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THE MINIMAL SUPPORT FOR A CONTINUOUS FUNCTIONAL ON A FUNCTION SPACE II

By

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Abstract. Let $C_k(X)$ be the function space with the compact-open topology over a Tychonoff space X and ξ a continuous real-valued function on $C_k(X)$. A closed subset S of X is called a *support* for ξ if $\xi(f)=\xi(g)$ holds for any pair (f, g) of functions in $C_k(X)$ such that $f|_S=g|_S$. It is proved that the minimal support for any real-valued continuous function on the space $C_k(X)$ exists.

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1. Introduction.

In this paper, we assume that all spaces are Tychonoff. Let $C(X)$ be the set of all real-valued continuous functions on X. We call a real-valued function on $C(X)$ a *functional*. $C_k(X)$, $C_n(X)$ and $C_p(X)$ denote function spaces over X with the compact-open topology, the sup-norm topology and the pointwise convergent topology respectively. For a family $\mathcal{A}$ of sets, we write $\cap\mathcal{A}=\cap\{A: A\in\mathcal{A}\}$ and $\cup\mathcal{A}=\cup\{A: A\in\mathcal{A}\}$. For a function f on X and a subset M of X, the restriction of f to M is denoted by $f|_M$. The symbol π_M denotes the map from $C_k(X)$ to $C_k(M)$ defined by $\pi_M(f)=f|_M$. We write

$$\langle f, K, \varepsilon\rangle=\{g\in C_k(X): |f(x)-g(x)|<\varepsilon \text{ for any } x\in K\}$$

for $f\in C_k(X)$, a compact subset K of X and a positive number ε. For a basic open set $U=\langle f, K, \varepsilon\rangle$ in $C_k(X)$, we set $supp(U)=K$. $\mathbf{R}$, ω and ω_1 denote the real line, the first infinite ordinal and the first uncountable ordinal, respectively. If we deal with ordinals as spaces, then we always consider usual order topologies on them. Other undefined terms can be found in [**1**].

Let ξ be a functional on $C(X)$. A closed subset S of X is called a *support* for ξ if $\xi(f)=\xi(g)$ holds for any pair (f, g) of functions in $C(X)$ such that

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$f|_S=g|_S$. We say that a support S for ξ is *minimal* if every support for ξ contains S. Let *Supp* ξ be the set of all supports for a continuous functional ξ on $C_k(X)$. In the previous paper [4], we proved that the minimal support for any continuous functional on $C_p(X)$ exists. Our purpose of this paper is to give a result for any continuous functional on $C_k(X)$:

THEOREM 1. *The minimal support for any continuous functional ξ on $C_k(X)$ exists.*

We note that Kundu, McCoy and Okuyama [3] proved that the compact minimal support for any linear continuous functional on $C_k(X)$ exists. In Section 3, we shall discuss almost σ-compactness of supports in general cases.

Our theorem is not valid for $C_n(X)$. A simple counterexample will be given in Section 4.

2. Proof of Theorem 1.

To prove the theorem, we show some results.

LEMMA 2. *Let K be a compact subset of X and F a closed subset of X. For any pair (f, g) of functions in $C_k(X)$ and a positive number ε, if $g\in\langle f, K\cap F, \varepsilon\rangle$ holds, then there exists a function h in $C_k(X)$ such that $h\in\langle f, K, \varepsilon\rangle$ and $h|_F=g|_F$.*

PROOF. Put $L=K\setminus\{x\in X: |f(x)-g(x)|<\varepsilon\}$. Then $L\cap F=\emptyset$ holds. Since L is compact, there exists a continuous function $r: X\to[0, 1]$ such that $r|_F=0$ and $r|_L=1$. Put $h=f\times r+g\times(1-r)$. Obviously $h|_F=g|_F$ holds. We have

$$|h(x)-f(x)|=|f(x)-g(x)|\,|r(x)-1|<\varepsilon$$

for any $x\in K$. The lemma is proved.

LEMMA 3. *Let F be a closed subset of X and π_F the restriction map from $C_k(X)$ into $C_k(F)$. Then π_F is an open map onto $\pi_F(C_k(X))$.*

PROOF. By Lemma 2, we have

$$\pi_F(\langle f, K, \varepsilon\rangle)=\{g\in C_k(F): |f(x)-g(x)|<\varepsilon \text{ for any } x\in K\cap F\}\cap\pi_F(C_k(X))$$

for any $f\in C_k(X)$, any compact subset K of X and any positive number ε. The lemma has been proved.

LEMMA 4. *For any pair (S, T) of elements of Supp ξ, $S\cap T$ belongs to Supp ξ.*

PROOF. Put $A=S\cap T$. Assume that there exist functions f and g in $C_k(X)$ such that $\xi(f)\neq\xi(g)$ and $f|_A=g|_A$. It is easily checked that the equality

$$\pi_S^{-1}(\pi_S(\xi^{-1}(\xi(g))))=\xi^{-1}(\xi(g))$$

holds for the restriction map π_S. From this, we have $f|_S\notin\pi_S(\xi^{-1}(\xi(g)))$. By Lemma 3, $\pi_S(\xi^{-1}(\xi(g)))$ is closed in $\pi_S(C_k(X))$. There exist a compact subset K of S and a positive number ε such that $\langle f, K, \varepsilon\rangle\cap\xi^{-1}(\xi(g))=\emptyset$. Since $K\cap T\subset A$ holds, by the assumption, we have $g\in\langle f, K\cap T, \varepsilon\rangle$. By Lemma 2, there exists h in $\langle f, K, \varepsilon\rangle$ such that $h|_T=g|_T$. This is a contradiction. The lemma is proved.

LEMMA 5. *$\cap$Supp ξ is a support for ξ.*

PROOF. Put $S=\cap$Supp ξ. Assume that there exist functions f and g in $C_k(X)$ such that $\xi(f)\neq\xi(g)$ and $f|_S=g|_S$. Since f does not belong to the closed subset $\xi^{-1}(\xi(g))$ of $C_k(X)$, there exist a compact subset K of X and a positive number ε such that $\langle f, K, \varepsilon\rangle\cap\xi^{-1}(\xi(g))=\emptyset$. Put $L=K\setminus\{x\in X: |f(x)-g(x)|<\varepsilon\}$. Then $L\cap S=\emptyset$ holds. Using Lemma 4, by the definition of S, we can find an element T of *Supp* ξ such that $L\cap T=\emptyset$. Since $g\in\langle f, K\cap T, \varepsilon\rangle$ holds, there exists h in $\langle f, K, \varepsilon\rangle$ such that $h|_T=g|_T$ by Lemma 2. This is a contradiction. The lemma is proved.

By Lemma 5, $\cap$*Supp* ξ is the minimal support for ξ, which proves Theorem 1.

3. Almost σ-compact minimal supports.

Let τ be a cardinal. A space X is said to be *almost τ-compact* if for any $\alpha<\tau$, there exists a compact subset K_α of X such that $X=\overline{\bigcup\{K_\alpha: \alpha<\tau\}}$. Almost ω-compact spaces are said to be *almost σ-compact*. The smallest cardinal τ such that X is almost τ-compact, is denoted by $cd(X)$. For function spaces with the pointwise convergent topology, any minimal support is separable (See [4]). In this section, we shall discuss almost σ-compactness of minimal supports.

DEFINITION. A space X has *property* (σ) if, for any continuous functional ξ on $C_k(X)$, the closed subset $\cap$*Supp* ξ of X is almost σ-compact.

First, we give a sufficient condition in order that X has property (σ).

LEMMA 6. *Let $\mathcal{D}$ be a dense subset of $C_k(S)$ which satisfies the countable* *chain condition and* ρ *a real-valued continuous function on* $\mathcal{D}$. *There exists a* σ-*compact subset* A *of* S *which satisfies the following condition* $(*)$;

$(*)$: *For any pair* (f, g) *of functions in* $\mathcal{D}$, *if* $f|_A=g|_A$ *holds, then* $\rho(f)=\rho(g)$ *holds.*

PROOF. Let $\{U_n: n\in\omega\}$ be a base for $\mathbf{R}$ and γ_n the maximal disjoint family of non-empty basic open subsets of $\mathcal{D}$ such that $\cup\gamma_n\subset\rho^{-1}(U_n)$. By the maximality of γ_n, $\cup\gamma_n$ is a dense subset of $\rho^{-1}(U_n)$. Put

$$A=\cup\{supp(U): U\in\cup\{\gamma_n: n\in\omega\}\}$$

(for the definition of $supp(U)$, see Section 1). Since $\mathcal{D}$ satisfies the countable chain condition, A is a σ-compact subset of S. To show that A satisfies the condition $(*)$, it is sufficient to show that, for any pair (f, g) of functions in $\mathcal{D}$, $g\in\overline{\cup\gamma_n}$ holds if $f|_A=g|_A$ and $f\in\overline{\cup\gamma_n}$ holds for some $n\in\omega$. Take a compact subset K of S and a positive number ε. Since $f\in\overline{\cup\gamma_n}$ holds, there exist an element U of γ_n and a function h in $\mathcal{D}$ such that $h\in\langle f, K, \varepsilon\rangle\cap U$. Since $h\in\langle g, K\cap supp(U), \varepsilon\rangle$ holds, by Lemma 2, there exists a function $\bar{h}$ in $C_k(S)$ such that $\bar{h}\in\langle g, K, \varepsilon\rangle$ and $\bar{h}|_{supp(U)}=h|_{supp(U)}$. Since $\mathcal{D}$ is dense in $C_k(S)$, the set $\langle g, K, \varepsilon\rangle\cap U$ is not empty. The lemma is proved.

THEOREM 7. *If the space* $C_k(X)$ *satisfies the countable chain condition, then* X *has property* (σ).

PROOF. Let ξ be a continuous functional on $C_k(X)$. Put $S=\cap Supp\ \xi$ and $\mathcal{D}=\pi_S(C_k(X))$. There exists a real-valued function ρ on $\mathcal{D}$ such that $\xi=\rho\circ\pi_S$. We can check that $\mathcal{D}$ and ρ satisfy the conditions in Lemma 6 easily. So there exists a σ-compact subset A of S which satifies the condition $(*)$ in Lemma 6. Obviously, $\bar{A}\in Supp\ \xi$ holds. By the minimality of $\cap Supp\ \xi$, we have $\cap Supp\ \xi=\bar{A}$. The theorem is proved.

Vidossich [6] and Nakhmanson [5] proved that $C_k(X)$ satisfies the countable chain condition if X is submetrizable (for the definition of submetrizability, see [1] Exercise 4.4.C. (b), page 286). We have the following corollary.

COROLLARY 8. *If* X *is submetrizable (in particular, metrizable), then* X *has property* (σ).

PROPOSITION 9. *The space* ω_1 *has property* (σ).

PROOF. It is sufficient to show that, for any continuous functional ξ on $C_k(\omega_1)$, there exists a countable support for ξ. To show this, we assume that there exists a positive number ε such that, for any $\alpha<\omega_1$, there exist functions f_α and g_α in $C_k(\omega_1)$ satisfying the following two conditions;

$$\text{(a)} \quad f_\alpha|_{[0,\alpha]}=g_\alpha|_{[0,\alpha]},$$

$$\text{(b)} \quad |\xi(f_\alpha)-\xi(g_\alpha)|>2\varepsilon.$$

Gul'ko [2] showed that $C_k(\omega_1)$ is Lindelöf. So the net $\{f_\alpha : \alpha<\omega_1\}$ has a cluster point h in $C_k(\omega_1)$. By the continuity of ξ, there exist a $\beta<\omega_1$ and a positive number δ such that, for any function f in $\langle h, [0, \beta], \delta\rangle$, $|\xi(f)-\xi(h)|<\varepsilon$ holds. Choose $\gamma<\omega_1$ such that $\beta<\gamma$ and $f_\gamma\in\langle h, [0, \beta], \delta\rangle$. Then we have $g_\gamma\in\langle h, [0, \beta], \delta\rangle$ by (a). This contradicts the condition (b). The proposition is proved.

REMARK. Nakhmanson [5] noted that $C_k(\omega_1)$ does not satisfy the countable chain condition.

In special cases, we have a necessary condition for a space to have property (σ). The following lemma is well-known (for example, see [1] Exercise 3.4.H. (b), page 166).

LEMMA 10. *Let K be a compact space. If $C_k(K)$ is separable, then K is metrizable.*

THEOREM 11. *Let X be a space which has a closed-and-open subset Y such that $cd(Y)=\omega_1$. If X has property (σ), then every compact subset of X is metrizable.*

PROOF. Suppose that a non-metrizable compact subset K of X exists. $C_k(K)$ is metrizable but not separable by Lemma 10. By Theorem 4.4.3 in [1], for any $\alpha<\omega_1$, there exists a non-constant continuous function $\xi_\alpha : C_k(K) \to[0, 1]$ such that $\{\xi_\alpha^{-1}((0, 1]) : \alpha<\omega_1\}$ is discrete. Since $cd(Y)=\omega_1$ holds, for any $\alpha<\omega_1$, there exists a non-empty compact subset K_α of Y such that $Y=\overline{\bigcup\{K_\alpha : \alpha<\omega_1\}}$. For any f in $C_k(X)$ and an $\alpha<\omega_1$, we put

$$s(f, \alpha)=\sup\{|f(x)| : x\in K_\alpha\}.$$

We define a functional ξ on $C_k(X)$ by;

$$\xi(f)=\sum\{s(f, \alpha)\times\xi_\alpha(\pi_K(f)) : \alpha<\omega_1\}$$

for any f in $C_k(X)$. Obviously ξ is continuous. We show that ξ has no almost σ-compact support. Let S be a support for ξ. Suppose that there exist a $\beta<\omega_1$ and a point x in K_β such that $x\notin S\cup K$. Take a function f in $C_k(X)$ such that $f|_K \in \xi_\beta^{-1}((0, 1])$. There exists a function g in $C_k(X)$ such that $f|_{S\cup K} = g|_{S\cup K}$ and $g(x) = s(f, \beta)+1$. We have

$$\xi(g)-\xi(f)=s(g, \beta)\times\xi_\beta(\pi_K(g))-s(f, \beta)\times\xi_\beta(\pi_K(f))\neq 0 .$$

This contradicts the fact $f|_S = g|_S$. So we have

$$Y=\overline{\cup\{K_\alpha:\ \alpha<\omega_1\}}\subset S\cup K .$$

If S is almost σ-compact, then $S\cup K$ is also obviously. Then Y is almost σ-compact because Y is a closed-and-open subset of $S\cup K$. This contradicts the fact $cd(Y)=\omega_1$. The theorem is proved.

Using Theorem 11, we have a space which does not have property (σ).

EXAMPLE. Let $D(\omega_1)$ be the discrete space whose cardinarity is ω_1. The space $D(\omega_1)\oplus(\omega_1+1)$ does not have property (σ).

REMARK. The above example shows that property (σ) is not preserved by topological sums in general. In fact, since $C_k(D(\omega_1))=C_p(D(\omega_1))$ holds, every continuous functional on $C_k(D(\omega_1))$ has the countable minimal support (See [4]). Obviously the space ω_1+1 has property (σ).

4. Remarks and comments.

I. Using the same idea in the proof of Theorem 1, we can prove the following lemma.

LEMMA 12. *Let $\mathcal{F}$ be a non-empty closed subset of $C_k(X)$. We put*

$$Supp\,\mathcal{F}=\{S\subset X:\ S \text{ is closed in } X,\ \pi_S^{-1}(\pi_S(\mathcal{F}))=\mathcal{F}\} .$$

Then the set $\cap Supp\,\mathcal{F}$ belongs to $Supp\,\mathcal{F}$.

This lemma gives a result on the minimal support.

THEOREM 13. *Let ξ be a continuous functional on $C_k(X)$. For $r\in\xi(C_k(X))$, we put $S_r=\cap Supp\,\xi^{-1}(r)$. Then we have*

$$\cap Supp\,\xi=\overline{\cup\{S_r:\ r\in\xi(C_k(X))\}} .$$

The proof of Theorem 13 is similar to the proof of Theorem 5 in [4].

II. Lemma 5 and Lemma 12 are not valid for $C_n(X)$. For any f in $C_n(\omega_1)$, $\bar{f}$ denotes the unique extention of f to ω_1+1. We define a functional ξ on $C_n(\omega_1)$ by the rule $\xi(f)=\bar{f}(\omega_1)$ for any f in $C_n(\omega_1)$. Then ξ is continuous ovbiously. Since $[\alpha, \omega_1)\in Supp\,\xi$ holds for any $\alpha<\omega_1$, we have $\cap Supp\,\xi=\emptyset$.

Put $\mathcal{F}=\{f\in C_n(\omega_1):\ \bar{f}(\omega_1)=0\}$. Then $\mathcal{F}$ is a non-empty closed subset of $C_n(\omega_1)$. Similarly, we have $\cap Snpp\,\mathcal{F}=\emptyset$ also.

References

[1] Engelking, R., General Topology, Heldermann Verlag, Berlin, 1989.

[2] Gul'ko, S.P., On properties of some function spaces, Soviet Math. Dokl., Vol. 19, 1978, pp. 1420-1424.

[3] Kundu, S., McCoy, R.A. and Okuyama, A., Spaces of continuous linear functionals on $C_k(X)$, Mathematica Japonica, 34, 1989, pp. 775-787.

[4] Morishita, K., The minimal support for a continuous functional on a function space, Proc. Amer. Math. Soc., Vol. 114, No. 2, 1992, pp. 585-587.

[5] Nakhmanson, L.B., The Souslin number and caliber of rings of continuous functions, Izvestiya VUZ. Mathematika, Vol. 28, 1984, pp. 49-55.

[6] Vidossich, G., Function spaces which are pseudo-$\aleph$-compact spaces, preprint 1972.

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