspaces in which the elements $k \in K$ act respectively as $\sigma(k)$ and $\overline{\sigma(k)}$; we have $\dim_K V_K = 2n$ and $\dim_K V_\pm = n$. By the action of K on $\bigwedge_{\boldsymbol{Q}}^2 V$, we have $\varphi_2^2 = d^2$ and can decompose the space as $\bigwedge_{\boldsymbol{Q}}^2 V = W_+ \oplus W_-$, where $W_\pm$ are the $+d$ and the $-d$ eigenspaces of φ_2. We showed that

$$\mathrm{Im}(i) \subset W_-$$

and for the equality it is sufficient to show that these spaces have the same dimension. Tensoring with K, we have $\bigwedge_{\boldsymbol{Q}}^2 V \otimes_{\boldsymbol{Q}} K = (W_+ \otimes_{\boldsymbol{Q}} K) \oplus (W_- \otimes_{\boldsymbol{Q}} K) = W_{+,K} \oplus W_{-,K}$, and looking at the decomposition of V_K and using the equality $\bigwedge_{\boldsymbol{Q}}^2 V \otimes_{\boldsymbol{Q}} K = \bigwedge_K^2 V_K$, we have also

$$\bigwedge_{\boldsymbol{Q}}^2 V \otimes_{\boldsymbol{Q}} K = \bigwedge_K^2 V_+ \oplus \bigwedge_K^2 V_- \oplus (V_+ \otimes_K V_-) .$$

It is easy to show that $\bigwedge_K^2 V_+ \subset W_{-,K}$, $\bigwedge_K^2 V_- \subset W_{-,K}$ and $V_+ \otimes_K V_- \subset W_{+,K}$, and hence we have

$$W_{-,K} = \bigwedge_K^2 V_+ \oplus \bigwedge_K^2 V_- , \quad W_{+,K} = V_+ \otimes_K V_- .$$

Now, using also the injectivity of the map i, we can compute the required dimensions:

$$\dim_K W_{-,K} = \binom{n}{2} + \binom{n}{2} = 2\binom{n}{2}, \quad \dim_K(\mathrm{Im}(i) \otimes_{\mathbf{Q}} K) = 2\dim_K \bigwedge_K^2 V = 2\binom{n}{2}.$$

□

3.2. Abelian varieties of Weil type. Let (X, K, E) be a polarized abelian fourfold of Weil type (where $K = \mathbf{Q}(\varphi)$ with $\varphi^2 = -d$); we have $H^1(X, \mathbf{Q}) \cong K^4$, $H^2(X, \mathbf{Q}) = \bigwedge_{\mathbf{Q}}^2 H^1(X, \mathbf{Q})$. Now, we study the sub-Hodge structures of the second cohomology group of a generic abelian fourfold of Weil type and show that, if $\mathrm{discr}(X, K, E) = 1$, there is a substructure of weight 2 and type $(1, 4, 1)$.

Let $S = i(\bigwedge_K^2 H^1(X, \mathbf{Q})) \subset H^2(X, \mathbf{Q})$. From Proposition 3.1 the map i is injective, so $S \cong \bigwedge_K^2 H^1(X, \mathbf{Q})$, and it is often useful to forget the inclusion. Then we have the following:

3.3. LEMMA. *Let (X, K, E) be an abelian fourfold of Weil type. Then the subspace S is a sub-Hodge structure of $H^2(X, \mathbf{Q})$ with* $\dim S^{2,0} = \dim S^{0,2} = 2$, $\dim S^{1,1} = 8$.

PROOF. We consider the automorphism φ_2 of $H^2(X, \mathbf{Q})$; we have $\varphi_2^2 = d^2$, and let H_+^2 and H_-^2 be its (respectively) $+d$ and $-d$ eigenspaces. Using Proposition 3.1, we have $S = H_-^2$, and hence it is a sub-Hodge structure of $H^2(X, \mathbf{Q})$. Let $H^1(X, \mathbf{C}) = V_+ \oplus V_-$ be the decomposition into the $i\sqrt{d}$ and the $-i\sqrt{d}$ eigenspaces of φ; since X is of Weil type $(2, 2)$, we have $V_\pm = V_\pm^{1,0} \oplus V_\pm^{0,1}$, where $V_\pm^{i,j} = V_\pm \cap H^{i,j}$ and $\dim_{\mathbf{C}} V_\pm^{i,j} = 2$. Then we have

$$\dim S^{2,0} = \dim \bigwedge_{\mathbf{C}}^2 V_+^{1,0} + \dim \bigwedge_{\mathbf{C}}^2 V_-^{1,0} = 2,$$

$$\dim S^{0,2} = \dim \bigwedge_{\mathbf{C}}^2 V_+^{0,1} + \dim \bigwedge_{\mathbf{C}}^2 V_-^{0,1} = 2,$$

$$\dim S^{1,1} = \dim V_+^{1,0} \otimes V_+^{0,1} + \dim V_-^{1,0} \otimes V_-^{0,1} = 8.$$

□

3.4. Hermitian form. We can extend H to a $\mathbf{Q}$-bilinear form $\tilde{H}$ on $\bigwedge_K^2 H^1(X, \mathbf{Q})$ by defining

$$\tilde{H}(a \wedge_K b, c \wedge_K d) := H(a, c)H(b, d) - H(a, d)H(b, c).$$

Let $\{v_1, w_1, v_2, w_2\}$ be a K-basis of $H^1(X, \mathbf{Q})$ in which the matrix of the Hermitian form is $\mathrm{diag}(a, 1, -1, -1)$. A direct computation shows that in the K basis of $\bigwedge_K^2 H^1(X, \mathbf{Q})$

$$a_1 = v_1 \wedge_K w_1, \quad a_2 = v_1 \wedge_K v_2, \quad a_3 = v_1 \wedge_K w_2,$$

$$b_1 = v_2 \wedge_K w_2, \quad b_2 = w_2 \wedge_K w_1, \quad b_3 = w_1 \wedge_K v_2,$$

and hence we have $\tilde{H} \cong \mathrm{diag}(a, -a, -a, 1, -1, -1)$.

Let now $\gamma : \bigwedge_K^4 H^1(X, \boldsymbol{Q}) \to K$ be the isomorphism sending $a_1 \wedge_K b_1$ to 1. Then we have the following

3.5. PROPOSITION. *There exists a φ-antilinear automorphism $t \in \mathrm{End}_{\mathrm{Hod}}(S)$ such that $t \circ t = a \cdot \mathrm{Id}$, and for all $v, w \in \bigwedge_K^2 H^1(X, \boldsymbol{Q})$ we have $\tilde{H}(v, w) = \gamma(t(w) \wedge v)$.*

PROOF. We define a K-linear isomorphism ρ

$$\rho : \bigwedge_K^2 H^1(X, \boldsymbol{Q}) \to \left(\bigwedge_K^2 H^1(X, \boldsymbol{Q})\right)^*, \quad x \mapsto [y \mapsto \gamma(x \wedge_K y)]$$

and a φ-antilinear bijection τ

$$\tau : \bigwedge_K^2 H^1(X, \boldsymbol{Q}) \to \left(\bigwedge_K^2 H^1(X, \boldsymbol{Q})\right)^*, \quad x \mapsto [y \mapsto \tilde{H}(y, x)].$$

The isomorphism t is defined as

$$t := \rho^{-1} \circ \tau : \bigwedge_K^2 H^1(X, \boldsymbol{Q}) \to \bigwedge_K^2 H^1(X, \boldsymbol{Q}).$$

Since $\tau(w) = \rho(t(w))$, we have $\tilde{H}(v, w) = \gamma(t(w) \wedge_K v)$ and the φ-antilinearity follows from the K-linearity of ρ^{-1} and the φ-antilinearity of τ. The representation $h : \boldsymbol{C}^* \mapsto GL(H^1(X, \boldsymbol{Q}))$ defining the Hodge structure of $H^1(X, \boldsymbol{Q})$ gives a representation

$$h_2 : \boldsymbol{C}^* \mapsto GL(S), \quad z \mapsto [v \wedge_K w \mapsto h(z)v \wedge_K h(z)w],$$

which defines a Hodge structure on S. Since $\bigwedge_K^4 H^1(X, \boldsymbol{Q}) \cong K$ is one-dimensional, we have that $h_4(z)(v \wedge_K w) = h_2(z)v \wedge_K h_2(z)w = |z|^4(v \wedge_K w)$ for all $v, w \in S$ and, by the properties of E and H, we also have that $\tilde{H}(h_2(z)v, h_2(z)w) = (z\bar{z})^2 \tilde{H}(v, w) = |z|^4 \tilde{H}(v, w)$. These observations show that for all $v, w \in S$

$$\begin{aligned}\gamma(t(h_2(z)(v)) \wedge_K h_2(z)w) &= \tilde{H}(h_2(z)w, h_2(z)v) = (z\bar{z})^2 \tilde{H}(w, v) \\ &= |z|^4 \gamma(t(v) \wedge_K w) = \gamma(h_2(z)t(v) \wedge_K h_2(z)w).\end{aligned}$$

Hence $t \circ h_2(z) = h_2(z) \circ t$, so $t \in \mathrm{End}_{\mathrm{Hod}}(S)$. If we write explicitly the action of $t = \rho^{-1}\tau$ on the elements of the basis $\{a_1, \ldots, b_3\}$, then we have

$$a_1 \mapsto a a_1^* \mapsto a b_1, \quad b_1 \mapsto b_1^* \mapsto a_1$$

(and similarly for the other elements of the basis), and hence $t \circ t = a \cdot \mathrm{Id}$ as required. □

3.6. COROLLARY. $\mathrm{End}_{\mathrm{Hod}}(S) \cong \boldsymbol{H}$, *where*

$$\boldsymbol{H} = \{\lambda_1 + \lambda_2 \varphi + \lambda_3 t + \lambda_4 \varphi \circ t; \varphi^2 = -d, t^2 = a, \varphi \circ t = -t \circ \varphi\}$$

is a quaternion algebra over $\boldsymbol{Q}$. In particular, if $a \notin \mathrm{Nm}_{K/\boldsymbol{Q}}(K^*)$, then $\boldsymbol{H}$ is a skew field and hence S is a simple Hodge structure.

PROOF. By Proposition 3.5 we have that $t \in \mathrm{End}_{\mathrm{Hod}}(S)$. Moreover,

$$h_2(\varphi x \wedge_K y) = h(\varphi x) \wedge_K h(y) = \varphi h(x) \wedge_K h(y) = \varphi h_2(x \wedge_K y),$$

so $K \hookrightarrow \mathrm{End}_{\mathrm{Hod}}(S)$. Since t is φ-antilinear, we have $\boldsymbol{H} \subset \mathrm{End}_{\mathrm{Hod}}(S)$.

Let $\mathrm{MT} \in GL(H^1(X, \boldsymbol{Q}))$ be the Mumford-Tate group of the Hodge structure $H^1(X, \boldsymbol{Q})$ (for the definitions and properties of the Mumford-Tate group, see [vG2, par. 6.4]). Then we have

$$\mathrm{End}_{\mathrm{Hod}}(S) \otimes \boldsymbol{C} = (\mathrm{End}_{\boldsymbol{C}}(S \otimes \boldsymbol{C}))^{\mathrm{MT}(\boldsymbol{C})} = \mathrm{End}\Big(\bigwedge_{\boldsymbol{C}}^{2} V_+ \oplus \bigwedge_{\boldsymbol{C}}^{2} V_-\Big)^{\mathrm{MT}(\boldsymbol{C})},$$

where V_+ and V_- are the standard and the dual representations of $\mathrm{MT}(\boldsymbol{C}) \cong SL(4, \boldsymbol{C})$ (the isomorphism $\mathrm{MT}(\boldsymbol{C}) \cong SL(2n, \boldsymbol{C})$ holds for a general polarized Abelian variety of Weil type of dimension $2n$, see [vG2]). Since $\bigwedge_{\boldsymbol{C}}^4 V_- \cong \boldsymbol{C}$, we have $\bigwedge_{\boldsymbol{C}}^2 V_+ \cong \bigwedge_{\boldsymbol{C}}^2 (V_-)^* \cong \bigwedge_{\boldsymbol{C}}^2 V_-$, so

$$\mathrm{End}_{\mathrm{Hod}}(S) \otimes \boldsymbol{C} \cong M_2\Big(\mathrm{End}\Big(\bigwedge_{\boldsymbol{C}}^{2} V_+\Big)^{\mathrm{MT}(\boldsymbol{C})}\Big) \cong M_2(\boldsymbol{C})$$

(the last isomorphism comes from Schur's lemma). Therefore, $\mathrm{End}_{\mathrm{Hod}}(S) \cong \boldsymbol{H}$ because they are of the same dimension. □

3.7. Hyperbolic lattice. The bilinear form associated to the lattice "hyperbolic plane" has a matrix $\mathrm{Hyp} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (see [B-P-V, p. 14]).

3.8. THEOREM. *Let (X, K, E) be an abelian fourfold of Weil type with discriminant one. Then we have*:

1. $S \cong T_+ \oplus T_-$, *where* $T_\pm = \mathrm{Ker}(t \pm \mathrm{Id})$ *are sub-Hodge structures.*
2. $\varphi : T_+ \xrightarrow{\sim} T_-$ *is an isomorphism of Hodge structures.*
3. *Let $T = T_+$. The polarization $\bigwedge^2 E$ of $H^2(X, \boldsymbol{Q})$ induces a polarization*

$$\bigwedge^2 E|_{T \times T} = -\frac{1}{2d} \tilde{H}|_{T \times T} : T \times T \to \boldsymbol{Q},$$

and we have $\tilde{H}|_{T \times T} = \mathrm{Hyp} \oplus \mathrm{Hyp} \oplus [-2] \oplus [-2d]$.

PROOF. The first result is obvious, since $t^2 = \mathrm{Id}$ by Proposition 3.5. The second follows from the φ-antilinearity of t: if $v \in T_+$, we have $t(v) = v$ and $t(\varphi v) = -\varphi t(v) = -\varphi v$, so $\varphi v \in T_-$. In order to prove the third result, we observe that $\tilde{H}$ is Hermitian, since

$$\begin{aligned}
\overline{\tilde{H}(a \wedge_K b, c \wedge_K d)} &= \overline{H(a, c)H(b, d)} - \overline{H(a, d)H(b, c)} \\
&= H(c, a)H(d, b) - H(d, a)H(c, b) \\
&= \tilde{H}(c \wedge_K d, a \wedge_K b).
\end{aligned}$$

On $T \times T$ the form $\tilde{H}$ is symmetric, since

$$\tilde{H}(v, a) = \gamma(t(a) \wedge_K v) = \gamma(a \wedge_K v) = \gamma(v \wedge_K a) = \tilde{H}(a, v),$$

and in particular $\mathrm{Im}(\tilde{H}) = 0$. By Proposition 3.1, the elements $a \wedge_K b$ of S can be written as $(1/2)a \wedge b - (1/2d)\varphi a \wedge \varphi b$, and an easy computation shows that

$$\bigwedge^2 E(a \wedge_K b, c \wedge_K d) = (1/2)[E(a,c)E(b,d) - E(a,d)E(b,c) \\ - (1/d)(E(\varphi a, c)E(\varphi b, d) - E(\varphi a, d)E(\varphi b, c))].$$

However,

$$\begin{aligned}\tilde{H}(a \wedge b, c \wedge d) &= H(a,c)H(b,d) - H(a,d)H(b,c) \\ &= (E(\varphi a, c) + \varphi E(a,c))(E(\varphi b, d) + \varphi E(b,d)) \\ &\quad - (E(\varphi a, d) + \varphi E(a,d))(E(\varphi b, c) + \varphi E(b,c)) \\ &= \{E(\varphi a, c)E(\varphi b, d) + \varphi^2 E(a,c)E(b,d) \\ &\quad - E(\varphi a, d)E(\varphi b, c) - \varphi^2 E(a,d)E(b,c)\} \\ &\quad + \varphi\{E(\varphi a, c)E(b,d) + E(a,c)E(\varphi b, d) \\ &\quad - E(\varphi a, d)E(b,c) - E(a,d)E(\varphi b, c)\},\end{aligned}$$

and on $T \times T$ (where $\mathrm{Im}(\tilde{H}) = 0$) we have $\bigwedge^2 E|_{T\times T} = -(1/2d)\tilde{H}|_{T\times T}$. A direct computation shows that $\tilde{H}|_{T\times T} \cong \mathrm{diag}(2, -2, 2d, -2d, -2, -2d)$. With an appropriate change of basis, we have $\tilde{H}|_{T\times T} = \mathrm{Hyp} \oplus \mathrm{Hyp} \oplus [-2] \oplus [-2d]$ as required. □

3.9. COROLLARY. *The Hodge structure T of* 3.8.3 *has type* $(1, 4, 1)$.

PROOF. We have from Lemma 3.3 that the type of S is $(2, 8, 2)$. By Theorem 3.8 we have that $S \cong T^{\oplus 2}$, and hence the type of T is $(1, 4, 1)$. □

4. Clifford algebras. We showed that there is a polarized sub-Hodge structure T of type $(1, 4, 1)$ with the quadratic form $Q \cong \mathrm{Hyp}\oplus\mathrm{Hyp}\oplus[-2]\oplus[-2d]$ contained in the second cohomology group of an abelian variety of Weil type with discriminant one. Now, using the Kuga-Satake construction, we construct an abelian variety from this Hodge structure and show that this is the variety of Weil type we started with.

4.1. DEFINITION (See [F-H, p. 301]). Let V be a vector space over $\mathbb{Q}$ of dimension n and ψ be a symmetric, nondegenerate bilinear form. The Clifford algebra C_n is the quotient of the tensor algebra by the two-sided ideal $I(\psi)$ generated by all elements of the form $v \otimes v - \psi(v, v)$.

4.2. NOTATION. We write simply the "product" $v_1 \cdots v_n$ for the class of $v_1 \otimes \cdots \otimes v_n$ in C_n.

4.3. DEFINITION. The even subalgebra C_n^+ is the algebra generated by all linear combinations of products of an even number of elements of V.

4.4. Dimensions. We have $\dim_{\mathbb{Q}} C_n = 2^n$ and $\dim_{\mathbb{Q}} C_n^+ = 2^{n-1}$.

5. Kuga-Satake varieties. Let (V, h, ψ) be a weight 2 polarized Hodge structure of type $(1, n-2, 1)$, and let $\{g_1, \ldots, g_n\}$ be a basis of V in which the symmetric bilinear form

$Q = -\psi$ is given by (see Lemma 2.4)

$$Q = d_1X_1^2 + d_2X_2^2 - d_3X_3^2 - \ldots - d_nX_n^2 \quad (d_i \in \boldsymbol{Q}_{>0}).$$

5.1. Complex torus. Let $J = (1/\sqrt{d_1d_2})g_1g_2$. Then we have $J^2 = -1$ and the left multiplication by J on C_n^+ (the even Clifford algebra constructed from (V, Q)) defines a complex structure on $C_{n,\boldsymbol{R}}^+ := C_n^+ \otimes_{\boldsymbol{Q}} \boldsymbol{R}$. Let now $C_{n,\boldsymbol{Z}}^+$ be the lattice of linear combinations of elements of the basis of C_n^+ with integer coefficients. Then

$$A_0 = (C_{n,\boldsymbol{R}}^+, J)/(C_{n,\boldsymbol{Z}}^+)$$

is a complex torus.

5.2. Polarization. Let $\mathrm{Tr}(x)$ be the trace of the map "right multiplication on C_n^+ by the element $x \in C_n^+$". We can define a polarization E on the complex torus A_0 (see [K-S], [vG1]) by setting $E(v, w) := \mathrm{Tr}(\alpha\iota(v)w)$, where ι is the canonical involution $\iota(g_1^{a_1} \ldots g_n^{a_n}) = g_n^{a_n} \ldots g_1^{a_1}$ and $\alpha \in C_n^+$ is an element such that $\iota(\alpha) = -\alpha$ and $E(v, Jv) > 0$ for all v.

5.3. Kuga-Satake variety. The abelian variety (A_0, E) is the Kuga-Satake variety associated to the Hodge structure (V, h, ψ) of type $(1, n-2, 1)$.

6. Abelian fourfolds.

6.1. Hodge structures of dimension 6. Let (V, h, ψ) be a weight 2 polarized Hodge structure of type $(1, 4, 1)$, and let $\{f_1, \ldots, f_6\}$ be a basis of V in which the bilinear form $Q = -\psi$ is given by the matrix $\mathrm{Hyp} \oplus \mathrm{Hyp} \oplus [l] \oplus [m]$ (with $l, m < 0$). In the basis

$$e_1 = f_1 + f_2, \quad e_2 = f_1 - f_2, \quad e_3 = f_3 + f_4, \quad e_4 = f_3 - f_4, \quad e_5 = f_5, \quad e_6 = f_6,$$

the matrix of Q is diagonal and we have $Q \cong \mathrm{diag}(2, -2, 2, -2, l, m)$. We construct the Kuga-Satake variety (A_0, E) associated to the Hodge structure, which is an abelian variety of dimension $\dim_{\boldsymbol{C}} A_0 = \dim_{\boldsymbol{C}} C_{6,\boldsymbol{R}}^+ = 16$. In order to examine the structure of this variety, the following is very important.

6.2. THEOREM. *Let V be a rational vector space of dimension 6 equipped with a symmetric bilinear form $Q = \mathrm{Hyp} \oplus \mathrm{Hyp} \oplus [l] \oplus [m]$ $(l, m < 0)$, and let C_6^+ be the even Clifford subalgebra generated by (V, Q). Then we have $C_6^+ \cong gl_4(\boldsymbol{Q}(\sqrt{-lm}))$.*

PROOF. Let $\{e_1, \ldots, e_6\}$ be the "diagonal" basis and C_4 be the Clifford algebra generated by $\{e_1, \ldots, e_4\}$. Let $z = e_1 \ldots e_6$. Then $\boldsymbol{Q}(z)$ is the center of C_6^+ and $z^2 = -16lm$. The map

$$\varphi : (C_4^+ \oplus e_5C_4^-) \otimes_{\boldsymbol{Q}} \boldsymbol{Q}(z) \to C_6^+, \quad a \otimes \lambda \mapsto a\lambda$$

is an isomorphism. Indeed, it is a surjective homomorphism between two $\boldsymbol{Q}$-algebras of the same dimension. We have

$$C_4^+ \cong \mathrm{End}\Big(\bigwedge^{\mathrm{ev}} W\Big) \oplus \mathrm{End}\Big(\bigwedge^{\mathrm{odd}} W\Big),$$

$$C_4^- \cong \mathrm{End}\Big(\bigwedge^{\mathrm{ev}} W, \bigwedge^{\mathrm{odd}} W\Big) \oplus \mathrm{End}\Big(\bigwedge^{\mathrm{odd}} W, \bigwedge^{\mathrm{ev}} W\Big),$$

where $W = \langle f_1, f_3 \rangle$ (cf. [F-H, p. 305]). Therefore we can construct the isomorphism

$$\mu : C_4^+ \oplus e_5 C_4^- \to gl_4(\boldsymbol{Q}), \quad (a, e_5 x) \mapsto \begin{pmatrix} a_{ee} & -l x_{oe} \\ x_{eo} & a_{oo} \end{pmatrix},$$

where $a_{ee} \in \mathrm{End}(\bigwedge^{\mathrm{ev}} W)$, $x_{oe} \in \mathrm{End}(\bigwedge^{\mathrm{odd}} W, \bigwedge^{\mathrm{ev}} W)$, $x_{eo} \in \mathrm{End}(\bigwedge^{\mathrm{ev}} W, \bigwedge^{\mathrm{odd}} W)$, and $a_{oo} \in \mathrm{End}(\bigwedge^{\mathrm{odd}} W)$. So, we have

$$C_6^+ \cong (C_4^+ \oplus e_5 C_4^-) \otimes_{\boldsymbol{Q}} \boldsymbol{Q}(z) \cong gl_4(\boldsymbol{Q}) \otimes_{\boldsymbol{Q}} \boldsymbol{Q}(z) \cong gl_4(\boldsymbol{Q}) \otimes_{\boldsymbol{Q}} \boldsymbol{Q}(\sqrt{-lm}) \cong gl_4(\boldsymbol{Q}(\sqrt{-lm})).$$

□

6.3. COROLLARY (Poincaré's decomposition). From Theorem 6.2 and [S] we have that, in the general case, $\mathrm{End}_0(A_0) \cong C_6^+ \cong gl_4(\boldsymbol{Q}(\sqrt{-lm}))$, and hence $A_0 \sim A^4$ (Poincaré's theorem), where A is a simple Abelian variety with $\mathrm{End}_0(A) \cong \boldsymbol{Q}(\sqrt{-lm})$.

6.4. THEOREM. *The abelian variety A is an abelian fourfold of Weil type over $K = \boldsymbol{Q}(\sqrt{-lm})$, and there exists a basis in which the Hermitian form $H = E(\varphi x, y) + \varphi E(x, y)$ is diagonal with $a = 1$ (that is, $H \cong \mathrm{diag}(1, 1, -1, -1)$).*

PROOF. Let $\beta = (1/4) f_1 f_2 f_3 f_4$. We have $\beta^2 = \beta$ and one can verify that the map

$$\phi : C_6^+ \to C_6^+, \quad x \mapsto x \cdot \beta$$

has kernel of dimension 24 over $\boldsymbol{Q}$. Hence the image has dimension 8 and we have that the image of $\mathrm{Im}\,\phi \otimes \boldsymbol{R}$ in the Kuga-Satake variety is isomorphic to A.

A direct computation shows that $\mathrm{Im}\,\phi$ has a basis

$$\varepsilon_1 = f_2 f_4, \quad \varepsilon_2 = \frac{1}{2} f_1 f_2 f_3 f_4, \quad \varepsilon_3 = f_2 f_3 f_4 f_5, \quad \varepsilon_4 = f_1 f_2 f_4 f_5,$$
$$\delta_1 = f_2 f_4 f_5 f_6, \quad \delta_2 = \frac{1}{2} f_1 f_2 f_3 f_4 f_5 f_6, \quad \delta_3 = f_2 f_3 f_4 f_6, \quad \delta_4 = f_1 f_2 f_4 f_6.$$

It is easy to show (using the "diagonal" basis) that

$$J\varepsilon_1 = -\varepsilon_2, \quad J\varepsilon_2 = \varepsilon_1, \quad J\varepsilon_3 = -\varepsilon_4, \quad J\varepsilon_4 = \varepsilon_3,$$
$$J\delta_1 = -\delta_2, \quad J\delta_2 = \delta_1, \quad J\delta_3 = -\delta_4, \quad J\delta_4 = \delta_3,$$

and

$$z\varepsilon_1 = 4\delta_1, \quad z\varepsilon_2 = 4\delta_2, \quad z\varepsilon_3 = 4l\delta_3, \quad z\varepsilon_4 = 4l\delta_4,$$
$$z\delta_1 = -4lm\varepsilon_1, \quad z\delta_2 = -4lm\varepsilon_2, \quad z\delta_3 = -4m\varepsilon_3, \quad z\delta_4 = -4m\varepsilon_4.$$

The i-eigenspace of the map $J|_{\mathrm{Im}\phi \otimes \boldsymbol{C}}$ is spanned by $-i\varepsilon_1 + \varepsilon_2$, $-i\varepsilon_3 + \varepsilon_4$, $-i\delta_1 + \delta_2$, $-i\delta_3 + \delta_4$, and the action of z on this subspace is given by the matrix

$$\begin{pmatrix} 0 & -4lm & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4m \\ 0 & 0 & 4l & 0 \end{pmatrix}.$$

The eigenvalues of the matrix are z and $\bar{z}$ both with multiplicity 2. Hence A is an abelian fourfold of Weil type over $\boldsymbol{Q}(z) \cong \boldsymbol{Q}(\sqrt{-lm})$.

We observe that the every admissible complex structure J' can be written as $J' = aJ\iota(a)$, where $a \in Spin(Q) := \{x \in C^+; x\iota(x) = 1, xV\iota(x) \subset V\}$ ($J' = e'_1 e'_3$ with $e_i'^2 > 0$; then e'_i is obtained from e_i by the action of $SO(Q)$ and $Spin(Q)$ is a 2 : 1 cover of $SO(Q)$). Since

$$\begin{aligned} E(x, J'x) &= \mathrm{Tr}(\alpha\iota(x)J') = \mathrm{Tr}(\alpha\iota(x)aJ\iota(a)x) \\ &= \mathrm{Tr}(\alpha\iota(\iota(a)x)J(\iota(a)x)) = \mathrm{Tr}(\alpha\iota(y)Jy)\,, \end{aligned}$$

we have that the choice of the element $\alpha \in C_6^+$ in the definition of the polarization does not depend on the choice of the complex structure ($E(x, Jx) > 0$ for all admissible J).

A direct computation shows that $\alpha = -f_1 f_3$ satisfies the "positivity condition" of E and that the matrix of the polarization in the basis $\{\varepsilon_i, \delta_j\}$ is

$$E = \begin{pmatrix} M & & & \\ & -2lM & & \\ & & lmM & \\ & & & -2mM \end{pmatrix}, \quad \text{where } M = \begin{pmatrix} 0 & -64 \\ 64 & 0 \end{pmatrix}.$$

On the K-basis $\{\varepsilon_1, \dots, \varepsilon_4\}$ we have that $H(x, y) = E(\varphi x, y) + \varphi E(x, y)$ has the matrix representation

$$H = \begin{pmatrix} \varphi M & \\ & -2\varphi M \end{pmatrix},$$

and this matrix can be diagonalized as $H \cong \mathrm{diag}(1, 1, -1, -1)$. □

We can now prove the following

6.5. THEOREM. *The abelian fourfold A occurring in the decomposition of the Kuga-Satake variety is isogenous to the Abelian fourfold X we started with.*

PROOF. The abelian fourfolds A and X are both of Weil type with discriminant equal to one, so we have only to show that they have the same complex structure. We consider the Hodge substructure $T \subset H^2(X, \boldsymbol{Q})$ defined in Theorem 3.8. We also have that $T \subset H^2(A, \boldsymbol{Q})$ (see [vG2]). Let $\mathrm{MT}(X)$ be the Mumford-Tate group of the abelian variety X, that is the Mumford-Tate group of the Hodge structure $H^1(X, \boldsymbol{Q})$. The subspace T is obviously a subrepresentation of $\mathrm{MT}(X)$ and its Hodge structure is given by a representation $h_+ : \boldsymbol{C}^* \to \mathrm{MT}(T)(\boldsymbol{R}) \subset GL(T_{\boldsymbol{R}}) \subset GL(H^2(X, \boldsymbol{R}))$. Now, we consider the map $h_X : \boldsymbol{C}^* \to \mathrm{MT}(X)(\boldsymbol{R}) \subset GL(H^1(X, \boldsymbol{R}))$ which gives the Hodge structure on $H^1(X, \boldsymbol{Q})$; we observe that the map $\mathrm{MT}(X)(\boldsymbol{R}) \to \mathrm{MT}(T)(\boldsymbol{R})$ is a 2 : 1 cover (over $\boldsymbol{C}$, and with A and X general, we have that it is the map $SL(4) \to SO(6) \cong SL(4)/\langle \pm\mathrm{Id}\rangle$) given by the action of $\mathrm{MT}(X)(\boldsymbol{R})$ on $\bigwedge^2 H^1(X, \boldsymbol{R})$. We have then the diagram

$$\begin{array}{lcll} h_+ : \boldsymbol{C}^* & \to & \mathrm{MT}(X)(\boldsymbol{R})/\langle \pm\mathrm{Id}\rangle = \mathrm{MT}(T)(\boldsymbol{R}) & \subset GL(H^2(X, \boldsymbol{R})) \\ & & \uparrow & \\ h_X : \boldsymbol{C}^* & \to & \mathrm{MT}(X)(\boldsymbol{R}) & \subset GL(H^1(X, \boldsymbol{R}))\,. \end{array}$$

The complex structure J_X on X is given by $h_X(i)$, which lies over $h_+(i)$, so we have two possible choices for J_X, J and $-J$. We repeat now the same argument using the inclusion

$T \subset H^2(A, \boldsymbol{Q})$; from $\mathrm{MT}(X) = \mathrm{MT}(A)$ (see [vG2]) we obtain the diagram

$$\begin{array}{llll} h_+ : \boldsymbol{C}^* & \to & \mathrm{MT}(X)(\boldsymbol{R})/\langle \pm \mathrm{Id} \rangle = \mathrm{MT}(T)(\boldsymbol{R}) & \subset GL(H^2(A, \boldsymbol{R})) \\ & & \uparrow & \\ h_A : \boldsymbol{C}^* & \to & \mathrm{MT}(A)(\boldsymbol{R}) = \mathrm{MT}(X)(\boldsymbol{R}) & \subset GL(H^1(A, \boldsymbol{R})) \end{array}$$

where h_A gives the Hodge structure on $H^1(A, \boldsymbol{Q})$. Then, for the complex structure on A, $J_A = h_A(i)$, we have the same two choices J and $-J$ as X since also $h_A(i)$ lies over $h_+(i)$. As A and X are of Weil type, we can identify the K-vector spaces $H^1(A, \boldsymbol{Q})$ and $H^1(X, \boldsymbol{Q})$, and thus $J_A = \pm J_X$. Since the polarization on an abelian fourfold of Weil type with discriminant equal to one is unique (see [vG2]), we have $E_X = E_A$. From the "positivity condition" of a polarization we conclude that A and X have the same complex structure. □

7. Higher dimensions. Now, we generalize the result of Theorem 6.4 and show that there exist other cases in which the Kuga-Satake construction gives abelian varieties of Weil type with discriminant one.

Let (V, h, Q) be a rational polarized weight 2 Hodge structure of $\dim_{\boldsymbol{Q}} V = n = 2m$, type $(1, n-2, 1)$, and let $Q = -\psi$ given by $Q = d_1X_1^2 + d_2X_2^2 - d_3X_3^2 - \ldots - d_nX_n^2$ ($d_i \in \boldsymbol{Q}_{>0}$) in a basis $\{g_1, \ldots, g_n\}$. Let $-d := (-1)^m d_1 \cdots d_n$ and $C := C_n$ be the Clifford algebra associated to (V, Q). We consider the case $n \equiv 2 \pmod 4$ equivalent to (see [vG1, Theorem 7.7]) $C^+ \cong M_{2^{m-1}}(\boldsymbol{Q}(\sqrt{-d}))$ and construct the Kuga-Satake variety A_0 associated to such a Hodge structure. Since $\mathrm{End}(A_0) \cong C^+ \cong M_{2^{m-1}}(\boldsymbol{Q}(\sqrt{-d}))$, we have from Poincare's theorem that $A_0 \sim A^{2^{m-1}}$, where A is an abelian variety with $\mathrm{End}(A) \cong \boldsymbol{Q}(\sqrt{-d})$. Obviously, we have that $\dim_{\boldsymbol{R}} C^+_{\boldsymbol{R}} = \dim_{\boldsymbol{Q}} C^+ = 2^{2m-1}$, and therefore $\dim(A_0) = \dim_{\boldsymbol{C}} C^+_{\boldsymbol{R}} = 2^{2m-2}$ and $\dim_{\boldsymbol{C}}(A) = 2^{m-1}$.

7.1. THEOREM. *Let (V, h, ψ) be a rational polarized weight 2 Hodge structure with dimension $\dim_{\boldsymbol{Q}} V = n = 2m$, type $(1, n-2, 1)$ and let $-d := (-1)^m d_1 \cdots d_n$. Then, if $n \equiv 2 \pmod 4$, A is an abelian variety of Weil type over $K = \boldsymbol{Q}(\sqrt{-d})$.*

PROOF. First, we consider the case in which $J = (1/\sqrt{d_1d_2})g_1g_2$ is the complex structure on $C^+_{\boldsymbol{R}}$. Let $z = g_1 \cdots g_n$. Then the element z is in the center of C^+ and we have

$$\begin{aligned} z^2 &= g_1 \cdots g_n g_1 \cdots g_n &&= (-1)^{(n-1)+(n-2)\cdots 1} g_1^2 \cdots g_n^2 \\ &= (-1)^{m(2m-1)} d_1 \cdots d_n &&= -d \end{aligned}$$

(since $2m-1$ is odd). Let $x \in C^+$ a generic element of the even subalgebra. We then define a map "left multiplication for x" by setting

$$l_x : C^+_{\boldsymbol{R}} \to C^+_{\boldsymbol{R}}, \quad v \mapsto xv.$$

The maps l_J and l_z are injective (e.g., $l_z(w_1) = l_z(w_2) \Rightarrow l_z^2(w_1) = l_z^2(w_2) \Rightarrow w_1 = w_2$) and both have complex eigenvalues. Therefore we have that, for all $v \in C^+_{\boldsymbol{R}}$, $l_J(v) \notin \langle v \rangle$ and $l_z(v) \notin \langle v \rangle$. Let $\boldsymbol{H} = \boldsymbol{Q}(J, z)$; $\boldsymbol{H}$ is an extension of degree 4 over $\boldsymbol{Q}$ and there are the obvious isomorphisms $H^1(A_0, \boldsymbol{R}) \cong C^+_{\boldsymbol{R}} \cong \boldsymbol{H}^{2m-3}$. Using these isomorphisms, we can construct a basis of $H^1(A_0, \boldsymbol{R})$ in the following way: let $\tilde{g}_1 = 1 \in C^+$. Using the "left multiplications", we obtain J, z and Jz (these elements are independent).

Now we choose an element $\tilde{g}_2$ not contained in the span of the previous elements and continue the construction. In this way we find the basis

$$\{\tilde{g}_1 = 1, J, z, Jz, \dots, \tilde{g}_t, J\tilde{g}_t, z\tilde{g}_t, Jz\tilde{g}_t\} \quad (t = 2^{2m-1}/4 = 2^{2m-3}).$$

It is easy to show that

$$\{i\tilde{g}_1 + J\tilde{g}_1, iz\tilde{g}_1 + Jz\tilde{g}_1, \dots, i\tilde{g}_t + J\tilde{g}_t, iz\tilde{g}_t + Jz\tilde{g}_t\}$$

is a basis of $H^{1,0}(A_0)$ (the i-eigenspace of J), and we can compute directly the action of z on this space:

$$\begin{aligned} z(ig_j + Jg_j) &= izg_j + Jzg_j \\ z(izg_j + Jzg_j) &= d(ig_j + Jg_j) \end{aligned}, \quad \text{and hence } z \cong \begin{pmatrix} 0 & -d & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & -d \\ & & & 1 & 0 \end{pmatrix}.$$

Since every

$$\begin{pmatrix} 0 & -d \\ 1 & 0 \end{pmatrix} \text{ can be diagonalized as } \begin{pmatrix} \sqrt{-d} & 0 \\ 0 & -\sqrt{-d} \end{pmatrix},$$

we see that A is of Weil type.

Now, we show that this result holds for all complex structures. Let $C_{\mathbf{C}}^+ = V_+ \oplus V_-$ be the decomposition of $C_{\mathbf{C}}^+$ in the $(\sqrt{-d})$- and $(-\sqrt{-d})$-eigenspaces of z. We observe that J commutes with z (z is contained in the center), so l_J respects this decomposition and it has t eigenvalues i and t eigenvalues $\bar{i}$ on each component (indeed, A is of Weil type and the condition "l_z has type (t, t) on $H^{1,0}(X)$" is obviously equivalent to the condition "l_J has type (t, t) on V_+"). Moreover, every complex structure can be written as gJg^{-1} with $g \in Spin(Q)$; we have $gz = zg$, so l_g respects the decomposition of $C_{\mathbf{C}}^+$, and therefore $l_{gJg^{-1}}$ respects this decomposition. Since the matrices of l_J and $l_{gJg^{-1}}$ have the same eigenvalues, $l_{gJg^{-1}}$ has type (t, t) and A is of Weil type for every complex structure. □

7.2. Discriminant. Let $E(x, y) = \mathrm{Tr}(\alpha\iota(x)y)$ be the polarization of A. We want to show that $\mathrm{discr}(A, \mathbf{Q}(\sqrt{-d}), E) = 1$.

7.3. THEOREM. *There exists a $\mathbf{Q}(\sqrt{-d})$ basis of $H_1(A, \mathbf{Q})$ in which*

$$H \cong \mathrm{diag}(1, \dots, 1, -1, \dots, -1).$$

PROOF. Using the decomposition

$$C^+ = (C_{n-2}^+ \oplus g_{n-1}C_{n-2}^+) \otimes_{\mathbf{Q}} \mathbf{Q}(z),$$

it is possible to construct a $\mathbf{Q}(z)$-basis $\{e_1, \dots, e_h\}$ (where $h = 2^{n-2}$) of $H_1(A, \mathbf{Q})$ in such a way that g_n does not appear in the elements e_i (as complex structure we can use, for example, the element J defined in 5.1). Using the Frobenius theorem, we can change the basis $\{e_i\}$ to a basis $\{\tilde{e}_i\}$ in which the matrix of the polarization has the form

$$E \sim \begin{pmatrix} 0 & a_1 & & & \\ -a_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & a_h \\ & & & -a_h & 0 \end{pmatrix} \quad (a_1 \in \boldsymbol{Q}).$$

Now, we decompose the even Clifford subalgebra as $C^+ = \boldsymbol{Q} \oplus C_0^+$; from [vG1] we have that $\mathrm{Ker}(\mathrm{Tr}) = C_0^+$. Then, we can choose the element $\alpha \in C_n^+$ of the polarization without terms containing g_n. Indeed, $\iota(x)y$ does not contain g_n, and if $\alpha = \alpha_1 + g_n\alpha_2$, we have

$$\mathrm{Tr}(\alpha\iota(x)y) = \mathrm{Tr}(\alpha_1\iota(x)y) + \mathrm{Tr}(g_n\alpha_2\iota(x)y) = \mathrm{Tr}(\alpha_1\iota(x)y) + 0$$

($g_n\alpha_2\iota(x)y$ contains necessarily g_n, so it cannot be a coefficient), so the term $g_n\alpha_2$ is useless.

By definition, $H(x, y) = E(zx, y) + \sqrt{-d}E(x, y)$, and we have $E(z\tilde{e}_i, \tilde{e}_j) = \mathrm{Tr}(a\iota(z\tilde{e}_i)z\tilde{e}_j) = 0$ (the argument contains g_n). Then

$$H \cong \begin{pmatrix} 0 & \sqrt{-d}a_1 & & & \\ -\sqrt{-d}a_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & \sqrt{-d}a_h \\ & & & -\sqrt{-d}a_h & 0 \end{pmatrix}.$$

On a $\boldsymbol{Q}(\sqrt{-d})$-basis, this matrix can be transformed as

$$H \cong \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & 1 & 0 \end{pmatrix}$$

and we have $H \cong \mathrm{diag}(1, \dots, 1, -1, \dots, -1)$ as required. □

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