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THE GENERALIZED VON NEUMANN–JORDAN CONSTANT AND NORMAL STRUCTURE IN BANACH SPACES

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Abstract. Recently, a new geometric constant called generalized von Neumann–Jordan constant was introduced. In this paper, the relationships between above constant and generalized García–Falset coefficient are given. In terms of this constant, the lower bounds for the weakly convergent sequence coefficient of a Banach space X are also shown. Moreover, some sufficient conditions which imply normal structure and uniform normal structure are presented.

1. Introduction and preliminaries

Let X be a real Banach space. For a nonempty bounded subset C of X, denote that

$$\operatorname{diam}(C) := \sup\{\| x - y \|: x, y \in C\} \text{ and } \operatorname{rad}(C) := \inf_{x\in C} \sup\{\| x - y \|: y \in C\}.$$

We say that X has *normal structure* if every bounded closed convex subset C of X with $\operatorname{diam}(C) > 0$, verifies $\operatorname{rad}(C) < \operatorname{diam}(C)$. X is said to have *uniform normal structure* if there exists $c \in (0, 1)$ such that, for all bounded closed convex subsets C of X with $\operatorname{diam}(C) > 0$, the inequality $\operatorname{rad}(C) < c\operatorname{diam}(C)$ holds.

A Banach space X is said to have the *fixed point property* if every nonexpansive mapping $T : C \to C$, *i.e.*,

$$\| Tx - Ty \| \leq \| x - y \|, \forall x, y \in C$$

acting on a nonempty bounded closed and convex subset C of X has a fixed point. It was proved by Kirk [9] that every reflexive Banach space with normal structure

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has the fixed point property. Many geometrical properties of Banach spaces implying normal structure and uniform normal structure have been studied.

The von Neumann–Jordan constant was defined by Clarkson as follows:

$$C_{NJ}(X) := \sup\{\frac{\| x+y \|^2 + \| x-y \|^2}{2(\| x \|^2 + \| y \|^2)} : x, y \in X, \text{not both zero}\}.$$

The *weakly convergent sequence coefficient* of X, $WCS(X)$, was defined in [3] as the supremum of the set of all numbers M with the property that for each weakly convergent sequence (x_n), there is some y in the closed convex hull of the sequence such that

$$M \limsup_{n\to\infty} \| x_n - y \| \leq \lim_{n\to\infty} \sup\{\| x_i - x_j \|: i, j \geq n\}.$$

It is clear that $1 \leq WCS(X) \leq 2$ and it is known that X has weak normal structure if $WCS(X) > 1$ [3].

The coefficient $R(a, X)$, which is a generalized García–Falset coefficient $R(X)$ [5], is introduced by Benavides [2]: For a given $a \geq 0$,

$$R(a, X) := \sup\{\liminf_{n\to\infty} \| x_n + x \|\},$$

where the supremum is taken over all $x \in X$ with $\| x \| \leq a$ and all weakly null sequences in the unit ball B_X such that $\lim_{n,m;n\neq m} \| x_n - x_m \| \leq 1$. It is clear that $R(1, X) = R(X)$. Benavides [2] also defined the coefficient $M(X)$ by

$$M(X) := \sup\{\frac{1+a}{R(a, X)} : a \geq 0\}.$$

The coefficient $RW(a, X)$ was introduced in [6]: For each $a \geq 0$,

$$RW(a, X) := \sup\{\min\{\liminf_{n\to\infty} \| x_n + x \|, \liminf_{n\to\infty} \| x_n - x \|\}\},$$

where the supremum is taken over all $x \in X$ with $\| x \| \leq a$ and all weakly null sequences in the unit ball B_X. It was proved that for any Banach space X, the inequality $RW(a, X) \geq R(a, X)$ holds [11].

Recall that

$$\rho'_X(0) := \lim_{t\to 0^+} \frac{\rho_X(t)}{t},$$

where $\rho_X[0, \infty) \to [0, \infty)$ is the *modulus of smoothness* of X defined by

$$\rho_X(t) := \sup\{\frac{\| x+ty \| + \| x-ty \|}{2} - 1 : x, y \in B_X\}.$$

The coefficient $\mu(X)$, defined by the infimum of the set of real numbers $r > 0$ such that

$$\limsup_{n\to\infty} \| x + x_n \| \leq r \limsup_{n\to\infty} \| x - x_n \|$$

for all $x \in X$ and all weakly null sequences (x_n) in X. It was proved that if the condition $\rho'_X(0) < \frac{1}{\mu(X)}$ holds, then X has normal structure [11].

The *modulus of convexity* of X is the function $\delta_X : [0, 2] \to [0, 1]$ defined by

$$\delta_X(\varepsilon) := \inf\{1 - \| \frac{x+y}{2} \|: x, y \in B_X, \| x-y \| \geq \varepsilon\}.$$

The *characteristic of convexity* of X is the number

$$\varepsilon_0(X) := \sup\{\varepsilon : \delta_X(\varepsilon) = 0\}.$$

In this paper, some relationships between the generalized von Neumann–Jordan constant and generalized García–Falset coefficient are given. The lower bounds for the weakly convergent sequence coefficient of a Banach space X are also shown. Moreover, some sufficient conditions for normal structure and uniform normal structure are presented.

2. The constant and the fixed point property

Recently, Cui, Huang, Hudzik and Kaczmarek [4] introduced a new geometric constant $C_{NJ}^{(p)}(X)$ called generalized von Neumann–Jordan constant, defined by

$$C_{NJ}^{(p)}(X) := \sup\{\frac{\| x+y \|^p + \| x-y \|^p}{2^{p-1}(\| x \|^p + \| y \|^p)} : x, y \in X, (x,y) \neq (0,0)\},$$

where $1 \leq p < \infty$.

The parametrized formula of this constant is the following

$$C_{NJ}^{(p)}(X) = \sup\{\frac{\| x+ty \|^p + \| x-ty \|^p}{2^{p-1}(1+t^p)} : x, y \in S_X, 0 \leq t \leq 1\},$$

where $1 \leq p < \infty$.

It was proved that the generalized von Neumann–Jordan constant satisfies the inequality $C_{NJ}^{(p)}(X) \leq 2$, and that Banach space X is uniformly non-square if and only if $C_{NJ}^{(p)}(X) < 2$ [4].

Next, we will show the relationships between the constant $C_{NJ}^{(p)}(X)$ and the coefficient $R(a, X)$, and we will obtain a sufficient condition for the fixed point property.

Proposition 2.1. *Let X be a Banach space. For any $a \geq 0$,*

$$C_{NJ}^{(p)}(X) \geq \frac{R(a,X)^p}{2^{p-2}(1+a^p)}.$$

Proof. Let $a \geq 0$. For any weakly null sequence (x_n) in B_X and any $x \in X$ with $\| x \| \leq a$, by the definition of $C_{NJ}^{(p)}(X)$, we have,

$$C_{NJ}^{(p)}(X) \geq \frac{\| x_n + x \|^p + \| x_n - x \|^p}{2^{p-1}(\| x_n \|^p + \| x \|^p)} \geq \frac{\min\{\| x_n + x \|^p, \| x_n - x \|^p\}}{2^{p-2}(\| x_n \|^p + \| x \|^p)}$$

Therefore,

$$C_{NJ}^{(p)}(X) \geq \frac{RW(a,X)^p}{2^{p-2}(1+a^p)}$$

and since the inequality $RW(a,X) \geq R(a,X)$ holds, we obtain

$$C_{NJ}^{(p)}(X) \geq \frac{R(a,X)^p}{2^{p-2}(1+a^p)}.$$

∎

Proposition 2.2. *If X is a Banach space, then*

$$C_{NJ}^{(p)}(X) \geq \frac{(R(X))^p(1+\frac{1}{(\mu(X))^p})}{2^{-p}}.$$

Proof. For any $\varepsilon > 0$, there exists $x \in S_X$ and (x_n) in B_X such that

$$\liminf_{n\to\infty} \| x_n + x \| \geq R(X) - \varepsilon.$$

We may extract a subsequence, still denoted by (x_n), such that $\liminf_{n\to\infty} \| x_n - x \|$ exists and

$$\liminf_{n\to\infty} \| x_n + x \| \geq R(X) - \varepsilon.$$

So we obtain,

$$\begin{aligned} 2^p C_{NJ}^{(p)}(X) &\geq \liminf_{n\to\infty} \| x_n + x \|^p + \liminf_{n\to\infty} \| x_n - x \|^p \\ &\geq \liminf_{n\to\infty} \| x_n + x \|^p \ (1 + \frac{1}{(\mu(X))^p}) \\ &\geq (R(X) - \varepsilon)^p (1 + \frac{1}{(\mu(X))^p}). \end{aligned}$$

By the arbitrariness of $\varepsilon > 0$, we get $C_{NJ}^{(p)}(X) \geq \frac{(R(X))^p(1+\frac{1}{(\mu(X))^p})}{2^{-p}}$. □

As a consequence of the preceding proposition, we have the following corollary.

Corollary 2.3. *If $C_{NJ}^{(p)}(X) < 1 + \frac{1}{(\mu(X))^p}$, then $R(X) < 2$, that is, such a Banach space X has the fixed point property.*

3. The generalized von Neumann–Jordan constant and normal structure

Now we show some inequalities on the generalized von Neumann–Jordan constant, and the constants $\varepsilon_0(X)$ and $\rho'_X(0)$. By these inequalities we obtain some sufficient conditions which imply normal structure.

Proposition 3.1. *For any Banach space X, we have the following:*

(1) $C_{NJ}^{(p)}(X) \geq 1 + \frac{\varepsilon_0(X)^p}{2^p}$,

(2) $C_{NJ}^{(p)}(X) \geq 1 + \rho'_X(0)^p$.

Proof. (1)Let $\varepsilon \in [0, 2]$. Suppose that there exists $x, y \in B_X$ such that $\| x - y \| \geq \varepsilon$. Then

$$C_{NJ}^{(p)}(X) \geq \frac{\| x+y \|^p + \| x-y \|^p}{2^{p-1}(\| x \|^p + \| y \|^p)} \geq \frac{\| x+y \|^p + \varepsilon^p}{2^p}$$

which is equivalent to

$$1 - \| \frac{x+y}{2} \| \geq 1 - \sqrt[p]{C_{NJ}^{(p)}(X) - (\frac{\varepsilon}{2})^p}.$$

By the definition of δ_X, we get

$$\delta_X(\varepsilon) \geq 1 - \sqrt[p]{C_{NJ}^{(p)}(X) - (\frac{\varepsilon}{2})^p},$$

that is,

$$C_{NJ}^{(p)}(X) \geq (\frac{\varepsilon}{2})^p + (1-\delta_X(\varepsilon))^p.$$

We conclude that

$$C_{NJ}^{(p)}(X) \geq \sup\{(\frac{\varepsilon}{2})^p + (1-\delta_X(\varepsilon))^p : \varepsilon \in [0,2]\}$$

and in particular, by $\varepsilon_0(X) = 2(1 - \lim_{\varepsilon\to 2}\delta_X(\varepsilon))$(see [7]),

$$C_{NJ}^{(p)}(X) \geq \lim_{\varepsilon\to 2^-}((\frac{\varepsilon}{2})^p + (1-\delta_X(\varepsilon))^p) = 1 + (\frac{\varepsilon_0(X)^p}{2^p}).$$

(2) By(i), and $\varepsilon_0(X^*) = 2\rho'_X(0)$(the consequence of Lindenstrauss' formulae [10]), we obtain that

$$C_{NJ}^{(p)}(X) = C_{NJ}^{(p)}(X^*) \geq 1 + (\frac{\varepsilon_0(X^*)^p}{2^p}) = 1 + \rho'_X(0)^p.$$

□

Corollary 3.2. *Let X be a Banach space. If $C_{NJ}^{(p)}(X) < 1 + \frac{1}{\mu(X)^p}$, then $\rho'_X(0) < \frac{1}{\mu(X)}$.*

Lemma 3.3. [11] *Let X be a Banach space. IF (x_n) be a weakly null sequence in S_X such that $\lim_{n,m\to\infty,n\neq m} \| x_n - x_m \| =: d$ exists, then there exist $(u_n),(v_n)$ and (w_n), weakly null sequence in S_X and (f_n) and (g_n), sequences in S_{X^*} for which*

$$\lim_{n\to\infty} f_n(-u_n) = \lim_{n\to\infty} g_n(u_n) = \frac{1}{d},$$

$$\lim_{n\to\infty} f_n(v_n) \geq \frac{a}{R(a,X)},$$

$$\lim_{n\to\infty} g_n(v_n) \geq \frac{1}{R(a,X)d}$$

and

$$\min\{\lim_{n\to\infty} f_n(w_n), \lim_{n\to\infty} g_n(w_n)\} \geq \frac{1}{\mu(X)d}.$$

Relationships between the generalized von Neumann–Jordan constant, the weakly convergent sequence coefficient, the coefficient $R(a,X)$ and the coefficient $M(X)$ is given in the following proposition.

Proposition 3.4.

(1) $C_{NJ}^{(p)}(X) \geq \frac{1}{WCS(X)^p} + \sup\{\frac{1}{2^p R(a,X)^p}(a + \frac{1}{WCS(X)})^p : a \geq 0\}$.

(2) $WCS(X)^p \geq \frac{1+(\frac{M(X)}{2})^p}{C_{NJ}^{(p)}(X)}$.

Proof. (1) Let(x_n) be a weakly null sequence in S_X such that $\lim_{n,m\to\infty} \| x_n - x_m \| = d$ exists. Put $a \geq 0$. By Lemma3.3 , there exist weakly null sequence (u_n) and (v_n) in S_X, sequence (f_n) and (g_n) in S_{X^*}, for which

$$\lim_{n\to\infty} f_n(-u_n) = \lim_{n\to\infty} g_n(u_n) = \frac{1}{d},$$

$$\lim_{n\to\infty} f_n(v_n) \geq \frac{a}{R(a,X)}$$

and

$$\lim_{n\to\infty} g_n(v_n) \geq \frac{1}{R(a,X)d}.$$

For each $n \geq 1$, $\| f_n - g_n \| \geq f_n(-u_n) + g_n(u_n)$ and $\| f_n + g_n \| \geq f_n(v_n) + g_n(v_n)$, so

$$\liminf_{n\to\infty} \| f_n - g_n \| \geq \frac{2}{d}$$

and

$$\liminf_{n\to\infty} \| f_n + g_n \| \geq \frac{1}{R(a,X)}(a + \frac{1}{d}).$$

Since, for each $n \geq 1$,

$$C_{NJ}^{(p)}(X) = C_{NJ}^{(p)}(X^*) \geq \frac{\| f_n + g_n \|^p + \| f_n - g_n \|^p}{2^p},$$

we obtain

$$C_{NJ}^{(p)}(X) \geq \frac{1}{d^p} + \frac{1}{2^p R(a,X)^p}(a + \frac{1}{d})^p.$$

By the equivalent expression of $WCS(X)$ [1], we conclude

$$C_{NJ}^{(p)}(X) \geq \frac{1}{WCS(X)^p} + \frac{1}{2^p R(a,X)^p}(a + \frac{1}{WCS(X)})^p.$$

Inequality (2) is a consequence of (i): for all $a > 0$,

$$\begin{aligned} C_{NJ}^{(p)}(X) &\geq \frac{1}{WCS(X)^p} + \frac{1}{2^p R(a,X)^p}(a + \frac{1}{WCS(X)})^p \\ &\geq \frac{1}{WCS(X)^p}(1 + (\frac{1+a}{2R(a,X)})^p. \end{aligned}$$

Thus

$$WCS(X)^p \geq \frac{1 + \frac{\sup\{\frac{1+a}{R(a,X)}:a\geq 0\}^p}{2^p}}{C_{NJ}^{(p)}(X)} \geq \frac{1 + (\frac{M(X)}{2})^p}{C_{NJ}^{(p)}(X)}$$

□

Corollary 3.5. *If the inequality $C_{NJ}^{(p)}(X) < 1 + (\frac{M(X)}{2})^p$ holds, then the Banach space X has normal structure.*

4. The generalized von Neumann–Jordan constant and uniform normal structure

$l_\infty(X)$ denotes that the subspace of the product space $\Pi_{n\in\mathbb{N}} X$ equipped with the norm $\| (x_n) \| := \sup_{n\in\mathbb{N}} \| x_n \| < \infty$. Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$ and let

$$N_{\mathcal{U}} = \{(x_n) \in l_\infty(X) : \lim_{\mathcal{U}} \| x_n \| = 0\}.$$

The ultrapower of X, denoted by $\tilde{X}$, is the quotient space $l_\infty(X)/N_{\mathcal{U}}$ equipped with the quotient norm, and $(\tilde{x}_n)$ denotes the elements of the ultrapower. Note that if $\mathcal{U}$ is non-trivial, then X can be embedded into $\tilde{X}$ isometrically. It was proved that if X is super-reflexive, that is $\tilde{X^*} = (\tilde{X})^*$, then X has uniform normal structure if and only if $\tilde{X}$ has normal structure [8]. We show a theorem about uniform normal structure.

Theorem 4.1. *Let X be a Banach space and the inequality*

$$C_{NJ}^{(p)}(X) < \frac{(\sqrt{4+t^2}+t)^p}{2^{2p-2}(1+t^p)}$$

holds for some $t \in (0,1]$. Then X has uniform normal structure.

Proof. Since X is uniformly non-square, so X is super-reflexive(see [13]), and consequently it is enough to show that X has normal structure. Suppose that X lacks normal structure. Then by Lemma 2 in [12], there exist $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \in S_{\tilde{X}}$ and $\tilde{f}_1, \tilde{f}_2, \tilde{f}_3 \in S_{\tilde{X}^*}$ satisfying:

(1) $\| \tilde{x}_i - \tilde{x}_j \| = 1$ and $\tilde{f}_i(\tilde{x}_j) = 0$ for all $i \neq j$.

(2) $\tilde{f}_i(\tilde{x}_i) = 1$ for $i = 1, 2, 3$.

(3) $\| \tilde{x}_3 - (\tilde{x}_2 + \tilde{x}_1) \| \geq \| \tilde{x}_2 + \tilde{x}_1 \|$.

Let $h(t) := \frac{(2-t+\sqrt{4+t^2})}{2}$ and we consider three possible cases.

Case 1: If $\| \tilde{x}_1 + \tilde{x}_2 \| \leq h(t)$. Let $\tilde{x} = \tilde{x}_1 - \tilde{x}_2$ and $\tilde{y} = \frac{(\tilde{x}_1+\tilde{x}_2)}{h(t)}$. Then $\tilde{x}, \tilde{y} \in B_{\tilde{X}}$, and

$$\begin{aligned}\| \tilde{x} + t\tilde{y} \| &= \| (1 + (\frac{t}{h(t)}))\tilde{x}_1 - (1 - (\frac{t}{h(t)}))\tilde{x}_2 \| \\ &\geq (1 + (\frac{t}{h(t)}))\tilde{f}_1(\tilde{x}_1) - (1 - (\frac{t}{h(t)}))\tilde{f}_1(\tilde{x}_2) \\ &= 1 + (\frac{t}{h(t)}),\end{aligned}$$

as well as

$$\begin{aligned}\| \tilde{x} - t\tilde{y} \| &= \| (1 + (\frac{t}{h(t)}))\tilde{x}_2 - (1 - (\frac{t}{h(t)}))\tilde{x}_1 \| \\ &\geq (1 + (\frac{t}{h(t)}))\tilde{f}_2(\tilde{x}_2) - (1 - (\frac{t}{h(t)}))\tilde{f}_2(\tilde{x}_1) \\ &= 1 + (\frac{t}{h(t)}).\end{aligned}$$

Case 2: If $\| \tilde{x}_1 + \tilde{x}_2 \| \geq h(t)$ and $\| \tilde{x}_3 + \tilde{x}_2 - \tilde{x}_1 \| \leq h(t)$. Let $\tilde{x} = \tilde{x}_2 - \tilde{x}_3$ and $\tilde{y} = \frac{(\tilde{x}_3+\tilde{x}_2-\tilde{x}_1)}{h(t)}$. Then $\tilde{x}, \tilde{y} \in B_{\tilde{X}}$, and

$$\begin{aligned}\| \tilde{x} + t\tilde{y} \| &= \| (1 + (\frac{t}{h(t)}))\tilde{x}_2 - (1 - (\frac{t}{h(t)}))\tilde{x}_3 - (\frac{t}{h(t)})\tilde{x}_1 \| \\ &\geq (1 + (\frac{t}{h(t)}))\tilde{f}_2(\tilde{x}_2) - (1 - (\frac{t}{h(t)}))\tilde{f}_2(\tilde{x}_3) - (\frac{t}{h(t)}))\tilde{f}_2(\tilde{x}_1) \\ &= 1 + (\frac{t}{h(t)}),\end{aligned}$$

as well as

$$\begin{aligned}\| \tilde{x} - t\tilde{y} \| &= \| (1 + (\frac{t}{h(t)}))\tilde{x}_3 - (1 - (\frac{t}{h(t)}))\tilde{x}_2 - (\frac{t}{h(t)})\tilde{x}_1 \| \\ &\geq (1 + (\frac{t}{h(t)}))\tilde{f}_3(\tilde{x}_3) - (1 - (\frac{t}{h(t)}))\tilde{f}_3(\tilde{x}_2) - (\frac{t}{h(t)}))\tilde{f}_3(\tilde{x}_1) \\ &= 1 + (\frac{t}{h(t)}).\end{aligned}$$

Case 3: If $\| \tilde{x}_1 + \tilde{x}_2 \| \geq h(t)$ and $\| \tilde{x}_3 + \tilde{x}_2 - \tilde{x}_1 \| \geq h(t)$. Let $\tilde{x} = \tilde{x}_3 - \tilde{x}_1$ and $\tilde{y} = \tilde{x}_2$. Then $\tilde{x}, \tilde{y} \in B_{\tilde{X}}$, and

$$\begin{aligned}\| \tilde{x} + t\tilde{y} \| &= \| \tilde{x}_3 + t\tilde{x}_2 - \tilde{x}_1 \| \\ &\geq \| \tilde{x}_3 + \tilde{x}_2 - \tilde{x}_1 \| - (1 - t) \\ &\geq h(t) + t - 1,\end{aligned}$$

as well as

$$\begin{aligned}\| \tilde{x} - t\tilde{y} \| &= \| \tilde{x}_3 - (t\tilde{x}_2 + \tilde{x}_1) \| \\ &\geq \| \tilde{x}_3 - (\tilde{x}_2 + \tilde{x}_1) \| - (1 - t) \\ &\geq h(t) + t - 1.\end{aligned}$$

Then, by the definition of $C_{NJ}^{(p)}(X)$ and $C_{NJ}^{(p)}(X) = C_{NJ}^{(p)}(\tilde{X})$, we obtain

$$\begin{aligned}C_{NJ}^{(p)}(X) &\geq \max\{\frac{(1 + \frac{t}{h(t)})}{2^{p-2}(1 + t^p)}, \frac{(h(t) + t - 1)^p}{2^{p-2}(1 + t^p)}\} \\ &= \frac{(\sqrt{4 + t^2} + t)^p}{2^{2p-2}(1 + t^p)}.\end{aligned}$$

This is a contradiction and thus the proof is complete. □

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