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BANDO-CALABI-FUTAKI CHARACTER OF COMPACT TORIC MANIFOLDS

Dedicated to Professor Tadao Oda on his sixtieth birthday

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Abstract. The Bando-Calabi-Futaki character of a compact Kähler manifold is an obstruction to the existence of Kähler metrics with constant scalar curvature, which is a generalization of the Futaki character of a Fano manifold. In this paper, we study the Bando-Calabi-Futaki character of a compact toric manifold. In particular, we shall prove that the Bando-Calabi-Futaki character of a compact toric manifold vanishes on the Lie algebra of the unipotent radical of the automorphism group.

1. Introduction. Let X be a compact connected r-dimensional complex manifold and $\eta \in H^2(X; \boldsymbol{R})$. We assume that η is *positive*, that is, there exists a Kähler metric g on X such that its Kähler form

$$\omega_g := \sqrt{-1}\sum_{i,j=1}^{r} g_{i\bar{j}}\, dz^i \wedge d\overline{z^j}$$

represents $2\pi\eta$ in the de Rham cohomology group $H^2_{DR}(X; \boldsymbol{R})$, where $(z^1, z^2, \dots, z^r)$ is a local holomorphic coordinate on X. We denote by $\mathrm{Aut}^\circ(X)$ the identity component of the group $\mathrm{Aut}(X)$ of holomorphic automorphisms of X, whose Lie algebra is identified with the Lie algebra $H^0(X; \mathcal{O}(T^{1,0}X))$ of holomorphic vector fields on X. Here $T^{1,0}X$ is the holomorphic vector bundle of tangent vectors of type $(1,0)$ on X. Recall that the Albanese map of X to the Albanese variety $\mathrm{Alb}(X)$ naturally induces a Lie group homomorphism

$$\alpha_X : \mathrm{Aut}^\circ(X) \to \mathrm{Aut}^\circ(\mathrm{Alb}(X)) \cong \mathrm{Alb}(X)\,.$$

Let G_X be the identity component of the kernel of the homomorphism α_X, and $\mathfrak{g}_X$ the corresponding Lie subalgebra of $H^0(X; \mathcal{O}(T^{1,0}X))$. Then, by a theorem of Fujiki [6], G_X has a natural structure of a linear algebraic group (defined over $\boldsymbol{C}$). We denote by U_X the unipotent radical of G_X. More generally, we consider a linear algebraic group G (defined over $\boldsymbol{C}$) and a homomorphism $\rho\colon G \to \mathrm{Aut}(X)$ of algebraic groups. By $\rho_*\colon \mathfrak{g} \to H^0(X; \mathcal{O}(T^{1,0}X))$, we denote the Lie algebra homomorphism induced from ρ, where $\mathfrak{g} := \mathrm{Lie}(G)$ is the Lie algebra of G.

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REMARK 1.1. (i) If X is Fano, i.e., the first Chern class $c_1(X)$ of X is positive, then $G_X = \mathrm{Aut}^\circ(X)$.

(ii) If X is an r-dimensional compact toric manifold, that is, X is an r-dimensional compact irreducible non-singular variety defined over $\boldsymbol{C}$ with an almost-homogeneous algebraic action of an r-dimensional algebraic torus $T_r := (\boldsymbol{C}^*)^r$, then $G_X = \mathrm{Aut}^\circ(X)$.

By Ric_g and s_g, we denote the *Ricci form* and the *scalar curvature* of g, respectively, namely, we put

$$\mathrm{Ric}_g = \sqrt{-1}\sum_{i,j=1}^{r} R_{i\overline{j}}\, dz^i \wedge d\overline{z^j} := -\sqrt{-1}\partial\overline{\partial}\log\det\bigl(g_{i\overline{j}}\bigr)\ ,$$

$$s_g := \sum_{i,j=1}^{r} g^{\overline{j}i} R_{i\overline{j}}\ ,$$

where $\bigl(g^{\overline{j}i}\bigr)$ is the inverse matrix of $\bigl(g_{i\overline{j}}\bigr)$. By means of the harmonic integration theory, there exists a real-valued C^∞ function $f \in C^\infty(X)_{\boldsymbol{R}}$ such that

$$s_g - r\mu_\eta = \Box_g f_g\,,$$

where $\Box_g := \sum_{i,j=1}^r g^{\overline{j}i}\bigl(\partial^2/\partial z^i\partial\overline{z^j}\bigr)$ is the complex Laplacian for functions on the Kähler manifold (X, g), and $\mu_\eta \in \boldsymbol{R}$ is the constant defined by

$$\mu_\eta := \frac{(c_1(X)\cup\eta^{r-1})[X]}{\eta^r[X]} = \frac{\displaystyle\int_X s_g\left(\frac{\omega_g}{2\pi}\right)^r}{\displaystyle r\int_X\left(\frac{\omega_g}{2\pi}\right)^r} \in \boldsymbol{R}\,. \tag{1.2}$$

Bando [2], Calabi [4] and Futaki [9] defined an obstruction to the existence of Kähler metrics with constant scalar curvature as follows:

DEFINITION 1.3 (Bando [2], Calabi [4] and Futaki [9]). A linear functional $F_X^\eta\colon H^0(X; \mathcal{O}(T^{1,0}X)) \to \boldsymbol{C}$ defined by

$$F_X^\eta(V) := \frac{1}{\sqrt{-1}}\int_X (Vf_g)\left(\frac{\omega_g}{2\pi}\right)^r, \quad V \in H^0(X; \mathcal{O}(T^{1,0}X))\,,$$

is called the *Bando-Calabi-Futaki character* of (X, η).

We now recall the following fundamental facts about the Bando-Calabi-Futaki characters:

FACT 1.4 (Bando [2], Calabi [4] and Futaki [9]). *Let X and η be as above. Then we have the following*:

(i) *F_X^η does not depend on the choice of g satisfying $[\omega_g] = 2\pi\eta$.*

(ii) *If X admits a Kähler metric g with constant scalar curvature satisfying $[\omega_g] = 2\pi\eta$, then F_X^η vanishes.*

(iii) *F_X^η is a Lie algebra character of $H^0(X;\mathcal{O}(T^{1,0}X))$, that is,*

$$F_X^\eta\,|_{[H^0(X;\mathcal{O}(T^{1,0}X)),H^0(X;\mathcal{O}(T^{1,0}X))]} \equiv 0\,.$$

REMARK 1.5. If η is the first Chern class $c_1(X)$ of X, then the Bando-Calabi-Futaki character $F_X^{c_1(X)}$ coincides with the original Futaki character, which was introduced in [8] as an obstruction to the existence of Einstein-Kähler metrics.

DEFINITION 1.6. Let $\pi_E\colon E\to X$ be a holomorphic vector bundle of rank k over X. We say that E is (G,ρ)-*linearized* if G acts on E biregularly in such a way that

(i) $\pi_E\circ\gamma=\rho(\gamma)\circ\pi_E$ for any $\gamma\in G$;

(ii) for any $\gamma\in G$ and $p\in X$,

$$\gamma|_{E_p}\colon E_p\to E_{\rho(\gamma)(p)}$$

is a $\boldsymbol{C}$-linear map, where $E_p:=\pi_E^{-1}(p)$ is the fiber of π_E at $p\in X$.

Furthermore, if G is a subgroup of $\mathrm{Aut}(X)$ and ρ is the inclusion map, then we simply say that E is G-*linearized*.

In [15], the author proved the following:

FACT 1.7 (Nakagawa [15]). *Let X and η be as above. We assume that there exists a holomorphic line bundle L over X such that L is G_X-linearized and $c_1(L)=\eta$, where $c_1(L)$ is the first Chern class of L. Then*

$$F_X^\eta\,|_{\mathfrak{u}_X}\equiv 0\,,$$

where $\mathfrak{u}_X:=\mathrm{Lie}(U_X)$ is the Lie algebra of U_X.

The main purpose of this paper is to generalize this fact to the case of a more general situation, that is, we shall prove the following theorem:

THEOREM 1.8. *Let X, η, G and ρ be as above. We assume that there exists a holomorphic line bundle L over X such that L is (G,ρ)-linearized and $c_1(L)=\eta$. Then*

$$(F_X^\eta\circ\rho_*)|_{\mathfrak{u}}\equiv 0$$

for any unipotent subgroup $U\subseteqq G$ with Lie algebra $\mathfrak{u}:=\mathrm{Lie}(U)$.

As an application of this theorem, we shall also prove the following theorem:

THEOREM 1.9. *Let X be an r-dimensional compact toric manifold. By definition, an r-dimensional algebraic torus $T_r:=(\boldsymbol{C}^*)^r$ acts on X biholomorphically; hence the Lie algebra $\mathfrak{t}_r:=\mathrm{Lie}(T_r)$ of T_r is regarded as a Lie subalgebra of $H^0(X;\mathcal{O}(T^{1,0}X))$. If $\eta\in H^2(X;\boldsymbol{Z})$ is positive, then the following are equivalent, without any assumptions concerning a linearization of the natural action of $\mathrm{Aut}(X)$ on X:*

(i) *F_X^η vanishes identically on $H^0(X;\mathcal{O}(T^{1,0}X))$.*

(ii) *F_X^η vanishes on $\mathfrak{t}_r$.*

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2. Bando-Calabi-Futaki characters as holomorphic invariants (Proof of Theorem 1.8). Throughout this section, we fix a compact connected r-dimensional complex manifold X, a positive class $\eta \in H^2(X;\boldsymbol{R})$, a linear algebraic group G (defined over $\boldsymbol{C}$) and a homomorphism $\rho\colon G \to \mathrm{Aut}(X)$ of algebraic groups. Let $\pi_E\colon E \to X$ be a holomorphic vector bundle of rank k over X. We assume that E is (G,ρ)-linearized. Then, for any $V \in \mathfrak{g}$, a *holomorphic action* (see [3])

$$\Lambda_V^E\colon A^0(E) \to A^0(E)\,,$$

of V on E is induced, that is, Λ_V^E satisfies the following properties:

(i) Λ_V^E is a $\boldsymbol{C}$-linear map.

(ii) For all $\psi \in C^\infty(X)_{\boldsymbol{C}}$ and $s \in A^0(E)$,

$$\Lambda_V^E(\psi s) = ((\rho_* V)\psi)s + \psi \Lambda_V^E s.$$

(iii) Λ_V^E commutes with $\overline{\partial}$, i.e., $\overline{\partial}\Lambda_V^E = \Lambda_V^E\,\overline{\partial}$.

Here we denote by $A^p(E)$ the space of E-valued p-forms on X for $p = 0, 1, \ldots, r$.

EXAMPLE 2.1. $E = T^{1,0}X$ is canonically $\mathrm{Aut}(X)$-linearized. In this case, $\Lambda_V^{T^{1,0}X}$ is the Lie differentiation L_V of vector fields with respect to a holomorphic vector field $V \in H^0(X;\mathcal{O}(T^{1,0}X))$ on X.

Let h be a Hermitian metric on E and $\nabla^h\colon A^0(E) \to A^1(E)$ the Hermitian connection of h (see for instance [12, p. 12]). We define the curvature Θ_h of ∇^h by

$$\Theta_h := \overline{\partial}(h^{-1}\partial h) \in A^2(\mathrm{End}(E))\,,$$

where $\mathrm{End}(E)$ is the endomorphism bundle of E over X. For each $V \in \mathfrak{g}$, we put $\mathcal{L}_V^{(E,h)} := \nabla^h_{\rho_* V} - \Lambda_V^E \in A^0(\mathrm{End}(E))$. Let $l \in \boldsymbol{Z}_{\geqq 0}$ be a non-negative integer and ϕ a $GL(k,\boldsymbol{C})$-invariant symmetric polynomial of degree $r+l$ on $\mathfrak{gl}(k,\boldsymbol{C})$ (see [10, p. 21]). For example, $c_1^{r+l} := ((\sqrt{-1}/2\pi)\,\mathrm{tr})^{r+l}$ is a $GL(k,\boldsymbol{C})$-invariant symmetric polynomial of degree $r+l$ on $\mathfrak{gl}(k,\boldsymbol{C})$. We now define a map $\mathcal{C}_E^\phi\colon \mathfrak{g} \to \boldsymbol{C}$ by

$$\mathcal{C}_E^\phi(V) := \int_X \phi\big(\mathcal{L}_V^{(E,h)} + \Theta_h\big)\,, \qquad V \in \mathfrak{g}\,.$$

For this map $\mathcal{C}_E^\phi$, we can prove the following facts:

FACT 2.2 (cf. Futaki and Morita [11]). *Let X, (E,h) and ϕ be as above. Then we have the following*:

(i) *$\mathcal{C}_E^\phi$ dose not depend on the choice of a Hermitian metric h on E, i.e., $\mathcal{C}_E^\phi$ is a holomorphic invariant of (X,E).*

(ii) *$\mathcal{C}_E^\phi$ is a G-invariant symmetric polynomial of degree l on $\mathfrak{g}$. In particular, if $l = 1$, then $\mathcal{C}_E^\phi$ is a character of the Lie algebra $\mathfrak{g}$.*

(iii) *For any* $V \in H^0(X; \mathcal{O}(T^{1,0}X))$,

$$F_X^{c_1(X)}(V) = -\frac{2\pi}{r+1}\mathcal{C}_{T^{1,0}X}^{c_1^{r+1}}(V) = -\frac{2\pi}{r+1}\mathcal{C}_{K_X^{-1}}^{c_1^{r+1}}(V)\,,$$

where $T^{1,0}X$ *and* $K_X^{-1} := \det T^{1,0}X = \bigwedge^r T^{1,0}X$ *are regarded as* $\mathrm{Aut}(X)$*-linearized bundles over* X *in terms of the canonical* $\mathrm{Aut}(X)$*-actions on them.*

Let g' be an arbitrary Hermitian metric on X. For $V \in \mathfrak{g}$, if a point $p \in X$ is a zero point of $\rho_* V \in H^0(X; \mathcal{O}(T^{1,0}X))$, then $\mathcal{L}_{\rho * V}^{(T^{1,0}X, g')}$ induces the linear map

$$\mathcal{L}_{\rho_* V, p}^{(T^{1,0}X, g')} = -(L_{\rho_* V})_p \colon T_p^{1,0}X \to T_p^{1,0}X\,.$$

$V \in \mathfrak{g}$ is said to be *non-degenerate* if the following two conditions hold:

(i) The zero set $\mathrm{Zero}(\rho_* V)$ of $\rho_* V$ is finite.

(ii) For each zero point $p \in \mathrm{Zero}(\rho_* V)$ of $\rho_* V$, the linear map

$$\mathcal{L}_{\rho_* V, p}^{(T^{1,0}X, g')} \colon T_p^{1,0}X \to T_p^{1,0}X$$

is non-singular.

The following localization formula for $\mathcal{C}_E^{\phi}$ allows us to calculate explicitly the Bando-Calabi-Futaki character of a compact toric manifold (see Corollary 4.6):

FACT 2.3 (Bott [3]). *Let* X, (E, h) *and* ϕ *be as above, and* $V \in \mathfrak{g}$ *a non-degenerate element. Then we have*

$$\mathcal{C}_E^{\phi}(V) = \sum_{p \in \mathrm{Zero}(V)} \frac{\phi\big(\mathcal{L}_{V,p}^{(E,h)}\big)}{\det \dfrac{\sqrt{-1}}{2\pi}\mathcal{L}_{\rho_* V, p}^{(T^{1,0}X, g')}}\,,$$

where g' *is an arbitrary Hermitian metric on* X.

Now, we assume that there exists a holomorphic line bundle L over X such that L is (G, ρ)-linearized and $c_1(L) = \eta$. Under this assumption, an argument similar to that in [17, Section 6] allows us to prove the following Tian's formula for the Bando-Calabi-Futaki character (see also [15, Section 3]):

THEOREM 2.4 (Tian [17]). *Let* X, η, G, ρ *and* L *be as above. Then, for any integer* $\delta \in \mathbf{Z}$ *and* $V \in \mathfrak{g}$, *we have*

$$\begin{aligned} F_X^{\eta}(\rho_* V) = &- \frac{2\pi}{2^r(r+1)!}\sum_{j=0}^{r}(-1)^j \binom{r}{j} \mathcal{C}_{K_X^{-1}\otimes L^{\delta+r-2j}}^{c_1^{r+1}}(V) \\ &+ 2\pi\left(\delta + \frac{r\mu_\eta}{r+1}\right)\mathcal{C}_L^{c_1^{r+1}}(V)\,, \end{aligned}$$

where $L^{\delta+r-2j} := L^{\otimes(\delta+r-2j)}$ *is the* $(\delta + r - 2j)$*-th tensor power of* L. *Here we regard* $K_X^{-1} \otimes L^{\delta+r-2j}$, $j = 0, 1, \ldots, r$, *as* (G, ρ)*-linearized line bundles by the canonical* $\mathrm{Aut}(X)$*-action on* K_X^{-1}.

Together with this Tian's formula, the following fact implies Theorem 1.8 by the same argument as that in [15, Section 4]:

FACT 2.5 (Mabuchi [13]). *Let X, G, ρ and L be as above. Then, for any unipotent subgroup U of G, $\mathcal{C}_L^{c_1^{n+1}}$ vanishes on the Lie algebra $\mathfrak{u} := \mathrm{Lie}(U)$ of U.*

3. Bando-Calabi-Futaki character of compact toric manifolds (Proof of Theorem 1.9). First, we recall some basic notions and facts concerning toric manifolds (see [16] for more details). Let $T_r := (\boldsymbol{C}^*)^r$ be an r-dimensional algebraic torus. We put $N := \boldsymbol{Z}^r$ and $M := \mathrm{Hom}_{\boldsymbol{Z}}(N, \boldsymbol{Z})(\cong \boldsymbol{Z}^r)$, where we regard elements of N and M as r-dimensional column vectors and row vectors, respectively. Let Σ be a *complete non-singular fan* in N (see [16] for the definition of a complete non-singular fan) and $\Sigma(i)$ the set of i-dimensional cones in Σ for $i = 0, 1, \ldots, r$. We denote by X_Σ the r-dimensional compact toric manifold associated with Σ. Then T_r acts on X_Σ biholomorphically, and X_Σ has an open dense T_r-orbit $\mathfrak{O}_\Sigma$ isomorphic to T_r.

FACT 3.1 (Cox [5]). *Let Σ be a complete non-singular fan in N and $d_\Sigma := \#\Sigma(1)$ the number of the one-dimensional cones in Σ. Then*:

(i) *There exists a $(d_\Sigma - r)$-dimensional algebraic subtorus H_Σ of $(\boldsymbol{C}^*)^{d_\Sigma}$ and an H_Σ-invariant open subset $\mathcal{W}_\Sigma$ of $\boldsymbol{C}^{d_\Sigma}$ such that H_Σ acts freely on $\mathcal{W}_\Sigma$ and*

$$X_\Sigma = \mathcal{W}_\Sigma / H_\Sigma \,.$$

Here the H_Σ-action on $\boldsymbol{C}^{d_\Sigma}$ is induced from the canonical $(\boldsymbol{C}^*)^{d_\Sigma}$-action on $\boldsymbol{C}^{d_\Sigma}$.

(ii) *Let $\widetilde{G}_\Sigma$ be the centralizer of H_Σ in $\mathrm{Aut}(\mathcal{W}_\Sigma)$. Then*

$$\mathrm{Aut}^\circ(X_\Sigma) \cong \widetilde{G}_\Sigma / H_\Sigma \,.$$

(iii) *$\widetilde{G}_\Sigma$ and $\mathrm{Aut}^\circ(X_\Sigma)$ are connected linear algebraic groups (defined over $\boldsymbol{C}$). Let $\widetilde{U}_\Sigma$ and U_Σ be the unipotent radicals of $\widetilde{G}_\Sigma$ and $\mathrm{Aut}^\circ(X_\Sigma)$, respectively. Then*

$$\rho_\Sigma|_{\widetilde{U}_\Sigma} : \widetilde{U}_\Sigma \to U_\Sigma$$

is an isomorphism, where $\rho_\Sigma : \widetilde{G}_\Sigma \to \mathrm{Aut}^\circ(X_\Sigma)$ is the natural projection induced by the isomorphism $\mathrm{Aut}^\circ(X_\Sigma) \cong \widetilde{G}_\Sigma / H_\Sigma$. Furthermore, there exists a reductive algebraic subgroup R_Σ of $\mathrm{Aut}^\circ(X_\Sigma)$ with T_r as a maximal algebraic torus such that

$$\mathrm{Aut}^\circ(X_\Sigma) = R_\Sigma \ltimes U_\Sigma \,.$$

EXAMPLE 3.2. A typical example of an r-dimensional compact toric manifold is the r-dimensional complex projective space $\boldsymbol{P}^r(\boldsymbol{C})$. If $X_\Sigma = \boldsymbol{P}^r(\boldsymbol{C})$, then we have:

$$\begin{aligned}
&d_\Sigma = r + 1 \,, \\
&H_\Sigma = \{(t, t, \ldots, t) \in (\boldsymbol{C}^*)^{r+1} ;\ t \in \boldsymbol{C}^*\} \cong \boldsymbol{C}^* \,, \\
&\mathcal{W}_\Sigma = \boldsymbol{C}^{r+1} \setminus \{0\} \,, \\
&\widetilde{G}_\Sigma = GL(r + 1, \boldsymbol{C}) \,, \\
&\mathrm{Aut}^\circ(X_\Sigma) = \mathrm{Aut}(X_\Sigma) = PGL(r + 1, \boldsymbol{C}) \,.
\end{aligned}$$

To each $\nu \in \Sigma(1)$, there corresponds a T_r-invariant Weil divisor D_ν on X_Σ. More generally, a map $\alpha\colon \Sigma(1) \to \boldsymbol{Z}$ defines a T_r-invariant Weil divisor $D(\alpha) := -\sum_{\nu\in\Sigma(1)} \alpha(\nu) D_\nu$, and we denote by L_α the T_r-linearized holomorphic line bundle over X_Σ corresponding to $D(\alpha)$, i.e., $L_\alpha = \mathcal{O}(D(\alpha))$.

EXAMPLE 3.3. Let Σ be a complete non-singular fan in N and X_Σ the compact toric manifold associated with Σ. Then the anti-canonical line bundle $K_{X_\Sigma}^{-1}$ of X_Σ corresponds to the map

$$\alpha_0\colon \Sigma(1) \ni \nu \mapsto -1 \in \boldsymbol{Z},$$

that is, $K_{X_\Sigma}^{-1}$ corresponds to the T_r-invariant Weil divisor $\sum_{\nu\in\Sigma(1)} D_\nu$.

If L_α is ample, that is, $c_1(L_\alpha) \in H^2(X_\Sigma; \boldsymbol{Z})$ is positive, then we say that α is *ample*. Let $\Sigma(1) = \{\nu_1, \nu_2, \ldots, \nu_{d_\Sigma}\}$ and put $\alpha_i := \alpha(\nu_i) \in \boldsymbol{Z}$ for $i = 1, 2, \ldots, d_\Sigma$. Then we define a character $\lambda_\alpha\colon (\boldsymbol{C}^*)^{d_\Sigma} \to \boldsymbol{C}^*$ of $(\boldsymbol{C}^*)^{d_\Sigma}$ by $\lambda_\alpha(s_1, s_2, \ldots, s_{d_\Sigma}) := s_1^{\alpha_1} s_2^{\alpha_2} \cdots s_{d_\Sigma}^{\alpha_{d_\Sigma}}$. H_Σ acts on $\mathcal{W}_\Sigma \times \boldsymbol{C}$ by

$$k\colon (z, \xi) \mapsto (k\cdot z, \lambda_\alpha(k)^{-1}\xi),$$

where $k \in H_\Sigma$, $z \in \mathcal{W}_\Sigma$ and $\xi \in \boldsymbol{C}$.

FACT 3.4 (cf. Audin [1, Chapter VI]). The projection $\mathcal{W}_\Sigma \longrightarrow X_\Sigma$ is a principal H_Σ-bundle. Furthermore, the T_r-linearized holomorphic line bundle L_α over X_Σ is given by

$$L_\alpha = \mathcal{W}_\Sigma \times_{\lambda_\alpha} \boldsymbol{C} := (\mathcal{W}_\Sigma \times \boldsymbol{C})/H_\Sigma.$$

PROPOSITION 3.5. *For α as above, L_α is the $(\widetilde{G}_\Sigma, \rho_\Sigma)$-linearized holomorphic line bundle over X_Σ.*

PROOF. The natural $\widetilde{G}_\Sigma$-action on $\mathcal{W}_\Sigma$ commutes with the H_Σ-action on $\mathcal{W}_\Sigma$. Then, by means of Fact 3.4, $\widetilde{G}_\Sigma$ acts on $L_\alpha = \mathcal{W}_\Sigma \times_{\lambda_\alpha} \boldsymbol{C}$ and L_α is $(\widetilde{G}_\Sigma, \rho_\Sigma)$-linearized. □

For any $\eta \in H^2(X_\Sigma; \boldsymbol{Z})$, in view of [7, Section 3.4], there exists a map $\alpha_\eta\colon \Sigma(1) \to \boldsymbol{Z}$ such that $c_1(L_{\alpha_\eta}) = \eta$. Therefore, Theorem 1.8 together with Fact 3.1 (iii) and Proposition 3.5 implies the following theorem:

THEOREM 3.6. *Let Σ be a complete non-singular fan in N and $\eta \in H^2(X_\Sigma; \boldsymbol{Z})$ a positive class. Then the Bando-Calabi-Futaki character $F_{X_\Sigma}^{\eta}$ of (X_Σ, η) vanishes on the Lie algebra $\mathfrak{u}_\Sigma := \mathrm{Lie}(U_\Sigma)$ of U_Σ.*

Recall that, for a reductive algebraic group R,

$$\mathrm{Lie}(R) = \mathrm{Lie}(\mathrm{Center}(R)) + [\mathrm{Lie}(R), \mathrm{Lie}(R)], \tag{3.7}$$

and $\mathrm{Lie}(\mathrm{Center}(R)) \subsetneqq \mathrm{Lie}(T)$ for every maximal algebraic torus T of R, where $\mathrm{Center}(R)$ is the center of R. Since R_Σ is reductive, Theorem 3.6 together with Fact 3.1 (iii) and (3.7) immediately implies Theorem 1.9.

4. A combinatorial formula for the Bando-Calabi-Futaki character of compact toric manifolds. In [14], the author established a combinatorial formula for the Futaki character of a toric Fano manifold. In this section, we shall also establish a combinatorial formula for the Bando-Calabi-Futaki character of a compact toric manifold by the same argument as in [14].

Throughout this section, we fix a complete non-singular fan Σ in $N := \boldsymbol{Z}^r$ and a positive class $\eta \in H^2(X_\Sigma; \boldsymbol{Z})$. We shall use the same notation as that in Section 3.

We define a basis $\{\tau_i := t^i(\partial/\partial t^i);\ i = 1, 2, \dots, r\}$ of the Lie algebra $\mathfrak{t}_r$ of T_r, where $(t^1, t^2, \dots, t^r)$ is the standard coordinate for $T_r = \left(\boldsymbol{C}^*\right)^r$. Note that we can regard $\mathfrak{t}_r$ as a complex Lie subalgebra of $H^0(X_\Sigma; \mathcal{O}(T^{1,0}X_\Sigma))$. For each $\sigma \in \Sigma(r)$ and $S \in GL(r, \boldsymbol{C})$, let

$$a_1(\sigma) = \begin{pmatrix} a_1^1(\sigma) \\ a_1^2(\sigma) \\ \vdots \\ a_1^r(\sigma) \end{pmatrix}, \dots, a_r(\sigma) = \begin{pmatrix} a_r^1(\sigma) \\ a_r^2(\sigma) \\ \vdots \\ a_r^r(\sigma) \end{pmatrix} \in N$$

be the generator of σ. We put

$$\begin{aligned} A(\sigma) &:= (a_1(\sigma), \quad a_2(\sigma), \quad \dots, \quad a_r(\sigma)) \\ &= \begin{pmatrix} a_1^1(\sigma) & a_2^1(\sigma) & \dots & a_r^1(\sigma) \\ a_1^2(\sigma) & a_2^2(\sigma) & \dots & a_r^2(\sigma) \\ \vdots & \vdots & \dots & \vdots \\ a_1^r(\sigma) & a_2^r(\sigma) & \dots & a_r^r(\sigma) \end{pmatrix} \in GL(r, \boldsymbol{Z}) \end{aligned}$$

and $Q(S; \sigma) = (q_j^i(S; \sigma)) := A(\sigma)^{-1}S \in GL(r, \boldsymbol{C})$. A non-singular matrix $S \in GL(r, \boldsymbol{C})$ is said to be *non-degenerate* if S satisfies $q_j^i(S; \sigma) \neq 0$ for all $i, j = 1, 2, \dots, r$, and $\sigma \in \Sigma(r)$.

EXAMPLE 4.1. For example, a non-singular matrix

$$S_0 := \begin{pmatrix} 1 & 1 & \dots & 1 \\ \pi & \pi^2 & \dots & \pi^r \\ \pi^2 & \pi^4 & \dots & \pi^{2r} \\ \vdots & \vdots & \dots & \vdots \\ \pi^{r-1} & \pi^{2(r-1)} & \dots & \pi^{r(r-1)} \end{pmatrix} \in GL(r, \boldsymbol{C})$$

is non-degenerate.

For $S = (s_i^j) \in GL(r, \boldsymbol{C})$ and $i = 1, 2, \dots, r$, we define a holomorphic vector field $V_i(S) := \sum_{i=1}^r s_i^j \tau_j$ on X_Σ. Then $\{V_i(S)\,;\ i = 1, 2, \dots, r\}$ is a basis of $\mathfrak{t}_r$. For a map $\alpha \colon \Sigma(1) \to \boldsymbol{Z}$, we define constants $\beta_i(S; \sigma, \alpha)$, $i = 1, 2, \dots, r$, by

$$\beta_i(S; \sigma, \alpha) := \sum_{j=1}^r \alpha(\langle a_j(\sigma) \rangle) q_i^j(S; \sigma)\,,$$

where $\langle a_i(\sigma) \rangle \in \Sigma(1)$ is the one-dimensional cone generated by $a_i(\sigma) \in N$. We put $b_i(\sigma, \alpha) := \beta(I_r; \sigma, \alpha)$, where $I_r \in GL(r, \boldsymbol{C})$ is the identity matrix.

In terms of the notation as above, we can establish the following combinatorial formula for $\mathcal{C}_{L_\alpha}^{c_1^{r+l}}(V_i(S))$:

THEOREM 4.2. *Let X_Σ be an r-dimensional compact toric manifold associated with a complete non-singular fan Σ and $S \in GL(r, \boldsymbol{C})$ a non-degenerate non-singular matrix. Then we have*

$$\mathcal{C}_{L_\alpha}^{c_1^{r+l}}(V_i(S)) = \left(\frac{\sqrt{-1}}{2\pi}\right)^l \sum_{\sigma \in \Sigma(r)} \frac{\beta_i(S;\sigma,\alpha)^{r+l}}{\prod_{j=1}^{r} q_i^j(S;\sigma)}$$

for any $\alpha\colon \Sigma(1) \to \boldsymbol{Z}$, $l \in \boldsymbol{Z}_{\geqq 0}$ and $i = 1, 2, \ldots, r$, where we regard L_α as a T_r-linearized holomorphic line bundle over X_Σ.

PROOF. For each $\sigma \in \Sigma(r)$, there exists a T_r-invariant open subset W_σ of X_Σ such that $W_\sigma \cong \boldsymbol{C}^r$ and $X_\Sigma = \bigcup_{\sigma \in \Sigma(r)} W_\sigma$. Let $(t^1, t^2, \ldots, t^r)$ and $(z^1(\sigma), z^2(\sigma), \ldots, z^r(\sigma))$ be the coordinate systems on $\mathfrak{O}_\Sigma \cong T_r = (\boldsymbol{C}^*)^r$ and $W_\sigma \cong \boldsymbol{C}^r$, respectively. The following system of identities is the coordinate transformation between these coordinates:

$$t^i = \prod_{j=1}^{r} z^j(\sigma)^{a_j^i(\sigma)}, \qquad i = 1, 2, \ldots, r\,.$$

From these identities, for every $\sigma \in \Sigma(r)$ and $i = 1, 2, \ldots, r$, we have

$$V_i(S) = \sum_{k=1}^{r} q_i^k(S;\sigma) z^k(\sigma) \frac{\partial}{\partial z^k(\sigma)}$$

on W_σ. In view of this expression of $V_i(S)$ and the non-degeneracy of S, we obtain

$$\mathrm{Zero}(V_i(S)) = \{\text{the origin } o(\sigma) \text{ of } W_\sigma \cong \boldsymbol{C}^r\ ;\ \sigma \in \Sigma(r)\}$$

for $i = 1, 2, \ldots, r$. For each $\sigma \in \Sigma(r)$, the T_r-linearized holomorphic line bundle L_α over X_Σ is trivialized on W_σ. In terms of this trivialization, the T_r-action on $L_\alpha|_{W_\sigma} = W_\sigma \times \boldsymbol{C}$ is given by

$$t\colon W_\sigma \times \boldsymbol{C} \ni (z, \xi) \mapsto \left(t \cdot z, \prod_{i=1}^{r} (t^i)^{-b_i(\sigma,\alpha)} \xi\right) \in W_\sigma \times \boldsymbol{C}\,,$$

where $t = (t^1, t^2, \ldots, t^r) \in T_r$ (see [16, p. 69]). Hence, for $\sigma \in \Sigma(r)$ and $i = 1, 2, \ldots, r$, we have

$$\mathcal{L}_{V_i(S), o(\sigma)}^{(L_\alpha, h)} = \sum_{j=1}^{r} b_j(\sigma,\alpha) s_i^j = \beta_i(S;\sigma,\alpha)\,, \tag{4.3}$$

where h is an arbitrary Hermitian metric on L_α. Moreover we also have, for $\sigma \in \Sigma(r)$ and $i = 1, 2, \ldots, r$,

$$\mathcal{L}^{g'}_{V_i(S),o(\sigma)} = \begin{pmatrix} q_i^1(S;\sigma) & & & \\ & q_i^2(S;\sigma) & & 0 \\ & & \ddots & \\ 0 & & & q_i^r(S;\sigma) \end{pmatrix}, \tag{4.4}$$

with respect to a basis $\{(\partial/\partial z^1(\sigma))_{o(\sigma)}, \dots, (\partial/\partial z^r(\sigma))_{o(\sigma)}\}$ of $T^{1,0}_{o(\sigma)}X_P$, where g' is an arbitrary Hermitian metric on X_Σ. Together with (4.3) and (4.4), Fact 2.3 immediately implies the theorem. □

As a corollary of this theorem, we obtain the following:

COROLLARY 4.5. *Let X_Σ be an r-dimensional compact toric manifold associated with a complete non-singular fan Σ, $S \in GL(r, \boldsymbol{C})$ a non-degenerate non-singular matrix, $\alpha_a \colon \Sigma(1) \to \boldsymbol{Z}$, $a = 1, 2, \dots, k$, and $b_1, b_2, \dots, b_k \in \boldsymbol{N}$ with $b_1 + b_2 + \cdots + b_k = r$. Then we have*

$$\begin{aligned} &(c_1(L_{\alpha_1})^{b_1} \cup c_1(L_{\alpha_2})^{b_2} \cup \cdots \cup c_1(L_{\alpha_k})^{b_k})\,[X_\Sigma] \\ &= \sum_{\sigma\in\Sigma(r)} \frac{\prod_{a=1}^{k} \beta_i(S;\sigma,\alpha_a)^{b_a}}{\prod_{j=1}^{r} q_i^j(S;\sigma)} \end{aligned} \tag{4.5.1}$$

for any $i = 1, 2, \dots, r$. In particular, for $\alpha \colon \Sigma(1) \to \boldsymbol{Z}$ and $i = 1, 2, \dots, r$, we have

$$\mu_{c_1(L_\alpha)} = \frac{\displaystyle\sum_{\sigma\in\Sigma(r)} \frac{\beta_i(S;\sigma,\alpha)^{r-1} \sum_{j=1}^{r} q_i^j(S;\sigma)}{\prod_{j=1}^{r} q_i^j(S;\sigma)}}{\displaystyle\sum_{\sigma\in\Sigma(r)} \frac{\beta_i(S;\sigma,\alpha)^{r}}{\prod_{j=1}^{r} q_i^j(S;\sigma)}}. \tag{4.5.2}$$

PROOF. Applying Theorem 4.2 to $\mathcal{C}^{c_1^r}_{L_{\alpha_1}^{\lambda_1}\otimes L_{\alpha_2}^{\lambda_2}\otimes\cdots\otimes L_{\alpha_k}^{\lambda_k}}(V_i(S))$, we obtain

$$\sum_{b_1+\cdots+b_k=r}\frac{r!}{b_1!\cdots b_k!}\lambda_1^{b_1}\cdots\lambda_k^{b_k}(c_1(L_{\alpha_1})^{b_1}\cup\cdots\cup c_1(L_{\alpha_k})^{b_k})[X_\Sigma]$$
$$=\sum_{\sigma\in\Sigma(r)}\frac{\left(\sum_{a=1}^{k}\lambda_a\beta_i(S;\sigma,\alpha_a)\right)^r}{\prod_{j=1}^{r}q_i^j(S;\sigma)}.$$

By comparing the coefficients of $\lambda_1^{b_1}\lambda_2^{b_2}\cdots\lambda_k^{b_k}$ in the equation above, we obtain the formula (4.5.1). The formula (4.5.2) is straightforward from the definition (1.2) of μ_η and the formula (4.5.1). □

In view of Theorems 2.4 and 4.2 and Corollary 4.5 combined with the equalities

$$\sum_{j=0}^{r}(-1)^j\binom{r}{j}(r-2j)^k=\begin{cases}0 & \text{if } k=0,1,\ldots,r-1,r+1\,,\\ 2^r r! & \text{if } k=r\,,\end{cases}$$

we can prove the following combinatorial formula for the Bando-Calabi-Futaki character of a compact toric manifold:

COROLLARY 4.6. *Let X_Σ be an r-dimensional compact toric manifold associated with a complete non-singular fan Σ and $S\in GL(r,\boldsymbol{C})$ a non-degenerate non-singular matrix. If $\alpha\colon\Sigma(1)\to\boldsymbol{Z}$ is ample, then, for $i=1,2,\ldots,r$, we have*

$$\sqrt{-1}F_{X_\Sigma}^{c_1(L_\alpha)}(V_i(S))=\sum_{\sigma\in\Sigma(r)}\frac{\beta_i(S;\sigma,\alpha)^r\sum_{j=1}^{r}q_i^j(S;\sigma)}{\prod_{j=1}^{r}q_i^j(S;\sigma)}$$
$$-\frac{r}{r+1}\left\{\frac{\displaystyle\sum_{\sigma\in\Sigma(r)}\frac{\beta_i(S;\sigma,\alpha)^{r-1}\sum_{j=1}^{r}q_i^j(S;\sigma)}{\prod_{j=1}^{r}q_i^j(S;\sigma)}}{\displaystyle\sum_{\sigma\in\Sigma(r)}\frac{\beta_i(S;\sigma,\alpha)^r}{\prod_{j=1}^{r}q_i^j(S;\sigma)}}\right\}\sum_{\sigma\in\Sigma(r)}\frac{\beta_i(S;\sigma,\alpha)^{r+1}}{\prod_{j=1}^{r}q_i^j(S;\sigma)}.$$

REMARK 4.7. (i) Let X_Σ, α and S be as in Corollary 4.6. Then we have

$$\begin{aligned}&\big(F_{X_\Sigma}^{c_1(L_\alpha)}(\tau_1),\ \ F_{X_\Sigma}^{c_1(L_\alpha)}(\tau_2),\ \ \ldots,\ \ F_{X_\Sigma}^{c_1(L_\alpha)}(\tau_r)\big)\\ &\quad = \big(F_{X_\Sigma}^{c_1(L_\alpha)}(V_1(S)),\ \ F_{X_\Sigma}^{c_1(L_\alpha)}(V_2(S)),\ \ \ldots,\ \ F_{X_\Sigma}^{c_1(L_\alpha)}(V_r(S))\big)S^{-1}\,.\end{aligned}$$

Therefore, in view of Corollary 4.6, we can calculate $F_{X_\Sigma}^{c_1(L_\alpha)}(\tau_i)$ for all $i = 1, 2, \ldots, r$.

(ii) Let X_Σ and α be as in Corollary 4.6. Then by means of Theorem 1.9, Corollary 4.6 and the identity in (i), we can obtain the entire information about the Bando-Calabi-Futaki character $F_{X_\Sigma}^{c_1(L_\alpha)}$ of $(X_\Sigma, c_1(L_\alpha))$.

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