TSUKUBA J. MATH.
Vol. 23 No. 2 (1999), 225–228

AN ULTRAPOWER WHICH DOES NOT PRESERVE THE TRUTH OF A Π_2 SENTENCE

By

Nobutaka TSUKADA

Abstract. We construct a 'counterexample' to Łoś' theorem in the ordered Mostowski model for set theory ZFA.

The proof of the fundamental theorem of ultraproducts, as is well known, uses ***AC*** (the axiom of choce). Howard [2] showed that it is necessary even if for proving its special case: ultrapowers. In fact, he showed how to construct an ultrapower, which does not preserve some Π_2 sentence, in a model for ***BPI*** (the Boolean Prime Ideal Theorem) $+\neg\boldsymbol{AC}$. In this paper, we give another such ultrapower in the ordered Mostowski model for ***ZFA*** (Zermelo-Fraenkel set theory with atoms, see Jech [1]).

Let I be a non empty set, let U be an ultrafilter on I and let $\mathfrak{A}$ be a model for the first order language $\mathscr{L}$. Let A be the universe set of $\mathfrak{A}$. Consider the equivalence relation $\equiv$ over the set A^I defined by:

$$f \equiv g \Leftrightarrow \{i \in I \mid f(i) = g(i)\} \in U \quad \text{for } f, g \in A^I.$$

If $f \in A^I$, let $[f]$ denote the equivalence class of f ($[f] = \{g \in A^I \mid f \equiv g\}$). The ultrapower $\mathfrak{A}^I/U$ is the model for $\mathscr{L}$ described as follows:

(i) The universe of $\mathfrak{A}^I/U$ is $A^I/U = \{[f] \mid f \in A^I\}$

(ii) Let P be an n-placed predicate symbol of $\mathscr{L}$. The interpretation of P in $\mathfrak{A}^I/U$ is the relation R such that $R([f_1],[f_2],\ldots,[f_n])$ iff

$$\{i \in I \mid \mathfrak{A} \models P(f_1(i), f_2(i), \ldots, f_n(i))\} \in U \quad (f_1, f_2, \ldots, f_n \in A^I).$$

Then Łoś' Theorem reads (see [3]):

For each formula ϕ *of* $\mathscr{L}$, *and for each* $f_1, f_2, \ldots, f_n \in A^I$

$$\mathfrak{A}^I \models \phi([f_1],[f_2],\ldots,[f_n]) \quad \text{iff } \{i \in I \mid \mathfrak{A} \models \phi(f_1(i), f_2(i), \ldots, f_n(i))\} \in U.$$

This theorem is proved by using ***AC***. We can prove without ***AC*** easily the following

Received April 1, 1998.
Revised January 20, 1999.

PROPOSITION. *Let σ be a Σ_2 sentence. If σ is true in a model $\mathfrak{A}$, then σ is true in every ultrapower of $\mathfrak{A}$.* □

So, the least possible hierarchy of sentences whose truth is not preserved is Π_2. In fact, Howard [2] showed that

*If every ultrapower preserves the truth of every Π_2 sentence and if **BPI** holds, then the axiom of choice holds.*

In this paper, we give another ultrapower which does not preserve the truth of a Π_2 sentence in the ordered Mostowski model for ***ZFA***. For the ***ZF*** model which is translated by P. J. Cohen (see Jech [1], 5.5.), we can obtain the same result.

Recall the ordered Mostowski model M for ***ZFA***. Let N be a model for $\boldsymbol{ZFA}+\boldsymbol{AC}$ with countable atoms. Since the set of atoms A of N is countable, we can endow dense linear ordering to A by an isomorphism: $\langle \boldsymbol{Q}, < \rangle \to \langle A, <_A \rangle$. Consider the automorphism group $\mathfrak{G}$ of $\langle A, <_A \rangle$. Each automorphism $\pi \in \mathfrak{G}$ can be extended to an automorphism of N by the recursion: $\pi(0) = 0$, $\pi(x) = \{\pi(y) \mid y \in x\}$. For $x \in N$,

x is *symmetric* if there is a finite subset E of A such that

$$\forall \pi \in \mathfrak{G}[\forall e \in E(\pi(e) = e) \Rightarrow \pi(x) = x] \quad \text{(such an } E \text{ is called a } \textit{support} \text{ of } x).$$

Let M be the class of all the *hereditarily symmetric* elements of N. Then M is a model for ***ZFA***, which contains all the elements of $A \cup \{A\} \cup \{<_A\} \cup \{\langle A, <_A \rangle\} \cup N_0$, where N_0 is the class of hereditarily atomless elements of N. In M, $\langle A, <_A \rangle$ is a dense linearly ordered set without endpoints, and A cannot be well-ordered, *a fortiori*, A has no countably infinite subset (Jech [1], p. 50 and p. 52).

LEMMA. (1) *In M, every subset of A is a finite union of intervals of A of the form $(\leftarrow, a)$ $\{a\}$ (a, b) $(a, \rightarrow)$ where $a, b \in A$, and where $(\leftarrow, a) = \{x \in A \mid x <_A a\}$, similarly for others.*

(2) *In M, only the non-principal ultrafilters on A are*

$$\{x \subset A \mid \exists a \in A\ (a, \rightarrow) \subset x\} \quad \textit{and} \quad \{x \subset A \mid \exists a \in A\ (\leftarrow, a) \subset x\}.$$

PROOF. (1) Trivial. (2) Let $U_0 = \{x \subset A \mid \exists a \in A\ (a, \rightarrow) \subset x\}$ and $U_1 = \{x \subset A \mid \exists a \in A\ (\leftarrow, a) \subset x\}$. First we prove U_0 is a non-principal ultrafilter in M. As $\langle A, <_A \rangle$ is a linearly ordered set without largest element in M, U_0 is a non-principal filter in M. If $x \in M$ and $x \subset A$, then there is an $a \in A$ such that $(a, \rightarrow) \subset x$ or $(a, \rightarrow) \subset A - x$ by (1), so exactly one of x and $A - x$ is in U_0. So

U_0 is an ultrafilter in M. Similarly for U_1. Next we consider in N, to determine non-principal ultrafilters in $M \cap \mathscr{P}(A)$. Let U be a non-principal ultrafilter in $M \cap \mathscr{P}(A)$ (U may be not in M). First assume that non of bounded intervals of A belong to U. Then by (1), it is clear that U is of either form given in (2). So in the following, we assume U contain a bounded interval as an element, and lead to a contradiction. Since U is a filter, U contains a bounded *closed* interval. Let $\psi : \langle \boldsymbol{Q}, < \rangle \to \langle A, <_A \rangle$ be the isomorphism which endows the dense linear ordering to A. Fix a bounded closed interval $I_0 = [a_0, b_0] \in U$. Using Lemma (1), by induction on $n < \omega$, we can make $I_n = [a_n, b_n]$ in such a way that the following conditions hold:

(i) $I_n \in U$,

(ii) the sequence $\{I_n\}$ is strictly descending,

(iii) $\lim_{n \to \infty} |\psi^{-1}(I_n)| = 0$, where $|\ |$ represents the length of interval.

Hence, there is a real α such that $\bigcap_{n \in \omega} \psi^{-1}(I_n) = \{\alpha\}$. Then

$$U = \{x \subset A \mid \exists a, b \in A(\psi^{-1}(a) < \alpha < \psi^{-1}(b) \wedge (a, b) \subset x)\}$$

If α is a rational, then U is a principal filter, contradicting our assumption. So α is an irrational. Now, assume that U is in M, and fix a support S of U. Since α is an irrational and S is finite, by (ii) and (iii), there is an $m < \omega$ such that $I_m \cap S = 0$. Take an n such that $m < n$ and $a_m <_A a_n$. Let π be an order automorphism such that if either $x \leqq_A a_m$ or $b_m \leqq_A x$, $\pi(x) = x$ and such that $\alpha < \psi^{-1}(\pi(a_n)) < \psi^{-1}(\pi(b_n))$. Then $\pi(U) = U$, for every member of a support S of U is preserved by π, and any member of $\psi^{-1}(\pi(I_n))$ is larger than α, and so $\neg \pi(I_n) \in U$, which is a contradiction. □

Now, we state our theorem. (Note that the statement "an ordered set has no end points" is Π_2.)

THEOREM. *In M, let U be a non-principal ultrafilter on A. Then $\langle A, <_A \rangle^A / U$ is a dense linearly ordered set with an end point. So, $\langle A, <_A \rangle$ and $\langle A, <_A \rangle^A / U$ are not elementarily equivalent.*

PROOF. CLAIM. $A^A / U = \{[c_a] \mid a \in A\} \cup \{[i_A]\}$, where c_a is the constant function with the value a and i_A is the identity function on A.

Firstly, assuming the CLAIM, we prove our theorem: That $\langle A, <_A \rangle^A / U$ is a dense linearly ordered set is obvious. Now, if U is $\{x \subset A \mid \exists a \in A\ (a, \rightarrow) \subset x\}$, then $[i_A]$ is the largest element of $\langle A, <_A \rangle^A / U$. If U is $\{x \subset A \mid \exists a \in A\ (\leftarrow, a) \subset x\}$, then $[i_A]$ is the least element of $\langle A, <_A\rangle^A/U$. Whereas $\langle A, <_A\rangle$ has no end points.

PROOF OF THE CLAIM. We consider only the case where $U = \{x \subset A \mid \exists a \in A\ (a, \rightarrow) \subset x\}$, another case is proved similarly. Let $f : A \rightarrow A$ be in M. First we assume $[f] < [i_A]$, i.e. $\{x \in A \mid f(x) <_A x\} \in U$ and prove $[f] = [c_a]$ for some $a \in A$. From the choice of U, there is an a_0 such that

$$(a_0, \rightarrow) \subset \{x \in A \mid f(x) <_A x\}.$$

Fix a support of f whose maximum element a^* is larger than a_0. Fix a_1 with $a^* <_A a_1$. Then $f(a_1) <_A a_1$. It suffices to show that if $a_1 <_A a$, then $f(a) = f(a_1)$, for letting $f(a_1) = b$, we have $[f] = [c_b]$. To show this, fix an arbitral a with $a_1 <_A a$. As $f(a_1) <_A a_1$ and $a^* <_A a_1$ we can take an order automorphism π of A such that if $x \leqq_A f(a_1)$ or $x \leqq_A a^*$ then $\pi(x) = x$, and $\pi(a_1) = a$. Since π preserves the support of f, $\pi f = f$, so

$$f(a) = (\pi f)(a) = (\pi f)(\pi(a_1)) = \pi(f(a_1)) = f(a_1).$$

Next, assume that $[i_A] < [f]$, i.e. $\{x \in A \mid x <_A f(x)\} \in U$. Again we prove that $[f] = [c_a]$ for some $a \in A$. From the choice of U, there is an a_0 such that

$$(a_0, \rightarrow) \subset \{x \in A \mid x <_A f(x)\}.$$

Fix a support of f whose maximum element a^* is larger than a_0. Fix a_1 with $a^* <_A a_1$. Then $a_1 <_A f(a_1)$. It suffices to show that if $f(a_1) <_A a$, then $f(a) = f(a_1)$. To show this, fix an a with $f(a_1) <_A a$. As $a_1 <_A f(a_1)$ and $a_1 <_A f(a)$, we can take an order automorphism π of A such that if $x \leqq {}_A a_1$, then $\pi(x) = x$ and $\pi(f(a_1)) = f(a)$. Since π preserves the support of f, $\pi f = f$ and so

$$f(a) = \pi(f(a_1)) = (\pi f)(\pi(a_1)) = f(a_1). \qquad \square$$

References

[1] J. Thomas, Jech: The Axiom of Choice, North-Holland, 1973.
[2] P. E. Howard: Łoś' Theorem and the Boolean Prime Ideal Theorem Imply the Axiom of Choice, Proc. Amer. Math. Soc. **49** (1975), 426–428.
[3] C. C. Chang & H. J. Keisler: Model Theory, North Holland, 1973.

Institute of Mathematics
University of Tsukuba
1-1-1 Tennodai, Tsukuba-shi,
Ibaraki 305-8571, Japan