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C^*-REFLEXIVITY DOESN'T PASS TO QUOTIENTS

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Abstract. Using a recently obtained criterion of C^*-reflexivity for commutative C^*-algebras, we show that the C^*-algebra of continuous functions on the Higson corona is not C^*-reflexive. This implies that C^*-reflexivity doesn't pass to quotient C^*-algebras.

1. Introduction and preliminaries

A unital C^*-algebra A is *C^*-reflexive* if any Hilbert C^*-module M over A is reflexive, i.e. if the second dual M'' of M coincides with M. In [5, 3] C^*-reflexivity of some classes of C^*-algebras was established, and in [2] the following criterion of C^*-reflexivity was obtained for commutative C^*-algebras.

Theorem 1.1 ([2]). *A commutative unital C^*-algebra A is not C^*-reflexive if and only if there exists a sequence $\{I_i\}_{i\in\mathbb{N}}$ of non-intersecting C^*-subalgebras $I_i \subset A$ such that the canonical inclusion $\oplus_{i=1}^{\infty} I_i \subset A$ extends to a $*$-homomorphism $\prod_{i=1}^{\infty} I_i \to A$.*

Note that the $*$-homomorphism $\prod_{i=1}^{\infty} I_i \to A$, if exists, has to be injective because $\oplus_{i=1}^{\infty} I_i$ is an essential ideal of $\prod_{i=1}^{\infty} I_i$.

Let A be the C^*-subalgebra of l^∞ that consists of all bounded sequences $\{a_n\}_{n\in\mathbb{N}}$, $a_n \in \mathbb{C}$, such that $\lim_{n\to\infty} |a_{n+1} - a_n| = 0$. This C^*-algebra is the algebra of all continuous functions on the *Higson compactification* $\nu\mathbb{N}$ of $\mathbb{N}$ [4]. Note that the subset $c_0 = C_0(\mathbb{N}) \subset A$ of all sequences vanishing at infinity (i.e.

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$\lim_{n\to\infty} a_n = 0$) is an ideal in A. The quotient C^*-algebra $B = C(\nu\mathbb{N})/C_0(\mathbb{N}) = C(\nu\mathbb{N} \setminus \mathbb{N})$ is the algebra of continuous functions on the *Higson corona*.

It was proved in [2] that $A = C(\nu\mathbb{N})$ is C^*-reflexive. Our aim is to study Hilbert C^*-modules over the quotient C^*-algebra $B = C(\nu\mathbb{N})/C_0(\mathbb{N}) = C(\nu\mathbb{N}\setminus\mathbb{N})$. Hilbert C^*-modules over quotient C^*-algebras are interesting to study in view of the theory of extensions for Hilbert C^*-modules recently developed by D. Bakić and B. Guljaš [1]. Higson corona provides a simple but non-trivial example of quotient C^*-algebras. In contrast with $A = C(\nu\mathbb{N})$, it turns out that $B = C(\nu\mathbb{N})/C_0(\mathbb{N}) = C(\nu\mathbb{N} \setminus \mathbb{N})$ is not C^*-reflexive.

2. Main results

Theorem 2.1. *The C^*-algebra $B = C(\nu\mathbb{N} \setminus \mathbb{N})$ is not C^*-reflexive.*

Proof. Let $n_1 < n_2 < \dots$ and $k_1 \le k_2 \le \dots$ be two sequences of positive integers such that

$$\lim_{i\to\infty} k_i = \infty \tag{2.1}$$

and

$$n_i + k_i < n_{i+1} \quad \text{for each} \quad i \in \mathbb{N}. \tag{2.2}$$

Let

$$U_i = \{n_i, n_i+1, n_i+2, \dots, n_i+k_i\}, \quad i \in \mathbb{N},$$

be a sequence of disjoint (due to (2.2)) subsets in $\mathbb{N}$. Fix a bijection $\alpha : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ and set $V_i^j = U_{\alpha(i,j)}$.

For each $i \in \mathbb{N}$, let $I_i \subset A$ be the C^*-subalgebra of sequences $\{a_n\}_{n\in\mathbb{N}}$ such that $a_n = 0$ when $n \notin \cup_{j\in\mathbb{N}} V_i^j$. Let $J_i = I_i/(c_0 \cap I_i)$ be a C^*-subalgebra in B. J_i is non-zero for each $i \in \mathbb{N}$ due to (2.1). Since $I_i \cap I_k = \{0\}$ when $i \ne k$, the same holds for the quotients J_i and J_k.

Let $a^{(i)} \in I_i$, and let $\dot{a}^{(i)} \in J_i$ denote the class of $a^{(i)}$. Assume that $\sup_{i\in\mathbb{N}} \|\dot{a}^{(i)}\|$ is bounded. Then the sequence $(\dot{a}^{(1)}, \dot{a}^{(2)}, \dots)$ represents an element of $\prod_{i\in\mathbb{N}} J_i$.

For a sequence $a = \{a_n\}_{n\in\mathbb{N}}$ and for a set $U = \{m, m+1, \dots, m+k\}$ define $L(a|_U)$ by

$$L(a|_U) = \max_{n=m-1}^{m+k} |a_{n+1} - a_n|.$$

As $a^{(i)} \in A$ for each $i \in \mathbb{N}$, so, by definition, we have $\lim_{j\to\infty} L(a^{(i)}|_{V_i^j}) = 0$. So, for each $i \in \mathbb{N}$, we can find an integer m_i such that

$$\sup_{j\ge m_i} L(a^{(i)}|_{V_i^j}) \le \frac{1}{i}$$

and set $b_n^{(i)} = \begin{cases} a_n^{(i)}, & \text{if } \ n \in \cup_{j=m_i}^{\infty} V_i^j \\ 0, & \text{otherwise.} \end{cases}$

As $a^{(i)} - b^{(i)}$ is finite for each $i \in \mathbb{N}$, so $\dot{b}^{(i)} = \dot{a}^{(i)}$.

Define a sequence $c = \{c_n\}_{n\in\mathbb{N}}$ by $c_n = \sum_{i=1}^{\infty} b_n^{(i)}$ and note that this sum contains no more than one non-trivial summand for each $n \in \mathbb{N}$, since all V_i^j are disjoint.

Lemma 2.2. (1) *$c \in A$;*

(2) *$c_n = b_n^{(i)}$ when $n \in V_i^j$ for some $j \in \mathbb{N}$;*

(3) *$\dot c \in B$ depends only on $\dot a^{(i)} \in B$, $i \in \mathbb{N}$, but not on their representatives in A.*

Proof. The claim directly follows from the construction of c.

□

The above construction determines a $*$-homomorphism

$$\bar\iota : \prod_{i=1}^{\infty} J_i \to B \quad \text{by} \quad \bar\iota(\dot a^{(1)}, \dot a^{(2)}, \dots) = \dot c.$$

It is clear that $\bar\iota$ makes the diagram

$$\begin{tikzcd} \oplus_{i=1}^{\infty} J_i \arrow[rr, hook] \arrow[dr, hook, "\iota"'] & & \prod_{i=1}^{\infty} J_i \arrow[dl, hook', "\bar\iota"] \\ & B & \end{tikzcd}$$

commuting, where $\iota_i : J_i \subset B$ is the canonical inclusion and $\iota = \oplus_{i=1}^{\infty} \iota_i$. Hence, by C^*-reflexivity criterion, B is not C^*-reflexive.

□

As the Higson corona is a closed subspace of the Higson compactification, so we get the following corollary.

Corollary 2.3. *C^*-reflexivity is not a hereditary property.*

This result suggests the following definition.

Definition 2.4. A compact Hausdorff space X is called *hereditarily C^*-reflexive* if $C(Y)$ is C^*-reflexive for any compact subset $Y \subset X$.

Let $\beta\mathbb{N}$ denote the Stone–Čech compactification of integers.

Theorem 2.5. *X is hereditarily C^*-reflexive if and only if no closed subspace of X is homeomorphic to $\beta\mathbb{N}$.*

Proof. If X is not hereditarily C^*-reflexive then there exists a closed subspace $Y \subset X$ such that $C(Y)$ is not C^*-reflexive. Then, by Theorem 4.1 of [2], $\beta\mathbb{N}$ can be embedded in Y (hence, in X too) as a closed subspace. In the opposite direction, if $Y = \beta\mathbb{N} \subset X$ is a closed subspace then X cannot be hereditarily C^*-reflexive since $C(\beta\mathbb{N}) = l^\infty$ is not C^*-reflexive.

□

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