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BANACH PARTIAL $*$-ALGEBRAS: AN OVERVIEW

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Abstract. A Banach partial $*$-algebra is a locally convex partial $*$-algebra whose total space is a Banach space. A Banach partial $*$-algebra is said to be of type (B) if it possesses a generating family of multiplier spaces that are also Banach spaces. We describe the basic properties of these objects and display a number of examples, namely, L^p-like function spaces and spaces of operators on Hilbert scales or lattices. Finally we analyze the important cases of Banach quasi $*$-algebras and CQ^*-algebras.

1. Introduction

The notion of locally convex partial $*$-algebra stems from the desire to exploit the simultaneous presence of the algebraic structure of a partial $*$-algebra and its topological structure in such a way that the two match perfectly [1, 2]. The resulting notion covers and unifies a variety of cases that have been discussed in the literature. We may mention, for instance, topological quasi $*$-algebras and CQ^*-algebras; partial $*$-algebras of functions, such as the scale of the L^p spaces on [0,1] or the lattice generated by the family $\{L^p(\mathbb{R}), 1 \leq p \leq \infty\}$; or partial $*$-algebras of operators, such as partial O^*-algebras or sets of operators on a PIP-space [4], in particular, operators on a lattice or a scale of Hilbert spaces.

In this chapter, we will review the important case of *Banach partial $*$-algebras*, as it is suitable in a special issue dedicated to Stefan Banach's birthday. In particular, we will analyze a distinguished class among them, that we call Banach

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partial $*$-algebras *of type* (B). In accordance with the spirit of the theory of PIP-spaces [4], the latter are characterized by the fact that each element of a generating family of multiplier spaces is a Banach space with respect to a norm topology that is compatible in a natural way with the underlying multiplication structure (section 5). The simplest examples of this kind of locally convex partial $*$-algebras are the chain $\{L^p\}$ and related function spaces and spaces of operators on a Hilbert scale or lattice.

Preliminary results on Banach partial $*$-algebras were already contained in the monograph [2]. Here, following mostly [5] and related papers of ours, we will go deeper and also modify some of the definitions, in the light of the new results. In particular, the structure called here Banach partial $*$-algebra of type (B) was called simply Banach partial $*$-algebra in [2]. We prefer to reserve the name to the larger class of locally convex partial $*$-algebras for which the total space is a Banach space and give a new name to the more sophisticated structure.

The chapter is organized as follows. After a quick reminder of the basic definitions on partial $*$-algebras (section 2), we introduce in section 3 the new definition of Banach partial $*$-algebraand discuss some consequences. In section 4, the analysis of the various topologies that may arise on multiplier spaces leads us, in section 5, to the new concept of Banach partial $*$-algebra of type (B). section 6 is devoted to examples. First, Banach partial $*$-algebras of functions, which have been described at length in [1] and [2]. Then, following mostly [5], we discuss Banach partial $*$-algebras of operators on a scale or a lattice of Hilbert spaces. In both cases, we show that the corresponding Banach partial $*$-algebras are indeed of type (B). Then, in section 7, we analyze in some detail the important case of Banach quasi $*$-algebras [26, 25] and, in particular, the $*$-semisimple ones, which are free of some pathologies. Finally, section 8 is devoted to a rather brief discussion of the descendants of Banach quasi $*$-algebras, the so-called CQ^*-algebras, which are in fact a generalization of the familiar C^*-algebras [10, 8]. The Appendix collects the basic facts about PIP-spaces, as used in the text [4].

2. Basic definitions on partial $*$-algebras

In order to keep the paper reasonably self-contained, we summarize in this section the basic facts on partial $*$-algebras and on their topological structure. Further details and proofs may be found in [1] or in the monograph [2].

A *partial $*$-algebra* is a complex vector space $\mathfrak{A}$, endowed with an involution $x \mapsto x^*$ (that is, a bijection such that $x^{**} = x$) and a partial multiplication defined by a set $\Gamma \subset \mathfrak{A} \times \mathfrak{A}$ (a binary relation) such that

(i) $(x, y) \in \Gamma$ implies that $(y^*, x^*) \in \Gamma$;

(ii) $(x, y_1), (x, y_2) \in \Gamma$ imply that $(x, \lambda y_1 + \mu y_2) \in \Gamma$ for all $\lambda, \mu \in \mathbb{C}$;

(iii) for any $(x, y) \in \Gamma$, there is defined a product $x \cdot y \in \mathfrak{A}$, which is distributive over the addition and satisfies the relation $(x \cdot y)^* = y^* \cdot x^*$.

We shall assume the partial $*$-algebra $\mathfrak{A}$ contains a unit e; that is, $e^* = e$, $(e, x) \in \Gamma$, for all $x \in \mathfrak{A}$, and $e \cdot x = x \cdot e = x$ for all $x \in \mathfrak{A}$. If $\mathfrak{A}$ has no unit, it may always be embedded into a larger partial $*$-algebra with unit, in the standard fashion. Notice that the partial multiplication is *not* required to be associative (and often it is not).

Given the defining set Γ, spaces of multipliers are defined in the following obvious way:

$$\begin{aligned}(x,y)\in\Gamma &\iff x\in L(y) \text{ or } x \text{ is a left multiplier of } y\\ &\iff y\in R(x) \text{ or } y \text{ is a right multiplier of } x.\end{aligned}$$

For any subset $\mathfrak{N}\subset\mathfrak{A}$, we write

$$L\mathfrak{N}=\bigcap_{x\in\mathfrak{N}}L(x),\quad R\mathfrak{N}=\bigcap_{x\in\mathfrak{N}}R(x),$$

and, of course, the involution exchanges the two

$$(L\mathfrak{N})^*=R\mathfrak{N}^*,\quad (R\mathfrak{N})^*=L\mathfrak{N}^*.$$

Clearly all these multiplier spaces are vector subspaces of $\mathfrak{A}$, containing e.

The partial $*$-algebra is *abelian* if $L(x)=R(x)$, for all $x\in\mathfrak{A}$, then $x\cdot y=y\cdot x$ for all $x\in L(y)$. In that case, we write simply for the multiplier spaces $L(x)=R(x)=:M(x)$ and $L\mathfrak{N}=R\mathfrak{N}=:M\mathfrak{N}$ for $\mathfrak{N}\subset\mathfrak{A}$.

The crucial fact is that the couple of maps (L,R) defines a *Galois connection* on the complete lattice of all vector subspaces of $\mathfrak{A}$ (ordered by inclusion), which means that (i) both L and R reverse order; and (ii) both LR and RL are closures; that is, for any subset $\mathfrak{N}\subset\mathfrak{A}$, one has

$$\begin{aligned}\mathfrak{N}\subset LR\mathfrak{N}\quad &\text{and}\quad LRL=L\\ \mathfrak{N}\subset RL\mathfrak{N}\quad &\text{and}\quad RLR=R.\end{aligned}$$

Let us denote by $\mathcal{F}^L$ and $\mathcal{F}^R$, the set of all LR-closed and RL-closed subspaces of $\mathfrak{A}$, respectively,

$$\begin{aligned}\mathcal{F}^L&=\{\mathfrak{N}\subset\mathfrak{A};\ \mathfrak{N}=LR\mathfrak{N}\},\\ \mathcal{F}^R&=\{\mathfrak{N}\subset\mathfrak{A};\ \mathfrak{N}=RL\mathfrak{N}\},\end{aligned}$$

both ordered by inclusion. Then standard results from universal algebra [12] yield the full multiplier structure of $\mathfrak{A}$.

Theorem 2.1. *Let $\mathfrak{A}$ be a partial $*$-algebra and let $\mathcal{F}^L$ and $\mathcal{F}^R$ be the sets of all LR-closed and RL-closed subspaces of $\mathfrak{A}$, respectively, both ordered by inclusion. Then*

(1) $\mathcal{F}^L$ is a complete lattice with lattice operations

$$\mathfrak{M}\wedge\mathfrak{N}=\mathfrak{M}\cap\mathfrak{N},\quad \mathfrak{M}\vee\mathfrak{N}=LR(\mathfrak{M}+\mathfrak{N}).$$

The largest and smallest elements are $\mathfrak{A}$ and $L\mathfrak{A}$, respectively.

(2) $\mathcal{F}^R$ is a complete lattice with lattice operations

$$\mathfrak{M}\wedge\mathfrak{N}=\mathfrak{M}\cap\mathfrak{N},\quad \mathfrak{M}\vee\mathfrak{N}=RL(\mathfrak{M}+\mathfrak{N}).$$

The largest and smallest elements are $\mathfrak{A}$ and $R\mathfrak{A}$, respectively.

(3) Both $L:\mathcal{F}^R\to\mathcal{F}^L$ and $R:\mathcal{F}^L\to\mathcal{F}^R$ are lattice anti-isomorphisms,

$$L(\mathfrak{M}\wedge\mathfrak{N})=L\mathfrak{M}\vee L\mathfrak{N},\quad \textit{etc.}$$

(4) The involution $\mathfrak{N} \leftrightarrow \mathfrak{N}^*$ is a lattice isomorphism between $\mathcal{F}^L$ and $\mathcal{F}^R$.

In addition to the two lattices $\mathcal{F}^L$ and $\mathcal{F}^R$, it is useful to consider the subset $\mathcal{F}^\Gamma \subset \mathcal{F}^L \times \mathcal{F}^R$ consisting of *matching pairs*; that is,

$$\mathcal{F}^\Gamma = \{(\mathfrak{N}, \mathfrak{M}) \in \mathcal{F}^L \times \mathcal{F}^R : \mathfrak{N} = L\mathfrak{M} \text{ and } \mathfrak{M} = R\mathfrak{N}\}.$$

Indeed these pairs describe completely the partial multiplication of $\mathfrak{A}$, for we can write

$$(x, y) \in \Gamma \iff \exists\, (\mathfrak{N}, \mathfrak{M}) \in \mathcal{F}^\Gamma \text{ such that } x \in \mathfrak{N} \text{ and } y \in \mathfrak{M}.$$

The complete lattices $\mathcal{F}^R$ and $\mathcal{F}^L$ are often difficult to describe explicitly, but much less is needed in practice. Indeed, the following notion is sufficient and much more manageable.

Definition 2.2. A subset $\mathcal{I}^R$ of $\mathcal{F}^R$ is called a *generating family* if
(i) $R\mathfrak{A} \in \mathcal{I}^R$ and $\mathfrak{A} \in \mathcal{I}^R$.
(ii) $x \in L(y)$ if and only if there exists $\mathfrak{M} \in \mathcal{I}^R$ such that $y \in \mathfrak{M}$ and $x \in L\mathfrak{M}$.
A generating family for $\mathcal{F}^L$ or $\mathcal{F}^\Gamma$ is defined in a similar way.

Thus a generating family determines completely the partial multiplication. Clearly, if $\mathcal{I}^R$ is a generating family for $\mathcal{F}^R$, $\mathcal{I}^L = L\mathcal{I}^R = \{L\mathfrak{M} : \mathfrak{M} \in \mathcal{I}^R\}$ is generating for $\mathcal{F}^L$, and similarly $\mathcal{I}^R_* = \{\mathfrak{M}^* : \mathfrak{M} \in \mathcal{I}^R\}$, but these two have *a priori* nothing in common.

The following properties are obvious:

(i) if $\mathcal{I}^R$ is generating for $\mathcal{F}^R$, so is the sublattice $\mathcal{J}^R$ of $\mathcal{F}^R$ generated from $\mathcal{I}^R$ by *finite* lattice operations.
(ii) if $\mathcal{I}^R$ is generating, the *complete* lattice generated by $\mathcal{I}^R$ is $\mathcal{F}^R$ itself.

3. Banach partial $*$-algebras

Particularizing the general definition of locally convex partial $*$-algebra given in [1, 2], we obtain the following one (note this is different from Definition 6.2.7 there, here we rather follow [5]).

Definition 3.1. A partial $*$-algebra $\mathfrak{A}$ is said to be a *normed partial $*$-algebra* if it carries a norm $\|\cdot\|$ such that

(i) the involution $x \mapsto x^*$ is isometric; $\|x^*\| = \|x\|$ for all $x \in \mathfrak{A}$;
(ii) For every $a \in L\mathfrak{A}$, there exists a constant $\gamma_a > 0$ such that

$$\|ax\| \leq \gamma_a \|x\| \qquad \text{for all}\, x \in \mathfrak{A}.$$

$\mathfrak{A}[\|\cdot\|]$ is called a *Banach partial $*$-algebra* if, in addition,

(iii) $\mathfrak{A}[\|\cdot\|]$ is a Banach space.

Using (i), (ii), and the fact that $R\mathfrak{A} = L\mathfrak{A}^*$, we also have

(ii') For every $b \in R\mathfrak{A}$, there exists a constant $\gamma_b > 0$ such that

$$\|xb\| \leq \gamma_b \|x\| \qquad \text{for all}\, x \in \mathfrak{A}.$$

Whereas $\mathfrak{A}$ carries its defining norm $\|\cdot\|$, the universal multiplier spaces $R\mathfrak{A}$ and $L\mathfrak{A}$ carry their own characteristic norms, defined as follows. To every $a \in L\mathfrak{A}$, one may associate a bounded linear map $L_a : \mathfrak{A} \to \mathfrak{A}$ by

$$L_a(x) = ax, \qquad x \in \mathfrak{A}.$$

Then we define the norm of $a \in L\mathfrak{A}$ as $\|a\|^{\diamond}_{L\mathfrak{A}} := \|L_a\|$, the latter being the usual norm on bounded operators. Similarly, to every $b \in R\mathfrak{A}$, one associates a bounded linear map $R_b : \mathfrak{A} \to \mathfrak{A}$ and the norm $\|b\|_{R\mathfrak{A}} := \|R_b\| = \|L_{b^*}\| = \|b^*\|^{\diamond}_{L\mathfrak{A}}$.

The simplest example of a Banach partial $*$-algebra is given by a closed subspace of a Banach $*$-algebra. Let indeed $\mathfrak{A}$ be a Banach $*$-algebra, with norm $\|\cdot\|$, and let $\mathfrak{B}$ be a $*$-invariant subspace of $\mathfrak{A}$. Then $\mathfrak{B}$ is a partial $*$-algebra with the relation

$$\Gamma = \{(x, y) \in \mathfrak{B} : xy \in \mathfrak{B}\}.$$

Then, if $(x, y) \in \Gamma$, $\|xy\| \leq \|x\|\,\|y\|$. If $\mathfrak{B}$ is closed with respect to the norm $\|\cdot\|$, then $\mathfrak{B}$ is a Banach space and thus a Banach partial $*$-algebra.

Conversely, under appropriate conditions, a Banach partial $*$-algebra is in fact a genuine Banach $*$-algebra.

Proposition 3.2. *Let $\mathfrak{A}$ be a Banach partial $*$-algebra with norm $\|\cdot\|$. Assume that*

(i) $\|ab\| \leq \|a\|\,\|b\|$ *whenever $a \in L(b)$;*
(ii) $R\mathfrak{A}$ *is $\|\cdot\|$-dense in $\mathfrak{A}$.*

Then $\mathfrak{A}$ is a Banach $*$-algebra.

The result is immediate, since $\mathfrak{A}$ is the norm closure of $R\mathfrak{A}$ and by (i) the multiplication is jointly continuous on $R\mathfrak{A}$.

4. Topologies on multiplier spaces

From now on, $\mathfrak{A}$ denotes a normed partial $*$-algebra with unit. We consider arbitrary multiplier spaces of $\mathfrak{A}$ and define intrinsic topologies on them, on the model of $L\mathfrak{A}$ above. Let $\mathfrak{M} \in \mathcal{F}^R$. To every $a \in L\mathfrak{M}$, one may associate a linear map L_a from $\mathfrak{M}$ into $\mathfrak{A}$:

$$L_a(x) = ax, \quad x \in \mathfrak{M} \text{ and } a \in L\mathfrak{M}.$$

Then the topology $\rho_{\mathfrak{M}}$ on $\mathfrak{M}$ is defined as the weakest locally convex topology on $\mathfrak{M}$ for which all maps L_a, $a \in L\mathfrak{M}$, are continuous from $\mathfrak{M}$ into $\mathfrak{A}[\|\cdot\|]$. Thus the (projective) topology $\rho_{\mathfrak{M}}$ is characterized by the set of seminorms

$$x \in \mathfrak{M} \mapsto \|ax\|, \qquad a \in L\mathfrak{M}.$$

It follows that $\rho_{\mathfrak{M}}$ is finer than the topology induced on $\mathfrak{M}$ by the norm of $\mathfrak{A}$ (take $a = e$ in the seminorms above).

In the same way, the topology $\lambda_{\mathfrak{N}}$ on $\mathfrak{N} \in \mathcal{F}^L$ is the weakest locally convex topology on $\mathfrak{N}$ such that all maps $R_b : x \mapsto xb$, $b \in R\mathfrak{N}$, are continuous from $\mathfrak{N}$ into $\mathfrak{A}[\|\cdot\|]$. Thus, the topology $\lambda_{L\mathfrak{M}}$ on $L\mathfrak{M}$ is defined by the set of seminorms

$$a \in L\mathfrak{M} \mapsto \|ax\|, \qquad x \in \mathfrak{M}.$$

Notice that, by Lemma 6.1.2 of [2], the topologies $\rho_{\mathfrak{A}}$ and $\lambda_{\mathfrak{A}}$ on $\mathfrak{A}$ are both equivalent to the original norm topology.

Consider now an arbitrary $\mathfrak{M} \in \mathcal{F}^R$ and the corresponding $L\mathfrak{M} \in \mathcal{F}^L$. Apart from $\rho_{\mathfrak{M}}$ and $\lambda_{L\mathfrak{M}}$, other topologies can be defined on $\mathfrak{M}$ and $L\mathfrak{M}$, respectively, starting from the fact that $L\mathfrak{M}$ may be identified with a space of linear maps from $\mathfrak{M}$ into $\mathfrak{A}$. Let $\mathcal{G}$ be a bounded subset of $\mathfrak{M}[\rho_{\mathfrak{M}}]$ and $a \in L\mathfrak{M}$. We put

$$\|a\|_{\mathcal{G}} = \sup_{x \in \mathcal{G}} \|ax\|.$$

The family of seminorms defined in this way endows $L\mathfrak{M}$ with a topology $\Lambda_{L\mathfrak{M}}$ finer than $\lambda_{L\mathfrak{M}}$. Clearly, $\Lambda_{L\mathfrak{M}}$ coincides with the topology of uniform convergence on bounded sets of $\mathfrak{M}[\rho_{\mathfrak{M}}]$ on the set of continuous linear maps L_a, $a \in L\mathfrak{M}$. One defines in a similar way a topology $\mathrm{P}_{\mathfrak{M}}$ on $\mathfrak{M}$. In general these topologies are neither normable, nor Fréchet.

In order to proceed, we have to study the relationship between the various topologies on a given matching pair $(\mathfrak{M}, L\mathfrak{M})$. Let $\mathfrak{M} \in \mathcal{F}^R$, and let $\|\cdot\|_{\mathfrak{M}}$ be a norm on $\mathfrak{M}$. We say that $\|\cdot\|_{\mathfrak{M}}$ is *admissible* if

$$\rho_{\mathfrak{M}} \preceq \|\cdot\|_{\mathfrak{M}} \preceq \mathrm{P}_{\mathfrak{M}}\,. \tag{4.1}$$

The original norm $\|\cdot\|$ of $\mathfrak{A}$, the norm $\|\cdot\|_{R\mathfrak{A}}$ of $R\mathfrak{A}$, and the norm $\|\cdot\|^{\diamond}_{L\mathfrak{A}}$ of $L\mathfrak{A}$ are clearly admissible.

Assume now that the norm $\|\cdot\|_{\mathfrak{M}}$ on $\mathfrak{M}$ is such that every multiplication operator L_a, $a \in L\mathfrak{M}$, is continuous from $\mathfrak{M}[\|\cdot\|_{\mathfrak{M}}]$ into $\mathfrak{A}[\|\cdot\|]$; that is, there exists $\gamma_a > 0$ such that

$$\|L_a x\| = \|ax\| \leq \gamma_a \|x\|_{\mathfrak{M}}\,, \qquad x \in \mathfrak{M}. \tag{4.2}$$

This is true, in particular, if the norm $\|\cdot\|_{\mathfrak{M}}$ is admissible. Then, generalizing the norm $\|\cdot\|^{\diamond}_{L\mathfrak{A}}$ on $L\mathfrak{A}$, we can define a norm $\|\cdot\|^{\diamond}_{L\mathfrak{M}}$ on $L\mathfrak{M}$ by

$$\|a\|^{\diamond}_{L\mathfrak{M}} = \sup_{\|x\|_{\mathfrak{M}} \leq 1} \|ax\|.$$

Since the unit ball of $\mathfrak{M}[\|\cdot\|_{\mathfrak{M}}]$ is bounded in $\mathfrak{M}[\rho_{\mathfrak{M}}]$, it follows that $\|\cdot\|^{\diamond}_{L\mathfrak{M}}$ is admissible, in the sense that

$$\lambda_{L\mathfrak{M}} \preceq \|\cdot\|^{\diamond}_{L\mathfrak{M}} \preceq \Lambda_{L\mathfrak{M}}. \tag{4.3}$$

Moreover, it follows from the definition that

$$\|ax\| \leq \|a\|^{\diamond}_{L\mathfrak{M}}\, \|x\|_{\mathfrak{M}} \qquad \text{for all } a \in L\mathfrak{M} \text{ and } x \in \mathfrak{M}.$$

In a similar way, we can define a new norm $\|\cdot\|^{\diamond\diamond}_{\mathfrak{M}}$ on $\mathfrak{M}$ by

$$\|x\|^{\diamond\diamond}_{\mathfrak{M}} = \sup_{\|a\|^{\diamond}_{L\mathfrak{M}} \leq 1} \|ax\|.$$

It is easily seen that $\|x\|^{\diamond\diamond}_{\mathfrak{M}} \leq \|x\|_{\mathfrak{M}}$, for every $x \in \mathfrak{M}$, and that $\|x\|^{\diamond\diamond}_{\mathfrak{M}}$ is admissible. Moreover,

$$\|ax\| \leq \|a\|^{\diamond}_{L\mathfrak{M}}\, \|x\|^{\diamond\diamond}_{\mathfrak{M}} \qquad \text{for all } a \in L\mathfrak{M} \text{ and } x \in \mathfrak{M},$$

which is closely reminiscent of the Hölder inequality.

If $\|\cdot\|^{\diamond\diamond}_{\mathfrak{M}}$ is strictly weaker than $\|\cdot\|_{\mathfrak{M}}$, then we can start the procedure again and define a new norm $\|\cdot\|^{\diamond\diamond\diamond}_{L\mathfrak{M}}$ on $L\mathfrak{M}$. We expect that, exactly as for semireflexive, but nonreflexive, Banach spaces, this procedure will never stop. First, it is easily seen that $\|\cdot\|^{\diamond\diamond\diamond}_{L\mathfrak{M}} \leq \|\cdot\|^{\diamond}_{L\mathfrak{M}}$. But we cannot go further, the procedure stops there.

Proposition 4.1. *If $\|\cdot\|_{\mathfrak{M}}$ is admissible, then one has $\|\cdot\|^{\diamond\diamond\diamond}_{L\mathfrak{M}} = \|\cdot\|^{\diamond}_{L\mathfrak{M}}$.*

A proof may be found in [5, Prop.4.2].

Following the pattern familiar for von Neumann algebras,[1] we define a distinguished class of norms.

Definition 4.2. An admissible norm $\|\cdot\|_{\mathfrak{M}}$ on $\mathfrak{M}$ is said to be *reproducing* if $\|\cdot\|^{\diamond\diamond}_{\mathfrak{M}}$ is equivalent to $\|\cdot\|_{\mathfrak{M}}$. Then $\mathfrak{M}[\|\cdot\|_{\mathfrak{M}}]$ itself is said to be reproducing.

Clearly, if $\mathfrak{M}$ carries a norm $\|\cdot\|_{\mathfrak{M}}$ that satisfies condition (4.2), then it also carries an admissible and a reproducing norm, namely, $\|\cdot\|^{\diamond\diamond}_{\mathfrak{M}}$. Moreover, a norm $\|\cdot\|_{\mathfrak{M}}$ can be reproducing only if it is admissible.

As we said above, the topology $\rho_{\mathfrak{M}}$ on the multiplier space $\mathfrak{M}$ is in general not normable, nor even Fréchet. However, *sequential* completeness of $\mathfrak{M}[\rho_{\mathfrak{M}}]$ has nice consequences on $\mathfrak{M}[\|\cdot\|_{\mathfrak{M}}]$. Indeed one has the following result given in [5, Theor.4.7].

Theorem 4.3. *Let $\mathfrak{M}[\rho_{\mathfrak{M}}]$ be sequentially complete, and let $\|\cdot\|_{\mathfrak{M}}$ be an admissible norm on $\mathfrak{M}$. Then the following statements are equivalent:*

(i) $\|\cdot\|_{\mathfrak{M}}$ *is reproducing;*
(ii) $\mathfrak{M}[\|\cdot\|_{\mathfrak{M}}]$ *is a Banach space;*
(iii) $\|\cdot\|_{\mathfrak{M}}$ *is the unique (up to equivalence) admissible Banach norm on $\mathfrak{M}$.*

Remark 4.4. Note that the implication (ii) $\Rightarrow$ (iii) does *not* rely on the assumption of sequential completeness: if $\mathfrak{M}[\|\cdot\|_{\mathfrak{M}}]$ is a Banach space for an admissible norm, then $\mathfrak{M}[\|\cdot\|_{\mathfrak{M}}]$ has, at most, one Banach admissible norm.

We have introduced in (4.1) and (4.3) several, comparable, norms on $\mathfrak{M}$ and $L\mathfrak{M}$, respectively. The natural question is to ascertain when some of these norms are equivalent. The following results are easy (a detailed proof may be found in [5, Prop. 4.9]).

Proposition 4.5. *Given $\mathfrak{M} \in \mathcal{F}^R$, assume that $\mathfrak{M}[\|\cdot\|_{\mathfrak{M}}]$ is a Banach space. Then P_m is equivalent to $\|\cdot\|^{\diamond\diamond}_{\mathfrak{M}}$ and $\|\cdot\|_{\mathfrak{M}}$ is admissible if and only if it is reproducing. Similarly, if $L\mathfrak{M}[\|\cdot\|^{\diamond}_{L\mathfrak{M}}]$ is a Banach space, then $\Lambda_{L\mathfrak{M}}$ is equivalent to $\|\cdot\|^{\diamond}_{L\mathfrak{M}}$.*

Proof.

First one shows that, if $\mathfrak{M}[\|\cdot\|_{\mathfrak{M}}]$ is a Banach space, one has $\rho_{\mathfrak{M}} \preceq \|\cdot\|^{\diamond\diamond}_{\mathfrak{M}} \sim \mathrm{P}_{\mathfrak{M}} \preceq \|\cdot\|_{\mathfrak{M}}$, which proves that $\|\cdot\|_{\mathfrak{M}}$ is admissible if and only if it is reproducing. The statement concerning $L\mathfrak{M}$ is proven in the same way. $\square$

[1] For any set $\mathfrak{A}$ of bounded operators containing the identity, the commutants satisfy the relations $\mathfrak{A}''' = \mathfrak{A}', \mathfrak{A} \subset \mathfrak{A}''$; then $\mathfrak{A}$ is a von Neumann algebra if $\mathfrak{A}'' = \mathfrak{A}$.

5. Banach partial $*$-algebras of type (B)

The preceding considerations show clearly that there is a deep analogy between partial $*$-algebras and PIP-spaces [4], the exchange under L or R replacing duality. For the convenience of the reader, we have collected in the Appendix the basic facts concerning PIP-spaces.

In the case of a PIP-space V, we have a complete involutive lattice $(V, \#)$, with involution $V_r \leftrightarrow V_{\bar{r}} = (V_r)^{\#}$. In addition, the whole structure can be reconstructed from a *generating* involutive sublattice $\mathcal{J}$ of $\mathcal{F}(V, \#)$, indexed by J, which means that

$$f \# g \iff \text{there exists } r \in J \text{ such that } f \in V_r \text{ and } g \in V_{\bar{r}}\,.$$

In the present case, we have two complete lattices $\mathcal{F}^R$ and $\mathcal{F}^L$, which are exchanged under L and R, respectively. Here too, the whole multiplication structure may be recovered from a generating family; that is, a subset $\mathcal{I}^R$ of $\mathcal{F}^R$ such that $x \in L(y)$ if and only if there is an element $\mathfrak{M} \in \mathcal{I}^R$ such that $y \in \mathfrak{M}$ and $x \in L\mathfrak{M}$,

$$x \in L(y) \iff \text{there exists } \mathfrak{M} \in \mathcal{I}^R \text{ such that } y \in \mathfrak{M} \text{ and } x \in L\mathfrak{M}\,.$$

Now, in the case of a PIP-space, an interesting (and practically sufficient) situation is obtained when all the elements of the generating sublattice are reflexive Banach spaces or Hilbert spaces in duality (LBS or LHS) (see the Appendix). By analogy, we are led to impose a perfect symmetry between left and right multipliers of our Banach partial $*$-algebra, and thus to require that the two spaces of a pair of matching subspaces $(\mathfrak{M}, L\mathfrak{M})$ be *both* Banach spaces for an admissible norm. These norms are then automatically reproducing and coincide with $\|\cdot\|_{\mathfrak{M}} \sim \|\cdot\|^{\diamond\diamond}_{\mathfrak{M}}$ and $\|\cdot\|^{\diamond}_{L\mathfrak{M}}$, respectively.

Our aim is to obtain a object in which the algebraic and the topological structures fit perfectly. To that effect, it is necessary to require that the multiplier spaces $\mathfrak{M} \in \mathcal{I}^R$, where $\mathcal{I}^R$ is a generating family, be *complete* in a natural norm $\|\cdot\|_{\mathfrak{M}}$, and similarly for the corresponding $L\mathfrak{M}$. Indeed, these spaces are completely determined by the partial multiplication (i.e., the set Γ). If one of them, say $\mathfrak{M}$, would be noncomplete, it could be embedded into its completion $\widetilde{\mathfrak{M}}$ with respect to $\|\cdot\|_{\mathfrak{M}}$, but nothing guarantees that the latter is still contained in $\mathfrak{A}$, and thus there is *a priori* no way of extending the partial multiplication to $\widetilde{\mathfrak{M}}$. This is exactly the same philosophy as that governing the construction of lattices of Hilbert spaces or, more generally, indexed PIP-spaces [4]: the elements of a generating family are always supposed to be complete; that is, Banach or Hilbert spaces, but no assumption is made on the global space V. Thus, on $\mathfrak{A}$ itself, the completion condition may be dispensed of; so that we can start both from a normed partial $*$-algebra and from a Banach partial $*$-algebra.

The condition that multiplier spaces be Banach has the further advantage to ensure the proper behavior of natural embeddings. Clearly, if $\mathfrak{M}_1 \subset \mathfrak{M}_2$ and both spaces carry their $\rho_{\mathfrak{M}}$ topology, then the embedding $\mathfrak{M}_1 \to \mathfrak{M}_2$ is continuous. If both spaces are Banach and carry their natural norm topology, then the embedding $\mathfrak{M}_1[\|\cdot\|^{\diamond\diamond}_{\mathfrak{M}_1}] \to \mathfrak{M}_2[\|\cdot\|^{\diamond\diamond}_{\mathfrak{M}_2}]$ is continuous as well. Indeed, since $\mathfrak{M}_1 \subset \mathfrak{M}_2$, one has $L\mathfrak{M}_2 \subset L\mathfrak{M}_1$ and $\lambda_{\mathfrak{M}_2}$ is finer than the topology induced on $L\mathfrak{M}_2$ by $\lambda_{\mathfrak{M}_1}$. Thus every $\lambda_{\mathfrak{M}_2}$-bounded subset of $L\mathfrak{M}_2$ is $\lambda_{\mathfrak{M}_1}$-bounded, and therefore, $\mathrm{P}_{\mathfrak{M}_1}$ is finer than $\mathrm{P}_{\mathfrak{M}_2}$, which means that $\|\cdot\|_{\mathfrak{M}_2} \leq \|\cdot\|_{\mathfrak{M}_1}$, as announced.

Therefore, following the pattern of PIP-spaces, we impose the Banach condition on the elements a generating family. Thus we introduce the following class of Banach partial $*$-algebras.

Definition 5.1. A normed partial $*$-algebra or a Banach partial $*$-algebra $\mathfrak{A}[\|\cdot\|]$ is said to be *of type* (B) if there exists a generating family $\mathcal{I}^R$ such that, for each pair of matching subspaces $\mathfrak{M} \in \mathcal{I}^R$ and $L\mathfrak{M} \in \mathcal{I}^L$, both spaces are Banach spaces for a reproducing norm.

Remarks 5.2. (1) $\mathfrak{A}$ itself has a reproducing norm, namely, $\|\cdot\|^{\diamond\diamond}$, *a priori* weaker than the original norm. However, since $\rho_{\mathfrak{A}}$ is equivalent to the original norm topology, one has always $\|\cdot\|^{\diamond\diamond} \sim \|\cdot\|$, whether $\mathfrak{A}$ is complete or not.

(2) We remind the reader that completeness of $\mathfrak{M}$ does not imply that of $L\mathfrak{M}$; thus we have to impose both explicitly.

As usual, one may consider the lattice obtained from the generating family under finite lattice operations. In the present case, all elements of that lattice, which is, of course, generating as well, are Banach spaces, with the norms borrowed from interpolation theory.

- $\mathfrak{M} \wedge \mathfrak{N} = \mathfrak{M} \cap \mathfrak{N}$, which is a Banach space with the projective norm $\|f\|_{\mathfrak{M}\wedge\mathfrak{N}} = \|f\|_{\mathfrak{M}} + \|f\|_{\mathfrak{N}}$.
- $\mathfrak{M} \vee \mathfrak{N} = RL(\mathfrak{M} + \mathfrak{N})$; now $\mathfrak{M} + \mathfrak{N}$ is a Banach space with the inductive norm $\|f\|_{\mathfrak{M}\vee\mathfrak{N}} = \inf\left(\|g\|_{\mathfrak{M}} + \|h\|_{\mathfrak{N}}\right)$, $f = g + h$, $g \in \mathfrak{M}$, $h \in \mathfrak{N}$, and it remains to show that it belongs to $\mathcal{F}^R$; that is, $\mathfrak{M} + \mathfrak{N} = RL(\mathfrak{M} + \mathfrak{N})$.

Then one can build the complete lattice $\mathcal{F}^R$, by applying arbitrary lattice operations, but the additional spaces so obtained are no longer Banach spaces in general.

As mentioned in section 3, one may obtain a Banach partial $*$-algebra simply by considering a closed $*$-invariant subspace $\mathfrak{B}$ of a Banach $*$-algebra $\mathfrak{A}$. We distinguish two situations.

Case 1 : Assume that $\mathfrak{B}$ is closed with respect to the norm $\|\cdot\|$. Then $\mathfrak{B}$ is a Banach space. Let $\mathfrak{M} \in \mathcal{F}^R := \mathcal{F}^R(\mathfrak{B})$ be the multiplier lattice of $\mathfrak{B}$. Then, for every $a \in L\mathfrak{M}$, the map $L_a : x \in \mathfrak{M} \mapsto ax \in \mathfrak{A}$ is continuous and hence closed. The restriction $\|\cdot\|_{\restriction\mathfrak{M}}$ of the norm $\|\cdot\|$ to $\mathfrak{M}$ has clearly the property (4.2). We prove that $\mathfrak{M}$ is closed with respect to $\|\cdot\|$. Let indeed $x_n \in \mathfrak{M} = RL\mathfrak{M}$ be with $x_n \to x \in \mathfrak{A}$. Then, for any $b \in L\mathfrak{M}$, one has $x_n b \to xb$. Since $x_n b \in \mathfrak{B}$ and $\mathfrak{B}$ is closed, we get $xb \in \mathfrak{B}$; thus $x \in RL\mathfrak{M} = \mathfrak{M}$. Therefore, each multiplier space is Banach under $\|\cdot\|$. This implies that, for every $\mathfrak{M} \in \mathcal{F}^R$, $\|\cdot\|^{\diamond\diamond}_{\mathfrak{M}} \sim \|\cdot\|_{\restriction\mathfrak{M}}$, since both are Banach norms and $\|\cdot\|^{\diamond\diamond}_{\mathfrak{M}} \preceq \|\cdot\|_{\restriction\mathfrak{M}}$. So $\mathfrak{B}$ is a Banach partial $*$-algebra of type (B).

Case 2 : $\mathfrak{B}$ is not closed with respect to the norm $\|\cdot\|$, but it carries another norm $\|\cdot\|_{\mathfrak{B}}$ that makes it into a Banach space. If each $\mathfrak{M} \in \mathcal{F}^R$ is Banach for a norm $\|\cdot\|_{\mathfrak{M}}$ satisfying the condition (4.1), then this norm is necessarily reproducing, by Theorem 4.3, taking into account that right and left multiplications are continuous maps. Thus, in this case too, $\mathfrak{B}$ is a Banach partial $*$-algebra of type (B).

6. Examples of Banach partial $*$-algebras of type (B)

In [5] and in the monograph [2], one may find a whole family of examples of Banach partial $*$-algebras of functions and Banach partial $*$-algebras of operators acting on a lattice of Hilbert spaces. We will review some of these examples here, without too much detail. In some cases, we will show how these examples illustrate the propositions above about the equivalence of the various topologies on multiplier spaces.

6.1. **Partial $*$-algebras of functions.**

6.1.1. *L^p spaces on a finite interval.* The simplest example of an abelian partial $*$-algebra is the space $L^1([0,1], \mathrm{d}x)$, equipped with the partial multiplication,

$$f \in M(g) \;\Leftrightarrow\; \exists\, q \in [1,\infty] \;\text{ such that } f \in L^q,\, g \in L^{\overline{q}},\; 1/q + 1/\overline{q} = 1.$$

Thus we consider as generating family the chain of Banach spaces $\mathcal{I}_0 = \{L^p([0,1], \mathrm{d}x), 1 \le p \le \infty\}$, with $L^p \subset L^q$, $p > q$. The lattice completion of $\mathcal{I}_0$, denoted $\mathcal{F}$, is obtained by adding the so-called “nonstandard” spaces

$$L^{p-} = \bigcap_{1\le q<p} L^q, \quad L^{p+} = \bigcup_{p<q\le \infty} L^q.$$

Then, for $1 < p \le \infty$, L^{p-}, with the projective topology, is a non-normable reflexive Fréchet space. For $1 \le p < \infty$, L^{p+}, with the inductive topology, is a nonmetrizable complete DF-space [17, 21].

We note the strict inclusions,

$$L^{p+} \;\subset\; L^p \;\subset\; L^{p-} \;\subset\; L^{q+} \qquad (1 < q < p < \infty)$$

in which all embeddings are continuous and have dense range.

As a consequence of the Hölder inequality, the multiplier spaces are

$$ML^p = L^{\overline{p}}, \quad ML^{p-} = L^{\overline{p}+}, \quad ML^{p+} = L^{\overline{p}-}$$

As for topologies, take first the spaces L^p, $1 \le p < \infty$. The following result is standard [4] or [28, Chap.15]:

$$\|f\|^{\diamond}_{\overline{p}} = \sup_{\|g\|_p\le 1} \int_0^1 |fg|\,\mathrm{d}x = \sup_{\|g\|_p\le 1} \left|\int_0^1 fg\,\mathrm{d}x\right| = \|f\|_{\overline{p}}\,, \qquad 1 \le p < \infty.$$

By the same argument, $\|f\|^{\diamond\diamond}_p = \|f\|_p$. Combining this result with Proposition 4.5, we obtain

$$\rho_{L^p} \preceq \|\cdot\|^{\diamond\diamond}_p = \|\cdot\|_p \sim \mathrm{P}_{L^p}, \qquad 1 \le p < \infty.$$

One can show [5] that every L^p, $1 < p \le \infty$, is sequentially complete for ρ_{L^p}. For $p = 1$, one can prove directly that ρ_{L^1} coincides with the usual norm topology (as it should!), using the fact that the function $f_0(x) \equiv 1$ belongs to L^1 with

$\|f_0\|_1 = 1$. However, contrary to what is said in [1] and in [2], the topology ρ_{L^p} does *not* coincide with the $\|\cdot\|_p$-norm topology for $p > 1$.

The "nonstandard" multiplier spaces $L^{p\pm}$ do not belong to the generating family; so we don't have to take them into consideration.

In conclusion, $\mathfrak{A} = L^1[(0,1), \mathrm{d}x]$ is an abelian Banach partial $*$-algebra of type (B), and it is tight, which means that $R\mathfrak{A} = L^\infty[(0,1), \mathrm{d}x]$ is dense in every multiplier space L^p.

6.1.2. *The spaces* $L^p(\mathbb{R}, \mathrm{d}x)$*.* We turn now to the spaces $L^p(\mathbb{R}, \mathrm{d}x)$ on the whole line, discussed in full generality in [4] and also in [5]. Hence we will be brief here. The difference with the previous case is that these no longer form a chain, no two of them being comparable. We have only

$$L^p \cap L^q \subset L^s \text{ for all } s \text{ such that } p < s < q.$$

Hence we take the lattice generated by $\mathcal{I} = \{L^p(\mathbb{R}, \mathrm{d}x), 1 \leq p \leq \infty\}$, that we call $\mathcal{J}$.

At this stage, it is convenient to introduce a unified notation:

$$L^{(p,q)} = \begin{cases} L^p \wedge L^q, & \text{if } p \geq q, \\ L^p \vee L^q, & \text{if } p \leq q. \end{cases}$$

Thus, for $1 < p, q < \infty$, each space $L^{(p,q)}$ is a reflexive Banach space, with conjugate dual $L^{(\overline{p},\overline{q})}$. The modifications when p, q equal 1 or ∞ are obvious.

Following [4, Sec.4.1.2], we represent the space $L^{(p,q)}$ by the point $(1/p, 1/q)$ of the unit square $\mathrm{J} = [0,1] \times [0,1]$. In this representation, the spaces L^p are on the main diagonal, intersections $L^p \cap L^q$ above it and sums $L^p + L^q$ below, the duality is $[L^{(s}]^\times = L^{(\overline{s})}$, where $s = (p,q)$ and $\overline{s} = (\overline{p}, \overline{q})$; that is, symmetry with respect to $L^2 = (\frac{1}{2}, \frac{1}{2})$. Hence, $L^{(p,q} \subset L^{(p',q')}$ if $(1/p, 1/q)$ is on the left and/or above $(1/p', 1/q')$; that is,

$$L^{(p,q)} \subset L^{(p',q')} \iff (p,q) \leq (p',q') \iff p \geq p' \text{ and } q \leq q'. \tag{6.1}$$

A figure representing the lattice $\mathcal{J}$ may be found, for instance, in [4, Fig.4.1] (and also on the cover page!).

The extreme spaces of the (complete) lattice are

$$V_{\mathrm{J}} = L_G := L^1 + L^\infty \quad \text{and} \quad V_{\mathrm{J}}^\# = L_G^\# = L^\infty \cap L^1$$

with their inductive and projective norms, respectively, which make them into nonreflexive Banach spaces (none of them is the dual of the other). Notice that the space L_G, known as the space of Gould [14], contains strictly all the L^p, $1 \leq p \leq \infty$. Here too, the lattice structure allows to give to V_{J} a structure of abelian Banach partial $*$-algebra of type (B). Notice that this partial $*$-algebra does have a unit, as we have assumed in general, namely, the function $f_0(x) \equiv 1$, which belongs to L^∞, but, of course, not to any space $L^p(\mathbb{R}, \mathrm{d}x)$, $p < \infty$.

The lattice operations on $\mathcal{J}$ are those familiar in interpolation theory

$$L^p \wedge L^q = L^p \cap L^q \text{ and } L^p \vee L^q = L^p + L^q,$$

which are Banach spaces under the projective and inductive norms, respectively, as indicated in section 5. Notice that the lattice $\mathcal{J}$ is already obtained at the first generation. one has, for example, $L^{(r,s)} \wedge L^{(a,b)} = L^{(r\vee a, s\wedge b)}$, where $L^{(r,s)} = L^r \wedge L^s$, if $r > s$ and $L^{(r,s)} = L^r \vee L^s$, if $r < s$. As for the lattice completion $\mathcal{F}_{\rm J}$, one can build an "enriched" or "nonstandard" square, exactly as in the previous section.

Now we endow $V_{\rm J}$ with the natural partial multiplication

$$f \in M(g) \iff fg \in V_{\rm J}, \text{ i.e., } fg \in L^s, \text{ for some } s,\ 1 \leq s \leq \infty.$$

Then the multipliers of the basic spaces are simple, namely, for $p > q$,

$$M(L^p \wedge L^q) = L^{\overline{p}} + L^\infty = L^{(\overline{p},\infty)}, \quad M(L^p \vee L^q) = L^{\overline{q}} + L^\infty = L^{(\overline{q},\infty)},$$

and thus

$$MM(L^p \wedge L^q) = L^p + L^\infty = L^{(p,\infty)}, \quad MM(L^p \vee L^q) = L^q + L^\infty = L^{(q,\infty)}.$$

Thus matching pairs are of the form $(L^{(p,\infty)}, L^{(\overline{p},\infty)})$. Since $L^{(\overline{q},\infty)} \subset L^{(\overline{p},\infty)}$ for $q < p$, these multiplier spaces form a chain of Banach spaces, isomorphic, as LBS, to the chain $\mathcal{I}_0 := \{L^p([0,1], \mathrm{d}x), 1 \leq p \leq \infty\}$.

Thus we may state the following proposition.

Proposition 6.1. *The space $L_G := L^1(\mathbb{R}, \mathrm{d}x) + L^\infty(\mathbb{R}, \mathrm{d}x)$ is a nonreflexive LBS, generated by the family $\mathcal{I} = \{L^p(\mathbb{R}, \mathrm{d}x), 1 \leq p \leq \infty\}$ and the corresponding compatibility $(L^p)^\# = L^{\overline{p}}$. In addition, L_G is an abelian Banach partial $*$-algebra of type (B), whose generating family $\mathcal{I}_1 := \{L^{(p,\infty)}, 1 \leq p \leq \infty\}$ is isomorphic, as LBS, to the chain $\mathcal{I}_0 = \{L^p([0,1], \mathrm{d}x), 1 \leq p \leq \infty\}$.*

6.1.3. *Amalgam spaces.* The lesson of the previous example is that an involutive lattice of (preferably reflexive) Banach spaces turns quite naturally into a (tight) Banach partial $*$-algebra of type (B) if it possesses a partial multiplication that verifies a (generalized) Hölder inequality. A whole class of examples is given by the so-called *amalgam spaces* [13]. The simplest ones are the spaces (L^p, ℓ^q) (sometimes denoted by $W(L^p, \ell^q)$) consisting of functions on $\mathbb{R}$ which are locally in L^p and have ℓ^q behavior at infinity, in the sense that the L^p norms over the intervals $(n, n+1)$ form an ℓ^q sequence. For $1 \leq p, q < \infty$, the norm

$$\|f\|_{p,q} = \left\{ \sum_{n=-\infty}^{\infty} \left[\int_n^{n+1} |f(x)|^p \, \mathrm{d}x \right]^{q/p} \right\}^{1/q}$$

makes (L^p, ℓ^q) into a Banach space. The same is true for the obvious extensions to p and/or q equal to ∞. Notice that $(L^p, \ell^p) = L^p$.

These spaces obey the following (immediate) inclusion relations, with all embeddings continuous.

- If $q_1 \leq q_2$, then $(L^p, \ell^{q_1}) \subset (L^p, \ell^{q_2})$.
- If $p_1 \leq p_2$, then $(L^{p_2}, \ell^q) \subset (L^{p_1}, \ell^q)$.

From this it follows that the smallest space is (L^∞, ℓ^1) and the largest one is (L^1, ℓ^∞), and therefore

- If $p \geq q$, then $(L^p, \ell^q) \subset L^p \cap L^q \subset L^s$ for all $q < s < p$.
- If $p \leq q$, then $(L^p, \ell^q) \supset L^p \cup L^q$.

Once again, Hölder's inequality is satisfied. Whenever $f \in (L^p, \ell^q)$ and $g \in (L^{\overline{p}}, \ell^{\overline{q}})$, then $fg \in L^1$ and one has

$$\|fg\|_1 \leq \|f\|_{p,q} \, \|g\|_{\overline{p},\overline{q}}.$$

Therefore, one has the expected duality relation,

$$(L^p, \ell^q)^* = (L^{\overline{p}}, \ell^{\overline{q}}) \text{ for } 1 \leq q, p < \infty.$$

The interesting fact is that, for $1 \leq p, q \leq \infty$, the set $\mathcal{J}_{\mathrm{W}}$ of all amalgam spaces $\{(L^p, \ell^q)\}$ may be represented by the points (p, q) of the *same* unit square J as in the previous example, with the *same* order structure. In particular, $\mathcal{J}_{\mathrm{W}}$ is a lattice with respect to the order (6.1):

$$\begin{aligned} (L^p, \ell^q) \wedge (L^{p'}, \ell^{q'}) &= (L^{p \vee p'}, \ell^{q \wedge q'}) \\ (L^p, \ell^q) \vee (L^{p'}, \ell^{q'}) &= (L^{p \wedge p'}, \ell^{q \vee q'}), \end{aligned}$$

where again $\wedge$ means intersection with projective norm and $\vee$ means vector sum with inductive norm.

We turn now to the partial ∗-algebra structure of $\mathcal{J}_{\mathrm{W}}$. At first sight, the situation becomes different, because, whereas L^1 is a partial ∗-algebraand ℓ^∞ is an algebra under componentwise multiplication, $(a_n) \cdot (b_n) = (a_n b_n)$. The L^p component characterizes the local behavior. Hence,

$$M(L^p, \ell^q) \supset (L^{\overline{p}}, \ell^\infty), \text{ for all } q,$$

and since the latter are totally ordered, we obtain, exactly as in the case of the L^p spaces,

$$M(L^p, \ell^q) = (L^{\overline{p}}, \ell^\infty).$$

Thus the natural partial multiplication on $\mathcal{J}$ reads.

$$f \in M(g) \iff \exists\, p \in [1, \infty] \text{ such that } f \in (L^p, \ell^\infty) \text{ and } g \in (L^{\overline{p}}, \ell^\infty). \quad (6.2)$$

The rest is as before, including the identification of the complete lattice $\mathcal{F}_{\mathrm{J}}$ with the "enriched" interval $[1, \infty]$. Thus we may state the following proposition.

Proposition 6.2. *The amalgam space (L^1, ℓ^∞), with the partial multiplication defined by (6.2), is a tight commutative Banach partial ∗-algebra of type (B), generated by the family of amalgam spaces $\mathcal{J}_{\mathrm{W}} = \{(L^p, \ell^q), 1 \leq q, p \leq \infty\}$. This Banach partial ∗-algebra is isomorphic to the one generated by the spaces $\{L^p(\mathbb{R}, \mathrm{d}x), 1 \leq p \leq \infty\}$, described in Proposition 6.1. In particular, its generating family $\mathcal{I}_2 = \{(L^p, \ell^\infty), 1 \leq p \leq \infty\}$ is isomorphic, as LBS, to the chain $\mathcal{I}_0 = \{L^p([0,1], \mathrm{d}x), 1 \leq p \leq \infty\}$.*

6.2. **Partial ∗-algebras of operators.**

6.2.1. *Operators on a Hilbert scale.* Let $\mathcal{H}$ be a Hilbert space with inner product $\langle \cdot | \cdot \rangle$, and let $S \geq 1$ be a positive unbounded self-adjoint operator with dense domain $D(S)$. Thus, the subspace $D(S)$ becomes a Hilbert space, denoted by $\mathcal{H}_1$, with the (graph) inner product

$$\langle f | g \rangle_1 = \langle Sf | Sg \rangle.$$

Let $\mathcal{H}_{\overline{1}}$ denote the conjugate dual of $\mathcal{H}_1$. Then $\mathcal{H}_{\overline{1}}$ is itself a Hilbert space.

With this construction, we get in a canonical way a scale of Hilbert spaces

$$\mathcal{H}_1 \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{H}_{\overline{1}},$$

where both inclusions are continuous and have dense range.

For every $\alpha > 0$, S^α is still a self-adjoint positive operator and $S^\alpha \geq 1$. So we can construct for S^α also a scale of Hilbert spaces

$$\mathcal{H}_\alpha \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{H}_{\overline{\alpha}}. \tag{6.3}$$

If $\alpha, \beta \in (0,1)$, with $\beta > \alpha$, then it turns out that

$$\mathcal{H}_1 \hookrightarrow \mathcal{H}_\beta \hookrightarrow \mathcal{H}_\alpha \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{H}_{\overline{\alpha}} \hookrightarrow \mathcal{H}_{\overline{\beta}} \hookrightarrow \mathcal{H}_{\overline{1}}. \tag{6.4}$$

As for the norms, we notice that, if $f \in \mathcal{H}_1$, then

$$\|f\| \leq \|f\|_\alpha \;\leq\; \|f\|_\beta \;\leq\; \|f\|_1 \qquad \text{for all } \alpha \in (0,1). \tag{6.5}$$

Let $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_{\overline{1}})$ be the Banach space of bounded operators from $\mathcal{H}_1$ into $\mathcal{H}_{\overline{1}}$ with its natural norm $\|\cdot\|_{1,\overline{1}}$. In $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_{\overline{1}})$ define an involution $A \mapsto A^*$ by

$$< A^* f, g > = \overline{< Ag, f >}, \qquad \text{for all } \; f, g \in \mathcal{H}_1\,,$$

where $< \cdot, \cdot >$ is the form that puts $\mathcal{H}_1$ and $\mathcal{H}_{\overline{1}}$ in conjugate duality. If $\alpha, \beta \in (-1,1)$ we can also consider the Banach space $\mathcal{B}(\mathcal{H}_\alpha, \mathcal{H}_\beta)$ of bounded operators from $\mathcal{H}_\alpha$ into $\mathcal{H}_\beta$ with its natural norm $\|\cdot\|_{\alpha,\beta}$.

Because of (6.4), the restriction to $\mathcal{H}_1$ of an operator of $\mathcal{B}(\mathcal{H}_\alpha, \mathcal{H}_\beta)$ belongs to $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_{\overline{1}})$. Therefore,

$$\mathcal{B}(\mathcal{H}_\alpha, \mathcal{H}_\beta) \subset \mathcal{B}(\mathcal{H}_1, \mathcal{H}_{\overline{1}}) \qquad \text{for all } \alpha, \beta \in [-1,1].$$

Moreover, $\mathcal{B}(\mathcal{H}_\alpha, \mathcal{H}_\beta)^* = \mathcal{B}(\mathcal{H}_{\overline{\beta}}, \mathcal{H}_{\overline{\alpha}})$ for every $\alpha, \beta \in [-1,1]$.

We define now the partial multiplication in $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_{\overline{1}})$. Let $X, Y \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_{\overline{1}})$. We say that $X \in L(Y)$ if there exist $\alpha, \beta, \gamma \in [-1,1]$ such that $Y \in \mathcal{B}(\mathcal{H}_\alpha, \mathcal{H}_\beta)$ and $X \in \mathcal{B}(\mathcal{H}_\beta, \mathcal{H}_\gamma)$. In this case XY, the usual composition of the maps X and Y, is well-defined and belongs to $\mathcal{B}(\mathcal{H}_\alpha, \mathcal{H}_\gamma) \subset \mathcal{B}(\mathcal{H}_1, \mathcal{H}_{\overline{1}})$. It easily seen that, if XY is well-defined, then Y^*X^* is also well defined and belongs to $\mathcal{B}(\mathcal{H}_{\overline{\gamma}}, \mathcal{H}_{\overline{\alpha}})$. Moreover $(XY)^* = Y^*X^*$. As a result, $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_{\overline{1}})$ with this multiplication is a partial $*$-algebra.

Next we have to identify the spaces of multipliers. By the definition of multiplication given above, it follows that the family of spaces $\{\mathcal{B}(\mathcal{H}_\alpha, \mathcal{H}_\beta)\}$ is a generating sublattice for the lattice of left (or even right) multipliers. An easy calculation gives the following result.

Proposition 6.3. *For every $\alpha, \beta \in [-1,1]$, one has $L\mathcal{B}(\mathcal{H}_\alpha, \mathcal{H}_\beta) = \mathcal{B}(\mathcal{H}_\beta, \mathcal{H}_{\overline{1}})$ and $R\mathcal{B}(\mathcal{H}_\alpha, \mathcal{H}_\beta) = \mathcal{B}(\mathcal{H}_1, \mathcal{H}_\alpha)$.*

Thus we get the same structure for the multiplier spaces as in the case of the L^p spaces on the line discussed in section 6.1.2.

Lemma 6.4. *The family $\mathcal{I}^R = \{\mathcal{B}(\mathcal{H}_1, \mathcal{H}_\beta);\ \beta \in [-1,1]\}$ generates the lattice $\mathcal{F}^R$ of right multipliers. Consequently, the family $L\mathcal{I}^R = \{\mathcal{B}(\mathcal{H}_\beta, \mathcal{H}_{\overline{1}});\ \beta \in [-1,1]\}$ generates the lattice $\mathcal{F}^L$ of left multipliers.*

Proof. The product XY is well-defined if and only if there exist $\alpha, \beta, \gamma \in [-1, 1]$ such that $X \in \mathcal{B}(\mathcal{H}_\beta, \mathcal{H}_\gamma)$ and $Y \in \mathcal{B}(\mathcal{H}_\alpha, \mathcal{H}_\beta)$ and, in this case $XY \in \mathcal{B}(\mathcal{H}_\alpha, \mathcal{H}_\gamma)$. Of course X may be regarded as an element of $\mathcal{B}(\mathcal{H}_\beta, \mathcal{H}_{\bar{1}})$. On the other hand, the restriction $Y_{(1)}$ of Y to $\mathcal{H}_1$ is an element of $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_\beta)$. Clearly $XY = XY_{(1)}$. □

Now we turn to the topological structure. The topology $\rho_{\mathcal{B}(\mathcal{H}_\alpha,\mathcal{H}_\beta)}$ on $\mathcal{B}(\mathcal{H}_\alpha,\mathcal{H}_\beta)$ is defined by the family of seminorms,

$$A \in \mathcal{B}(\mathcal{H}_\alpha, \mathcal{H}_\beta) \mapsto \|(XA)\lceil_{\mathcal{H}_1}\|_{1,\bar{1}}, \quad X \in \mathcal{B}(\mathcal{H}_\beta, \mathcal{H}_{\bar{1}}).$$

Since

$$\|(XA)\lceil_{\mathcal{H}_1}\|_{1,\bar{1}} \leq \|XA\|_{\alpha,\bar{1}} \leq \|X\|_{\beta,\bar{1}}\|A\|_{\alpha,\beta},$$

it follows that $\rho_{\mathcal{B}(\mathcal{H}_\alpha,\mathcal{H}_\beta)}$ is coarser than the topology defined by $\|\cdot\|_{\alpha,\beta}$. Then we can start the procedure outlined in section 4 to construct admissible or reproducing norms. We start with considering a space $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_\beta)$ with $\beta \in [-1, 1]$; that is, an element of $\mathcal{I}^R$ and the corresponding set of left multipliers $L\mathcal{B}(\mathcal{H}_1, \mathcal{H}_\beta) = \mathcal{B}(\mathcal{H}_\beta, \mathcal{H}_{\bar{1}})$. Clearly, for $X \in \mathcal{B}(\mathcal{H}_\beta, \mathcal{H}_{\bar{1}})$ and $Y \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_\beta)$, we have

$$\|XY\|_{1,\bar{1}} \leq \|X\|_{\beta,\bar{1}}\|Y\|_{1,\beta},$$

which entails, in particular, that the norm $\|\cdot\|_{1,\beta}$ satisfies condition (4.2).

This implies that $\|X\|^\diamond_{L\mathcal{B}(\mathcal{H}_1,\mathcal{H}_\beta)} \leq \|X\|_{\beta,\bar{1}}$. On the other hand, taking $Y = S^{1-\beta} \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_\beta)$, we have $\|S^{1-\beta}\|_{1,\beta} = 1$ and $\|XS^{1-\beta}\|_{1,\bar{1}} = \|X\|_{\beta,\bar{1}}$. Therefore

$$\|X\|^\diamond_{L\mathcal{B}(\mathcal{H}_1,\mathcal{H}_\beta)} = \sup_{\|Y\|_{1,\beta}\leq 1} \|XY\|_{1,\bar{1}} \geqslant \|X\|_{\beta,\bar{1}}.$$

Thus $\|\cdot\|^\diamond_{L\mathcal{B}(\mathcal{H}_1,\mathcal{H}_\beta)} = \|\cdot\|_{\beta,\bar{1}}$. One can prove in a similar way that $\|\cdot\|^{\diamond\diamond}_{\mathcal{B}(\mathcal{H}_1,\mathcal{H}_\beta)} = \|\cdot\|_{1,\beta}$. Note, however, that the natural norm of a space $\mathcal{B}(\mathcal{H}_\alpha, \mathcal{H}_\beta)$, with $\alpha < 1$, is *not* reproducing, in general, as can be shown by an easy computation.

Thus the Banach partial ∗-algebra $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_{\bar{1}})$ has a generating family of Banach spaces, each of them endowed with a reproducing norm.

Proposition 6.5. *The space $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_{\bar{1}})$, with the structure described above, is a Banach partial ∗-algebra of type (B).*

Remarks 6.6. (1) As in the case of the L^p chain discussed in section 6.1.1, one may enrich the scale (6.4) by introducing "nonstandard" spaces $\mathcal{H}_{\beta^-}$ and $\mathcal{H}_{\beta^+}$ (which, of course, are no longer Banach spaces) and operators from/into them. A detailed analysis may be found in [5]. This is in fact an application of interpolation theory, and it was explicitly developed in the context of PIP-spaces by Karwowski and one of us in [3].

(2) In fact the situation described here is perfectly general. Indeed, if $\mathcal{H}_* \subset \mathcal{H} \subset \mathcal{H}^*$ is any scale of Hilbert spaces, then, by the second representation theorem for sesquilinear forms [16, VI.2.6], there exists a self-adjoint operator $S \geq 1$ such that $D(S) = \mathcal{H}_*$ and $\langle f|g\rangle_* = \langle Sf|Sg\rangle$ for all $f, g \in \mathcal{H}_*$.The same construction can be extended to an unbounded scale of Hilbert spaces. Then, however, the full ambient space is no longer a Banach space, but an inductive limit of Banach spaces. This suggests the extension of the partial algebraic structure to such situations as well.

In conclusion, we emphasize that the structure just analyzed, operators on a Hilbert scale, occurs frequently both in mathematics and in physics. Standard examples include.

- The scale of Sobolev spaces $W^2_s(\mathbb{R})$, $s \in \mathbb{R}$, where $f \in W^2_s(\mathbb{R})$ whenever its Fourier transform $\widehat{f}$ satisfies the condition $(1+|.|^2)^{s/2}\,\widehat{f} \in L^2(\mathbb{R})$. The corresponding norm reads as $\|f\|_s = \|(1+|.|^2)^{s/2}\,\widehat{f}\|, s \in \mathbb{R}$. Here the defining operator is $(A_{\rm m} f)(x) = (1 - \frac{d^2}{dx^2})^{1/2} f(x)$.
- The Fourier transform of the preceding scale, corresponding to the operator $(A_{\rm p} f)(x) = (1 + x^2)^{1/2} f(x)$.
- The scale of the quantum harmonic oscillator, corresponding to the operator $(A_{\rm osc} f)(x) = (1 + x^2 - \frac{d^2}{dx^2}) f(x)$.

(The notation is suggested by the operators of momentum, position, and harmonic oscillator energy in quantum mechanics, respectively).

6.2.2. *Operators on a Lattice of Hilbert spaces.* Actually a similar structure is obtained if one considers operators on a Lattice of Hilbert spaces (see the Appendix). Indeed, take an arbitrary LHS with a distinguished family of Hilbert subspaces $V_{\rm I} = \{\mathcal{H}_r,\, r \in {\rm I}\}$ for some index set I. Once again the topological and lattice structures coincide. $q < p$ implies that $\mathcal{H}_q \subset \mathcal{H}_p$ and the embedding is continuous with dense range. Similarly, $\mathcal{H}_{p\wedge q}$ and $\mathcal{H}_{\overline{p}\vee\overline{q}}$ are dual to each other. Moreover, $V^{\#}$ is dense in every $\mathcal{H}_r$, $r \in {\rm I}$. Thus the operators on $V_{\rm I}$ are generated by the sets of bounded operators $\{\mathcal{B}(\mathcal{H}_p, \mathcal{H}_q), p, q \in {\rm I}\}$, exactly as before. Thus here too we get a Banach partial $*$-algebra of type (B). Examples of such a LHS abound, for instance,

- Köthe sequence spaces, including (weighted) ℓ^2 spaces,
- the space $L^1_{\rm loc}(X,\, {\rm d}\mu)$ of locally integrable functions on a measure space (X, μ). The generating sublattice consists of weighted L^2 spaces,
- locally integrable functions or sequences of prescribed growth, with a similar generating sublattice,
- Köthe function spaces [28, Chap.15], generalizing the preceding two spaces,
- a lattice of Hilbert spaces of analytic functions around the Fock-Bargmann space.

All those LHS are described in great detail in [2, Chap.6] and [4, Chap.4]. Then, as before, the set of operators on them become normed or Banach partial $*$-algebras of type (B).

7. Banach quasi $*$-algebras

7.1. **quasi $*$-algebras.** A completely different type of partial $*$-algebras is that of quasi $*$-algebras, introduced initially by Lassner [19, 18]. The idea was to provide a reasonable mathematical environment for properly dealing with the thermodynamical limit of local observables of certain quantum statistical models that did not fit into the set-up of the algebraic formulation of quantum theories developed by Haag and Kastler [15].

A *quasi $*$-algebra* is a couple $(\mathfrak{X}, \mathfrak{A}_0)$, where $\mathfrak{X}$ is a vector space with involution, $\mathfrak{A}_0$ is a $*$-algebra and a vector subspace of $\mathfrak{X}$, and $\mathfrak{X}$ is an $\mathfrak{A}_0$-bimodule whose module operations and involution extend those of $\mathfrak{A}_0$. The simplest way to construct such an object consists in taking the completion of a locally convex $*$-algebra $(\mathfrak{A}_0, \tau)$ where the multiplication is separately but not jointly continuous. Of particular interest is, of course, the case where τ is a norm topology. Particularizing Definition 3.1, we define a Banach partial $*$-algebra as follows [26, 25].

Definition 7.1. A quasi $*$-algebra $(\mathfrak{A}, \mathfrak{A}_o)$ is called a *Banach quasi $*$-algebra* if $\mathfrak{A}$ is a Banach space under a norm $\|\cdot\|$ satisfying the following properties:

(i) $\|a^*\| = \|a\|$ for all $a \in \mathfrak{A}$;
(ii) $\mathfrak{A}_o$ is dense in $\mathfrak{A}[\|\cdot\|]$;
(iii) for every $x \in \mathfrak{A}_o$, the map $R_x : a \in \mathfrak{A}[\|\cdot\|] \mapsto ax \in \mathfrak{A}[\|\cdot\|]$ is continuous.

The continuity of the involution implies that

(iv) for every $x \in \mathfrak{A}_o$, the map $L_x : a \in \mathfrak{A}[\|\cdot\|] \mapsto xa \in \mathfrak{A}[\|\cdot\|]$ is continuous too.

We will suppose that $(\mathfrak{A}, \mathfrak{A}_o)$ has a unit e; that is, an element $e \in \mathfrak{A}_o$ such that $ae = ea = a$ for every $a \in \mathfrak{A}$.

If $(\mathfrak{A}, \mathfrak{A}_o)$ is a Banach quasi $*$-algebra, a norm topology can be defined on $\mathfrak{A}_o$ in the following way. For $x \in \mathfrak{A}_o$, the following functions

$$\|x\|_L = \sup_{\|a\| \le 1} \|ax\| \quad \text{and} \quad \|x\|_R = \sup_{\|a\| \le 1} \|xa\|, \quad x \in \mathfrak{A}_o,\ a \in \mathfrak{A}, \tag{7.1}$$

are well defined norms on $\mathfrak{A}_o$. It is easy to see that $\|x\|_L = \|x^*\|_R$ (and, of course, $\|x\|_R = \|x^*\|_L$) for every $x \in \mathfrak{A}_o$. Moreover, by (7.1) it follows that

$$\|ax\| \le \|a\|\|x\|_L \quad \text{and} \quad \|xa\| \le \|a\|\|x\|_R \qquad \text{for all } a \in \mathfrak{A},\ x \in \mathfrak{A}_o. \tag{7.2}$$

Again by (7.1) and together with (7.2), we deduce that

$$\|xy\|_L \le \|x\|_L\|y\|_L \quad \text{and} \quad \|xy\|_R \le \|x\|_R\|y\|_R \qquad \text{for all } x, y \in \mathfrak{A}_o.$$

Finally we put

$$\|x\|_0 := \max\{\|x\|_L, \|x\|_R\}. \tag{7.3}$$

Corollary 7.2. *If the Banach quasi $*$-algebra $(\mathfrak{A}, \mathfrak{A}_o)$ has a unit, then the $*$-algebra $\mathfrak{A}_o[\|\cdot\|_0]$ is a normed $*$-algebra; therefore we may suppose, without loss of generality, that $\|e\|_0 = 1$. Moreover,*

$$\begin{aligned} \|xy\| \le \|x\|\|y\|_0, \quad \|yx\| \le \|x\|\|y\|_0, &\quad \textit{for all } x, y \in \mathfrak{A}_o, \\ \text{and, for } x = e, \qquad \|y\| \le \|e\|\|y\|_0 &\quad \textit{for all } y \in \mathfrak{A}_o. \end{aligned}$$

Definition 7.3. A Banach quasi $*$-algebra $(\mathfrak{A}, \mathfrak{A}_o)$ is called a *BQ^*-algebra* if $\mathfrak{A}_o[\|\cdot\|_0]$ is a Banach $*$-algebra and a *proper CQ^*-algebra* if $\mathfrak{A}_o[\|\cdot\|_0]$ is a C^*-algebra (see Definition 8.1).

Example 7.4. Let $I = [0, 1]$. Then $(L^p(I), C(I))$, where $C(I)$ denotes the CQ^*-algebra of all continuous functions on I and $p \ge 1$, is a Banach quasi $*$-algebra (more precisely, a proper CQ^*-algebra [9], if $C(I)$ is endowed with the usual supremum norm $\|\cdot\|_\infty$; actually in this case, one has $\|\cdot\|_0 = \|\cdot\|_\infty$).

Example 7.5. Let $\mathfrak{M}$ be a von Neumann algebra, and let τ be a normal finite faithful trace [23] on $\mathfrak{M}$. Then the completion of $\mathfrak{M}$ with respect to the norm

$$\|X\|_p = \tau(|X|^p)^{1/p}, \qquad X \in \mathfrak{M},$$

is usually denoted by $L^p(\tau)$ [20, 22] and is a Banach space consisting of operators affiliated with $\mathfrak{M}$. Then $(L^p(\tau), \mathfrak{M})$ is a Banach quasi $*$-algebra with unit, more precisely, a CQ^*-algebra.

An important role is played by *bounded elements*, which are defined via the following two linear maps from $\mathfrak{A}_o$ into $\mathfrak{A}$:

$$\begin{aligned} x \in \mathfrak{A}_o &\mapsto L_a x = ax \in \mathfrak{A} \\ x \in \mathfrak{A}_o &\mapsto R_a x = xa \in \mathfrak{A}. \end{aligned} \tag{7.4}$$

An element $a \in \mathfrak{A}$ is called *bounded* if both L_a and R_a are bounded operators on $\mathfrak{A}_o$; that is, if there exists $\gamma > 0$ such that

$$\max\{\|ax\|, \|xa\|\} \leq \gamma \|x\| \qquad \text{for all } x \in \mathfrak{A}_o.$$

The set of bounded elements is denoted by $\mathfrak{A}_{\mathrm{b}}$, and it carries the following natural norm

$$\|a\|_{\mathrm{b}} := \max\{\|L_a\|, \|R_a\|\}, \qquad a \in \mathfrak{A}_{\mathrm{b}},$$

where the norms on the right hand side are those of bounded operators on $\mathfrak{A}$. It is clear that both L_a and R_a extend to $\mathfrak{A}$ (we denote by $\overline{L}_a$ and $\overline{R}$, respectively, these extensions); so that one can think of extending the multiplication by exploiting these extensions. For instance, if $a, b \in \mathfrak{A}_{\mathrm{b}}$, both $\overline{L}_a b$ and $\overline{R}_b a$ are well defined, but they need not be equal in general. Thus extending the multiplication is possible only under additional assumptions. This unpleasant feature does not appear, for instance, in the case of $*$-semisimple Banach quasi $*$-algebras. This notion is defined through the following family of sesquilinear forms. We denote by $\mathcal{S}_{\mathfrak{A}_o}(\mathfrak{A})$ the family of all sesquilinear forms $\varphi \in \mathfrak{A} \times \mathfrak{A}$ such that

(i) $\varphi(a, a) \geq 0$ for all $a \in \mathfrak{A}$;
(ii) $\varphi(ax, y) = \varphi(x, a^* y)$ for all $a \in \mathfrak{A}$ and $x, y \in \mathfrak{A}_o$;
(iii) $|\varphi(a, b)| \leq \|a\| \, \|b\|$ for all $a, b \in \mathfrak{A}$.

Definition 7.6. A Banach quasi $*$-algebra $(\mathfrak{A}, \mathfrak{A}_o)$ is said to be $*$-*semisimple* if $a \in \mathfrak{A}$ and $\varphi(a, a) = 0$, for each $\varphi \in \mathcal{S}_{\mathfrak{A}_o}(\mathfrak{A})$, imply that $a = 0$.

For instance, the Banach quasi $*$-algebras considered in Examples 7.4 and 7.5 can be shown to be $*$-semisimple if and only if $p \geq 2$ [9]. As mentioned before, for any pair a, b of bounded elements of a $*$-semisimple Banach quasi $*$-algebra $(\mathfrak{A}, \mathfrak{A}_o)$ one has $\overline{L}_a b = \overline{R}_b a$. Indeed, for every $\varphi \in \mathcal{S}_{\mathfrak{A}_o}(\mathfrak{A})$, we have

$$\begin{aligned} \varphi((\overline{L}_a b) z, z) &= \lim_{m \to \infty} \varphi((a y_m) z, z) \\ &= \lim_{m \to \infty} \varphi(a (y_m z), z) = \lim_{m \to \infty} \varphi(y_m z, a^* z) \\ &= \varphi(bz, a^* z), \end{aligned}$$

where $\{y_m\}$ is a sequence of elements of $\mathfrak{A}_o$ converging to b in $\mathfrak{A}$. Analogously, if $\{x_n\}$ is a sequence of elements of $\mathfrak{A}_o$ converging to a in $\mathfrak{A}$, we have

$$\begin{aligned}\varphi((\overline{R}_b a)z, z) &= \lim_{n\to\infty} \varphi((x_n b)z, z)\\ &= \lim_{n\to\infty} \varphi(x_n(bz), z) = \lim_{n\to\infty} \varphi(bz, x_n^* z)\\ &= \varphi(bz, a^* z).\end{aligned}$$

Therefore,

$$\varphi((\overline{L}_a b - \overline{R}_b a)z, z) = 0 \qquad \text{for all } \varphi \in \mathcal{S}_{\mathfrak{A}_o}(\mathfrak{A}) \text{ and } \ z \in \mathfrak{A}_o.$$

By [24, Lemma 3.13] it follows that $(\overline{L}_a R_b) - \overline{R}_b L_a)z = 0$ for every $z \in \mathfrak{A}_o$. Thus, $\overline{L}_a \overline{R}_b = \overline{R}_b \overline{L}_a$.

Hence the multiplication of $a, b \in \mathfrak{A}_{\rm b}$ can be defined by

$$a \bullet b := \overline{L}_a b = \overline{R}_b a.$$

Then we have the following result.

Proposition 7.7. *If $(\mathfrak{A}, \mathfrak{A}_o)$ is a $*$-semisimple Banach quasi $*$-algebra, then $\mathfrak{A}_{\rm b}[\|\cdot\|_{\rm b}]$ is a Banach $*$-algebra with respect to the multiplication $\bullet$.*

The notion of bounded element plays an important role also for introducing the notion of spectrum of an element of a Banach quasi $*$-algebra.

To that effect, we need a closer analysis of the linear maps L_a and R_a defined in (7.4). Elements of $\mathfrak{A} \setminus \mathfrak{A}_o$ are, in general, *unbounded* maps in the Banach space $\mathfrak{A}$. It is natural to deal with the problem of inverting an element $a \in \mathfrak{A}$ first by inverting L_a and R_a. As it is customary in the theory of unbounded operators, we will look for *bounded* inverses.

Definition 7.8. Let $(\mathfrak{A}, \mathfrak{A}_o)$ be a Banach quasi $*$-algebra with unit e. An element $a \in \mathfrak{A}$ is called *closable* if the linear maps

$$L_a : a \in \mathfrak{A}_o \mapsto ax \in \mathfrak{A}, \qquad R_a : a \in \mathfrak{A}_o \mapsto xa \in \mathfrak{A}$$

are closable in $\mathfrak{A}$.

If $a \in \mathfrak{A}$ we denote by $\overline{L}_a$ the closure of L_a; that is, the linear operator defined on the domain

$$D(\overline{L}_a) = \{b \in \mathfrak{A} : \exists \{x_n\} \subset \mathfrak{A}_o,\ \|b - x_n\| \to 0, \text{ and } \{ax_n\} \text{ is Cauchy}\}$$

by

$$\overline{L}_a b = \lim_{n\to\infty} ax_n.$$

Similarly, $\overline{R}_a$ and $D(\overline{R}_a)$ will denote the closure of R_a and its domain, respectively. The definitions are obvious modifications of the previous ones.

Definition 7.9. Let $(\mathfrak{A}, \mathfrak{A}_o)$ be a Banach quasi $*$-algebra with unit e, and let $a \in \mathfrak{A}$ be a closable element. We say that a has a *bounded inverse* if there exists $b \in \mathfrak{A}_{\rm b} \cap D(\overline{L}_a) \cap D(\overline{R}_a)$ such that $\overline{R}_b a = \overline{L}_b a = e$.

If $(\mathfrak{A}, \mathfrak{A}_o)$ is $*$-semisimple, then this element b, if any, is unique. In this case we denote the bounded inverse of a by a^{-1}.

For a Banach ($*$-)algebra, the existence of the inverse of an element a can be characterized through the invertibility of the corresponding maps L_a and R_a. A similar result does not hold for a Banach quasi $*$-algebra, but again the $*$-semisimple case is completely under control.

Proposition 7.10. *Let $(\mathfrak{A}, \mathfrak{A}_o)$ be a $*$-semisimple Banach quasi $*$-algebra with unit e. Then every element $a \in \mathfrak{A}$ is closable and the following statements are equivalent:*

(i) *The element a has a bounded inverse a^{-1}.*

(ii) *Both $\overline{L}_a$ and $\overline{R}_a$ possess everywhere defined (hence, bounded) inverses.*

Let $(\mathfrak{A}, \mathfrak{A}_o)$ be a $*$-semisimple Banach quasi $*$-algebra with unit e.

Definition 7.11. The *resolvent* $\rho(a)$ of $a \in \mathfrak{A}$ is the set

$$\rho(a) := \{\lambda \in \mathbb{C} : \ a - \lambda e \text{ has a bounded inverse}\}.$$

The set $\sigma(a) := \mathbb{C} \setminus \rho(a)$ is called the *spectrum* of a.

Proposition 7.12. *Let $a \in \mathfrak{A}$. The following statements hold:*

(i) *The resolvent $\rho(a)$ is an open subset of the complex plane.*

(ii) *The resolvent function $R_\lambda(a) := (a - \lambda e)^{-1} \in \mathfrak{A}_{\mathrm{b}}$, $\lambda \in \rho(a)$, is $\|\cdot\|_{\mathrm{b}}$-analytic on each connected component of $\rho(a)$.*

(iii) *For any two points $\lambda, \mu \in \rho(a)$, $R_\lambda(a)$ and $R_\mu(a)$ commute and*

$$R_\lambda(a) - R_\mu(a) = (\mu - \lambda) R_\mu(a) \bullet R_\lambda(a).$$

Example 7.13. Let us consider again the Banach quasi $*$-algebra $(L^p(I), C(I))$, and let $f \in L^p(I)$. Then it is easily seen that the spectrum $\sigma(f)$ of f coincides with its essential range; that is, the set of all $\lambda \in \mathbb{C}$ such that the set

$$\{x \in I : |f(x) - \lambda| < \epsilon\}$$

has positive Lebesgue measure for every $\epsilon > 0$.

Definition 7.14. Let $a \in \mathfrak{A}$. The non-negative number

$$r(a) := \sup\{|\lambda|, \lambda \in \sigma(a)\}$$

is called the *spectral radius* of a.

Remark 7.15. Of course, if $a \in \mathfrak{A}_{\mathrm{b}}$, then $\sigma(a)$ is the same set as that obtained regarding it as an element of the Banach $*$-algebra $\mathfrak{A}_{\mathrm{b}}$. For an arbitrary element $a \in \mathfrak{A}$, $\sigma(a)$, which is a nonempty closed set, could be an unbounded subset of $\mathbb{C}$. The next proposition shows that, if $a \in \mathfrak{A} \setminus \mathfrak{A}_{\mathrm{b}}$, then $\sigma(a)$ is necessarily unbounded.

Proposition 7.16. *Let $a \in \mathfrak{A}$. Then, $r(a) < \infty$ if and only if $a \in \mathfrak{A}_{\mathrm{b}}$.*

The aim of this section was to give the reader the flavor of the behavior of Banach quasi $*$-algebras. A series of other interesting results can be obtained by considering $*$-representations by means of (in general, unbounded) operators. We do not enter here into this topic, referring the reader to the original papers, for instance, [24] or to the forthcoming monograph [27]. As for Banach $*$-algebras, representations give a deep insight into this structure, at least in the case when they are sufficiently many ($*$-semisimple case). In this case also, bounded elements play a crucial role: they are in fact characterized by being represented by bounded operators under any $*$-representation of the given Banach quasi $*$-algebra.

8. CQ^*-ALGEBRAS

A significant example of a Banach quasi $*$-algebra is a CQ^*-algebra, discussed in [2, Sec.6.2.3]. This is a generalization of C^*-algebras, in the sense that a CQ^*-algebra can be viewed as the completion of a C^*-algebra with respect to a weaker norm [10, 8].

Definition 8.1. Let $(\mathfrak{A}, \mathfrak{A}_o)$ be a Banach quasi $*$-algebra with norm $\|\cdot\|$ and involution $*$. We say that $(\mathfrak{A}, \mathfrak{A}_o)$ is a *proper CQ^*-algebra* if

(i) $\mathfrak{A}_o$ is a C^*-algebra with norm $\|\cdot\|_0$ and involution $*$ inherited by that of $\mathfrak{A}$;
(ii) $\mathfrak{A}_o$ is dense in $\mathfrak{A}$ with respect to the norm $\|\cdot\|$;
(iii) $\|x\|_0 = \sup_{a\in\mathfrak{A}, \|a\|\leq 1} \|ax\|, \ x \in \mathfrak{A}_o$.

We have defined the norm $\|\cdot\|_0$ for a Banach quasi $*$-algebra $(\mathfrak{A}, \mathfrak{A}_o)$ by (7.1) and (7.3). Condition (iii) is exactly equivalent to the one given in section 7, because $\mathfrak{A}_o$ is supposed to be a C^*-algebra. Hence the two definitions 8.1 and 7.3 of a proper CQ^*-algebra coincide.

A proper CQ^*-algebra can be obtained by completing of a C^*-algebra $\mathfrak{A}_o[\|\cdot\|]_0$ with respect to a weaker norm. Indeed, we have the following proposition.

Proposition 8.2. *Let $\mathfrak{A}_o$ be a C^*-algebra with norm $\|\cdot\|_0$ and involution $*$. Let $\|\cdot\|$ be another norm on $\mathfrak{A}_o$ weaker than $\|\cdot\|_0$, in the sense that*

$$\|x\| \leq \|x\|_0, \qquad \text{for all } x \in \mathfrak{A}_o,$$

and satisfying the following conditions:

(i) $\|xy\| \leq \|x\| \, \|y\|_0$ *for all* $x, y \in \mathfrak{A}_o$;
(ii) $\|x^*\| = \|x\|$ *for all* $x \in \mathfrak{A}_o$.

Let $\mathfrak{A}$ denote the $\|\cdot\|$-completion of $\mathfrak{A}_o$. Then, $(\mathfrak{A}, \mathfrak{A}_o)$ is a proper CQ^*-algebra.

For C^*-algebras, the situation is completely clear, a commutative C^*-algebra with unit is isometrically $*$-isomorphic to the C^*-algebra $C(X)$ of all $\mathbb{C}$-valued continuous functions on the compact space X of characters of $C(X)$. This correspondence is the so-called *Gelfand transform.*

CQ^*-algebras do not behave so nicely: the first reason is that Proposition 8.2 allows the existence of non isomorphic CQ^*-algebras over $C(X)$; the second reason is that, as it is known already for Banach $*$-algebras, the Gelfand transform is not, in general, an isometric $*$-isomorphism.

However, as we shall see, any $*$-semisimple commutative CQ^*-algebra can be thought of as a CQ^*-algebra of functions. We remind the reader that, in the case of the L^p-spaces, $*$-semisimplicity occurs if and only if $p \geq 2$.

Let X be a compact Hausdorff space, and let $\mathcal{M} = \{\mu_\alpha, \alpha \in I\}$ be a family of Borel measures on X, for which there exists a constant $c > 0$ such that $\mu_\alpha(X) \leq c$ for all $\alpha \in I$. Let $\|\cdot\|_{p,\alpha}$ be the norm on $L^p(X, \mu_\alpha)$. The completion $L^p_I(X, M)$ of $C(X)$ with respect to $\|\cdot\|_{p,I}$ is a Banach space, where the norm $\|\cdot\|_{p,I}$ is defined as

$$\|\phi\|_{p,I} := \sup_{\alpha \in I} \|\phi\|_{p,\alpha}.$$

The norm $\|\cdot\|_{p,I}$ also satisfies the conditions of Proposition 8.2. Therefore, $(L^p_I(X, \mathcal{M}), C(X))$ is a commutative CQ^*-algebra. It is clear that $L^p_I(X, \mathcal{M})$ can be identified with a subspace of $L^p(X, \mu_\alpha)$ for all $\alpha \in I$. The next proposition describes how the Gelfand transform extends to the case of $*$-semisimple commutative CQ^*-algebras [8].

Proposition 8.3. *Let $(\mathfrak{A}, \mathfrak{A}_o)$ be a $*$-semisimple commutative CQ^*-algebra with identity e. Then, there exist a family $\mathcal{M}$ of Borel measures on the Hausdorff compact space X of the characters of $\mathfrak{A}_o$ and a map $\Phi : a \in \mathfrak{A} \mapsto \Phi(a) := \widehat{a} \in L^2_{\mathrm{I}}(X, \mathcal{M})$ with the following properties:*

(i) *Φ extends the Gelfand transform of elements of $\mathfrak{A}_o$ and $\Phi(\mathfrak{A}) \supset C(X)$;*
(ii) *Φ is linear and injective;*
(iii) *$\Phi(ax) = \Phi(a)\Phi(x)$ for all $a \in \mathfrak{A}$, $x \in \mathfrak{A}_o$;*
(iv) *$\Phi(a^*) = \Phi(a)^*$ for all $a \in \mathfrak{A}$.*

Thus $\mathfrak{A}$ can be identified with a subspace of $L^2_{\mathrm{I}}(X, \mathcal{M})$.

If $\mathfrak{A}$ is regular; that is, if

$$\|a\|^2 = \sup_{\varphi \in \mathcal{S}_{\mathfrak{A}_o}(\mathfrak{A})} \varphi(a, a), \; a \in \mathfrak{A}\,,$$

then Φ is an isometric $*$-isomorphism of $\mathfrak{A}$ onto $L^2_{\mathrm{I}}(X, \mathcal{M})$.

So far we have considered *proper* CQ^*-algebras. They can be understood as a particular case of a richer structure where three different involutions are involved.

Definition 8.4. Let $\mathfrak{A}_\#$ be a C^*-algebra with norm $\|\cdot\|_\#$ and involution $\#$. Let $\mathfrak{A}[\|\cdot\|]$ be a left Banach module over the C^*-algebra $\mathfrak{A}_\#$ with isometric involution $*$ and such that $\mathfrak{A}_\# \subset \mathfrak{A}$. Set $\mathfrak{A}_\flat = (\mathfrak{A}_\#)^*$. We say that $\{\mathfrak{A}, *, \mathfrak{A}_\#, \#\}$ is a *CQ^*-algebra* if

(i) $\mathfrak{A}_\#$ is dense in $\mathfrak{A}$ with respect to its norm $\|\cdot\|$;
(ii) $\mathfrak{A}_o := \mathfrak{A}_\# \cap \mathfrak{A}_\flat$ is dense in $\mathfrak{A}_\#$ with respect to its norm $\|\cdot\|_\#$;
(iii) $(xy)^* = y^* x^*$ for all $x, y \in \mathfrak{A}_o$;
(iv) $\|x\|_\# = \sup_{a \in \mathfrak{A}, \|a\| \leq 1} \|xa\|, \; x \in \mathfrak{A}_\#$.

Since $*$ is isometric, it is easy to see that the space $\mathfrak{A}_\flat$ itself is a C^*-algebra with respect the norm $\|x\|_\flat := \|x^*\|_\#$ and the involution $x^\flat := (x^*)^{\sharp *}$.

Remark 8.5. It is quite clear that we can restate the previous definition starting from a C^*-algebra $\mathfrak{A}_\flat$ and a right module $\mathfrak{A}$ over $\mathfrak{A}_\flat$, with $\mathfrak{A}_\flat \subset \mathfrak{A}$, satisfying the following properties:

(i') $\mathfrak{A}_\flat$ is dense in $\mathfrak{A}$ with respect to its norm $\|\cdot\|$;
(ii') $\mathfrak{A}_o := \mathfrak{A}_\flat \cap \mathfrak{A}_\#$ is dense in $\mathfrak{A}_\flat$ with respect to its norm $\|\cdot\|_\flat$;

(iii') $(xy)^* = y^*x^*$ for all $x, y \in \mathfrak{A}_o$;
(iv') $\|x\|_\flat = \sup_{a\in\mathfrak{A}, \|a\|\le 1} \|ax\|, \; x \in \mathfrak{A}_\flat$.

It is then also natural to adopt the notation $\{\mathfrak{A}, *, \mathfrak{A}_\flat, \flat\}$ for indicating a CQ^*-algebra as it has been done in many papers on this subject (see for instance [10] or [8]).

According to Definition 8.1, a proper CQ^*-algebra is then a CQ^*-algebra such that $\mathfrak{A}_\# = \mathfrak{A}_\flat = \mathfrak{A}_o$ and the involutions $*$ and $\#$ coincide on $\mathfrak{A}_o$.

Remark 8.6. If $\{\mathfrak{A}, *, \mathfrak{A}_\#, \#\}$ is a CQ^*-algebra, then $(\mathfrak{A}, \mathfrak{A}_o)$ is a Banach quasi $*$-algebra.

The main interest for the structure of CQ^*-algebra comes from the Tomita–Takesaki theory. We briefly discuss this matter. Further information may be found in [2, Chap.5] or the original papers quoted there.

We remind the reader that a $*$-algebra $\mathfrak{A}_o$ with involution $\#$ is called a *left Hilbert algebra* [23, Section 10.1] if it is a dense subspace in a Hilbert space $\mathcal{H}$ with inner product $\langle\cdot|\cdot\rangle$ satisfying the following conditions:

(i) For any $x \in \mathfrak{A}_o$ the map $y \in \mathfrak{A}_o \mapsto xy \in \mathfrak{A}_o$ is continuous;
(ii) $\langle xy|z\rangle = \langle y|x^\# z\rangle$ for all $x, y, z \in \mathfrak{A}_o$;
(iii) $\mathfrak{A}_o^2 := \{xy : x, y \in \mathfrak{A}\}$ is total in $\mathcal{H}$;
(iv) The involution $x \mapsto x^\#$ is closable in $\mathcal{H}$.

By (i), for any $x \in \mathfrak{A}_o$, we denote by L_x the unique continuous linear extension to $\mathcal{H}$ of the map $y \in \mathfrak{A}_o \mapsto xy \in \mathfrak{A}_o$; then, using (ii), it is easy to see that the map

$$L : x \in \mathfrak{A}_o \mapsto L_x \in \mathcal{B}(\mathcal{H})$$

is a bounded $*$-representation of $\mathfrak{A}_o$ on $\mathcal{H}$. We define

$$\mathfrak{L}(\mathfrak{A}_o) = \{L_x : x \in \mathfrak{A}_o\}''.$$

We denote by S the closure of the operator S_0 defined on the dense domain $\mathfrak{A}_o^2$ by

$$S_0 : y \in \mathfrak{A}_o^2 \mapsto y^\# \in \mathcal{H}. \tag{8.1}$$

Let $S = J\Delta^{1/2}$ be the polar decomposition of S. Then, J is an isometric involution on $\mathcal{H}$ and Δ is a nonsingular positive self-adjoint operator in $\mathcal{H}$ such that $S = J\Delta^{1/2} = \Delta^{-1/2}J$ and $S^* = J\Delta^{-1/2} = \Delta^{1/2}J$; J is called the *modular conjugation operator* of $\mathfrak{A}_o$ and Δ is called the *modular operator* of $\mathfrak{A}_o$.

We define the commutant $\mathfrak{A}_o'$ of $\mathfrak{A}_o$ as follows: For any $y \in \mathcal{D}(S^*)$, we put $R_y x = L_x y$, $x \in \mathfrak{A}_o$, and $\mathfrak{A}_o' = \{y \in \mathcal{D}(S^*) : R_y \text{ is bounded }\}$. Then, $\mathfrak{A}_o'$ is a *right* Hilbert algebra in $\mathcal{H}$ with involution $y \mapsto y^\flat := S^*y$ and multiplication $y_1y_2 := R_{y_2}y_1$, $y_1, y_2 \in \mathfrak{A}_o'$ (we do not give explicitly the definition of a right Hilbert algebra; we refer again to [23, Section 10.1]).

The commutant $\mathfrak{A}_o''$ of $\mathfrak{A}_o'$ is defined by

$$\mathfrak{A}_o'' = \{x \in \mathcal{D}(S) : y \in \mathfrak{A}_o' \mapsto xy \text{ is continuous }\}.$$

For any $x \in \mathfrak{A}_o''$, we denote by L_x the unique continuous linear operator on $\mathcal{H}$, such that $L_x y = R_y x$, $y \in \mathfrak{A}_o'$. Then, $\mathfrak{A}_o''$ is a left Hilbert algebra in $\mathcal{H}$ with involution S and multiplication $x_1x_2 := L_{x_1}x_2$, containing $\mathfrak{A}_o$. A left Hilbert algebra $\mathfrak{A}_o$ is said to be *full* if $\mathfrak{A}_o = \mathfrak{A}_o''$.

The Tomita fundamental theorem states that, for every $t \in \mathbb{R}$, $J\mathfrak{L}(\mathfrak{A}_o)''J = \mathfrak{L}(\mathfrak{A}_o)'$ and $\Delta^{it}\mathfrak{L}(\mathfrak{A}_o)''\Delta^{-it} = \mathfrak{L}(\mathfrak{A}_o)''$. Let $\mathfrak{A}_o$ be a full left Hilbert algebra in $\mathcal{H}$, and let

$$\mathfrak{A}_{00} := \{x \in \cap_{\alpha\in\mathbb{C}}\mathcal{D}(\Delta^\alpha) : \Delta^\alpha x \in \mathfrak{A}_o, \text{for all } \alpha \in \mathbb{C}\}.$$

Then, $\mathfrak{A}_{00}$ is a left Hilbert subalgebra in $\mathcal{H}$ such that $\mathfrak{A}_{00}'' = \mathfrak{A}_o$ and $J\mathfrak{A}_{00} = \mathfrak{A}_{00}$; $\{\Delta^\alpha : \alpha \in \mathbb{C}\}$ is a complex one-parameter group of automorphisms of $\mathfrak{A}_{00}$ such that

$$(\Delta^\alpha x)^\# = \Delta^{-\overline{\alpha}}x^\# \ \text{ and } \ (\Delta^\alpha x)^* = \Delta^{-\overline{\alpha}}x^* \qquad \text{for all } \alpha \in \mathbb{C} \ \text{ and } \ x \in \mathfrak{A}_{00}.$$

The left Hilbert subalgebra $\mathfrak{A}_{00}$ is called the *maximal Tomita algebra* of $\mathfrak{A}_o$.

Let now $\mathfrak{A}_o$ be a full left Hilbert algebra with identity e and involution $\#$ in $\mathcal{H}$. Then, as seen above, the commutant $\mathfrak{A}_o'$ of $\mathfrak{A}_o$ is a full right Hilbert algebra in $\mathcal{H}$ with (the same) identity and involution $\flat$. The involution in $\mathcal{H}$ is defined by the modular conjugation operator J. For shortness we put $\mathcal{H}_\flat = \mathfrak{A}_o'$ and $\mathcal{H}_\# = \mathfrak{A}_o$. We consider now the system $(\mathcal{H}, J, \mathcal{H}_\#, \#)$ and define a topological structure in it. For $y \in \mathcal{H}_\#$,

$$\|y\|_\# := \|L_y\| = \sup_{\|x\|\le 1} \|yx\|,$$

where L denotes the regular $*$-representation of $\mathfrak{A}_o$ in $\mathcal{B}(\mathcal{H})$. We also define $\|x\|_\flat := \|Jx\|_\#$ for every $x \in \mathcal{H}_\flat$.

The conditions (i) and (iv) of Definition 8.4 are obviously fulfilled, whereas condition (iii) follows from the known equality $(Jx)^\flat = Jx^\#$ for every $x \in \mathcal{H}_\#$. The C^*-property for the norm $\|\cdot\|_\#$ follows easily from the fact that the linear map $y \mapsto L_y$ is a $*$-representation of $\mathcal{H}_\#$ into $\mathcal{B}(\mathcal{H})$. The algebras $\mathcal{H}_\#[\|\cdot\|_\#]$ and $\mathcal{H}_\flat[\|y\|_\flat]$ are complete.

To conclude that $\{\mathcal{H}, J, \mathcal{H}_\#, \#\}$ is a CQ^*-algebra, we need to prove that $\mathcal{H}_\flat\cap\mathcal{H}_\#$ is dense in $\mathcal{H}_\#$ with respect to $\|\cdot\|_\#$. In this full generality, the question is still open. However, in order to give a partial answer to this problem, the notion of HCQ^*-algebra has been introduced in [7].

The starting point is a Hilbertian quasi $*$-algebra $(\mathfrak{A}, \mathfrak{A}_o)$, by which we simply mean a Banach quasi $*$-algebra whose norm is Hilbertian (that is, it satisfies the parallelogram law).

Definition 8.7. A Hilbertian quasi $*$-algebra $(\mathfrak{A}[\|\cdot\|], \mathfrak{A}_o)$ is said to be a *HCQ^*-algebra* if there is another involution $\#$ of $\mathfrak{A}$ such that $L_x^* = L_{x^\#}$ and $\|x\| \le \|L_x\|$ for each $x \in \mathfrak{A}_o$. Here we denote it by $(\mathfrak{A}[\|\cdot\|], \#)$.

Suppose that $(\mathfrak{A}[\|\cdot\|], \#)$ is a HCQ^*-algebra with involution operator $J_\mathfrak{A}$; that is, $J_\mathfrak{A}a = a^*, a \in \mathfrak{A}$. Then $\mathfrak{A}_o$ is a left Hilbert algebra in the Hilbert space $\mathcal{H} := \mathfrak{A}[\|\cdot\|]$, whose full left Hilbert algebra $\mathfrak{A}_o''$ has a unit u.

The identity map $i : \mathfrak{A}_o[\|\cdot\|_\#] \to \mathfrak{A}_o[\|\cdot\|]$ has a continuous extension $\widehat{i}$ from the completion $\mathfrak{A}_\#$ of $\mathfrak{A}_o[\|\cdot\|_\#]$ ($\mathfrak{A}_\#$ is, of course, a C^*-algebra) into $\mathfrak{A}[\|\cdot\|]$. We will suppose that the two norms $\|\cdot\|$ and $\|\cdot\|_\#$ are *compatible*; that is, that the map $\hat{i}^{-1} : \mathfrak{A}_o[\|\cdot\|] \to \mathfrak{A}_\#[\|\cdot\|_\#]$ is closable. In this case $\mathfrak{A}_\#$ is identified with a dense subspace of $\mathfrak{A}$ and the following conditions on $\mathfrak{A}_o$ are fulfilled:

(a.1) $\|x^{\#}x\|_{\#} = \|x\|_{\#}^2$ for all $x \in \mathfrak{A}_o$;
(a.2) $\|x\| \leq \|x\|_{\#}$ for all $x \in \mathfrak{A}_o$;
(a.3) $\|xy\| \leq \|x\|_{\#}\|y\|$ for all $x, y \in \mathfrak{A}_o$.

If $(\mathfrak{A}[\|\cdot\|], \#)$ is a HCQ^*-algebra, then $(\mathfrak{A}, J_{\mathfrak{A}}, \mathfrak{A}_{\#}, \#)$ is a CQ^*-algebra.

A HCQ^*-algebra is called *standard* if one of the following equivalent conditions is satisfied (here J denotes the modular conjugation defined after (8.1)):

(i) $J_{\mathfrak{A}} = J$.
(ii) $\langle x^{\#}|x^*\rangle \geq 0$, for each $x \in \mathfrak{A}_o$.

From the previous discussion it follows that a HCQ^*-algebra is a CQ^*-algebra constructed from a left Hilbert algebra. However, so far, we do not know if standard HCQ^*-algebras do really exist. The following theorem characterizes Hilbert spaces that can be regarded as standard HCQ^*-algebras.

Theorem 8.8. *Let $\mathcal{H}$ be a Hilbert space. The following statements are equivalent:*

(i) *$\mathcal{H}$ is a standard HCQ^*-algebra.*
(ii) *$\mathcal{H}$ contains a left Hilbert algebra with unit as dense subspace.*
(iii) *There exists a von Neumann algebra on $\mathcal{H}$ with a cyclic and separating vector.*

We have omitted the details of the whole construction which are quite heavy; dealing with them here goes beyond the scope of this review. We refer instead the interested reader to [7] or [27]. The conclusion that we have reached with Theorem 8.8 is that the family of HCQ^*-algebras is quite rich; it has as many members of the class of von Neumann algebras for which the construction of the Tomita–Takesaki theory can be performed.

9. Outcome

In the preceding sections, we have analyzed in detail individual Banach partial $*$-algebras, including those of type (B). The next step is to consider maps from one Banach partial $*$-algebrainto another one, in particular homomorphisms or isomorphisms. These include in particular the notion of representation; that is, a $*$-homomorphism from a given (Banach) partial $*$-algebra into the partial $*$-algebra $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$, the set of all (closable) linear operators X such that $D(X) = \mathcal{D}$, $D(X^*) \supseteq \mathcal{D}$. Here a $*$-homomorphism is a linear map $\rho : \mathfrak{A} \to \mathfrak{B}$ such that (i) $\rho(x^*) = \rho(x)^*$ for every $x \in \mathfrak{A}$ and (ii) whenever $x \in L(y)$ in $\mathfrak{A}$, then $\rho(x) \in L(\rho(y))$ in $\mathfrak{B}$ and $\rho(x)\,\rho(y) = \rho(x\,y)$. Now, a privileged role is played by the well-known GNS representation, and the latter is closely related to the notion of biweights. In the case of Banach partial $*$-algebras, these objects have been analyzed in [6] and the outcome is that being of type (B) does not bring much improvement. Therefore we will not pursue the subject in the present review and refer the reader to the original paper [6].

Appendix A. Partial inner product spaces

For the convenience of the reader, we have collected here the main features of partial inner product spaces, keeping only what is needed for reading the paper. Further information may be found in our monograph [4].

The general framework is that of a PIP-space V, corresponding to the linear compatibility $\#$; that is, a symmetric binary relation $f\#g$ which preserves linearity. We call *assaying subspace* of V a subspace S such that $S^{\#\#} = S$, and we denote by $\mathcal{F}(V,\#)$ the family of all assaying subspaces of V, ordered by inclusion. The assaying subspaces are denoted by $V_r, V_q, \ldots$, and the index set is F. By definition, $q \leq r$ if and only if $V_q \subseteq V_r$. Thus we may write

$$f\#g \iff \exists\, r \in F \text{ such that } f \in V_r, g \in V_{\overline{r}}\,.$$

General considerations [12] imply that the family $\mathcal{F}(V,\#) := \{V_r, r \in F\}$, ordered by inclusion, is a complete involutive lattice; that is, it is stable under the following operations, arbitrarily iterated:

. involution: $V_r \leftrightarrow V_{\overline{r}} := (V_r)^{\#}$,
. infimum: $V_{p\wedge q} := V_p \wedge V_q = V_p \cap V_q$, $\qquad (p,q,r \in F)$
. supremum: $V_{p\vee q} := V_p \vee V_q = (V_p + V_q)^{\#\#}$.

The smallest element of $\mathcal{F}(V,\#)$ is $V^{\#} = \bigcap_r V_r$, and the greatest element is $V = \bigcup_r V_r$.

By definition, the index set F is also a complete involutive lattice; for instance,

$$(V_{p\wedge q})^{\#} = V_{\overline{p\wedge q}} = V_{\overline{p}\vee\overline{q}} = V_{\overline{p}} \vee V_{\overline{q}}.$$

Given a vector space V equipped with a linear compatibility $\#$, a *partial inner product* on $(V, \#)$ is a Hermitian form $\langle\cdot|\cdot\rangle$ defined exactly on compatible pairs of vectors. A *partial inner product space* (PIP-space) is a vector space V equipped with a linear compatibility and a partial inner product.

From now on, we will assume that our PIP-space $(V, \#, \langle\cdot|\cdot\rangle)$ is *nondegenerate*; that is, $\langle f|g\rangle = 0$ for all $f \in V^{\#}$ implies $g = 0$. As a consequence, $(V^{\#}, V)$ and every couple $(V_r, V_{\overline{r}})$, $r \in F$, are a dual pair in the sense of topological vector spaces [17]. Next we assume that every V_r carries its Mackey topology $\tau(V_r, V_{\overline{r}})$; so that its conjugate dual is $(V_r)^\times = V_{\overline{r}}$, for all $r \in F$. Then, $r < s$ implies $V_r \subset V_s$, and the embedding operator $E_{sr} : V_r \to V_s$ is continuous and has dense range. In particular, $V^{\#}$ is dense in every V_r. In what follows, we also assume the partial inner product to be positive definite, $\langle f|f\rangle > 0$ whenever $f \neq 0$.

In fact, the whole structure can be reconstructed from a fairly small subset of $\mathcal{F}$, namely, a *generating* involutive sublattice $\mathcal{J}$ of $\mathcal{F}(V,\#)$, indexed by J, which means that

$$f\#g \iff \exists\, r \in J \text{ such that } f \in V_r, g \in V_{\overline{r}}\,.$$

The resulting structure is called an *indexed* PIP-*space* and denoted simply by $V_J := (V, \mathcal{J}, \langle\cdot|\cdot\rangle)$.

For practical applications, it is essentially sufficient to restrict oneself to the case of an indexed PIP-space satisfying the following conditions:

(i) every $V_r, r \in J$, is a Hilbert space or a reflexive Banach space; so that the Mackey topology $\tau(V_r, V_{\overline{r}})$ coincides with the norm topology;
(ii) there is a unique self-dual, Hilbert, assaying subspace $V_o = V_{\overline{o}}$.
(iii) for every $V_r \in \mathcal{J}$, the norm $\|\cdot\|_{\overline{r}}$ on $V_{\overline{r}} = V_r^\times$ is the conjugate of the norm $\|\cdot\|_r$ on V_r. In particular, the partial inner product $\langle\cdot|\cdot\rangle$ coincides with the inner product of V_o on the latter.

In that case, the *indexed* PIP*-space* $V_J := (V, \mathcal{J}, \langle\cdot|\cdot\rangle)$ is called, respectively, a *lattice of Hilbert spaces* (LHS) or a *lattice of Banach spaces* (LBS). This implies, in addition, that, for a LHS:

(i) for every pair $V_p, V_q \in \mathcal{J}$, the norm on $V_{p\wedge q} := V_p \cap V_q$ is equivalent to the projective norm, given by

$$\|f\|^2_{p\wedge q} = \|f\|^2_p + \|f\|^2_q,$$

(ii) for every pair $V_p, V_q \in \mathcal{J}$, the norm on $V_{p\vee q} := V_p + V_q$, the vector sum, is equivalent to the inductive norm

$$\|f\|^2_{p\vee q} = \inf_{f=g+h} \left(\|g\|^2_p + \|h\|^2_q\right), \; g \in V_p, f \in V_q\,.$$

Similar formulas are used in the LBS case, simply omitting the squares. These norms come from interpolation theory [11].

Note that $V^\#$ and V themselves usually do *not* belong to the family $\{V_r,\, r \in J\}$, but they can be recovered as

$$V^\# = \bigcap_{r\in J} V_r, \qquad V = \sum_{r\in J} V_r.$$

A standard, albeit trivial, example is that of a Rigged Hilbert space (RHS) $\Phi \subset \mathcal{H} \subset \Phi^\#$ (it is trivial because the lattice $\mathcal{F}$ contains only three elements).

Familiar concrete examples are sequence spaces with $V = \omega$, the space of *all* complex sequences $x = (x_n)$, and spaces of locally integrable functions with $V = L^1_{\rm loc}(\mathbb{R}, {\rm d}x)$, the space of Lebesgue measurable functions, integrable over compact subsets.

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