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WEIGHTED INEQUALITIES FOR A CLASS OF SEMIADDITIVE OPERATORS

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Abstract. We find necessary and sufficient conditions for the validity of weighted Hardy-type inequalities for a class of semiadditive operators.

1. Introduction

Let $I=(0,\infty)$, $0<\theta,q,p\le\infty$ and $\frac{1}{p}+\frac{1}{p'}=1$. Suppose that w, u and v are non-negative functions such that they, together with $v^{1-p'}$, are locally integrable on I.

We introduce the following operators:

$$T_\theta^+f(x)=\left(\int_0^x w(t)\left|\int_t^x f(s)ds\right|^\theta dt\right)^{\frac{1}{\theta}},\ T_\theta^-f(x)=\left(\int_x^\infty w(t)\left|\int_x^t f(s)ds\right|^\theta dt\right)^{\frac{1}{\theta}}.$$

The operators T_θ^+ and T_θ^- are superlinear for $0<\theta<1$ and sublinear for $\theta>1$. These operators become linear for $\theta=1$.

We consider the inequalities:

$$\left(\int_0^\infty u(x)\left|T_\theta^\pm f(x)\right|^q dx\right)^{\frac{1}{q}}\le C^\pm\left(\int_0^\infty v(t)|f(t)|^p dt\right)^{\frac{1}{p}}, \tag{1.1}$$

where $C^\pm$ are positive constants. Let us notice that the Hardy-type inequality with the operator T_θ^+ is directly equivalent to the inequality with the operator

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T_θ^-, via a simple change of variable in the integrals. Moreover, it suffices to study (1.1) for $f \geq 0$.

Let

$$\Delta_\theta^+ g(x) = \left(\int_0^x w(t) \left| g(x) - g(t) \right|^\theta dt \right)^{\frac{1}{\theta}}, \ \Delta_\theta^- g(x) = \left(\int_x^\infty w(t) \left| g(t) - g(x) \right|^\theta dt \right)^{\frac{1}{\theta}}$$

be the θ-mean deviations with the weight w of the value of a function g from $g(x)$ on the intervals $(0, x)$ and (x, ∞), respectively. Then inequality (1.1) is equivalent to the following inequality:

$$\left(\int_0^\infty u(x) \left| \Delta_\theta^\pm g(x) \right|^q dx \right)^{\frac{1}{q}} \leq C^\pm \left(\int_0^\infty v(t) |g'(t)|^p dt \right)^{\frac{1}{p}}.$$

Inequality (1.1) was investigated in [5], where necessary and sufficient conditions for its validity were found for $1 \leq p \leq q < \infty$ and $0 < \theta < \infty$.

In this work we study the case $0 < q < p$, $p \geq 1$ and $0 < \theta < \infty$. Here we prove the sufficiency part of the provided case and the general v. Necessity is derived for the case $0 < \theta, q < \infty$, $\max\{\theta, q\} < p$, $p > 1$ and the general v. Moreover, necessity is also derived for the case $\max\{\theta, q\} < 1 = p$ and $v \equiv 1$.

For $(\alpha, \beta) \subset I$ we assume

$$A^+(\alpha, \beta) = \sup_{\alpha < x < \beta} \left(\int_\alpha^x w(t) dt \right)^{\frac{1}{\theta}} \left(\int_x^\beta v^{1-p'}(s) ds \right)^{\frac{1}{p'}},$$

$$A^-(\alpha, \beta) = \sup_{\alpha < x < \beta} \left(\int_x^\beta w(t) dt \right)^{\frac{1}{\theta}} \left(\int_\alpha^x v^{1-p'}(s) ds \right)^{\frac{1}{p'}},$$

$$B^+(\alpha, \beta) = \left(\int_\alpha^\beta w(t) \left(\int_\alpha^t w(s) ds \right)^{\frac{\theta}{p-\theta}} \left(\int_t^\beta v^{1-p'}(s) ds \right)^{\frac{\theta(p-1)}{p-\theta}} dt \right)^{\frac{p-\theta}{p\theta}},$$

$$B^-(\alpha, \beta) = \left(\int_\alpha^\beta w(t) \left(\int_t^\beta w(s) ds \right)^{\frac{\theta}{p-\theta}} \left(\int_\alpha^t v^{1-p'}(s) ds \right)^{\frac{\theta(p-1)}{p-\theta}} dt \right)^{\frac{p-\theta}{p\theta}},$$

$$D^+(\alpha, \beta) = \left(\int_\alpha^\beta w(t) \left(\int_\alpha^t w(s) ds \right)^{\frac{\theta}{1-\theta}} (\underline{v}(t, \beta))^{\frac{\theta}{\theta-1}} dt \right)^{\frac{1-\theta}{\theta}},$$

$$D^-(\alpha,\beta)=\left(\int\limits_\alpha^\beta w(t)\left(\int\limits_t^\beta w(s)ds\right)^{\frac{\theta}{1-\theta}}(\underline{v}(\alpha,t))^{\frac{\theta}{\theta-1}}\,dt\right)^{\frac{1-\theta}{\theta}},$$

where $\underline{v}(\alpha,\beta)=\operatorname*{ess\,inf}_{\alpha<t<\beta} v(t)$.

Let $H^+(\alpha,\beta)$ and $H^-(\alpha,\beta)$ be the best constants of the following Hardy inequalities

$$\left(\int\limits_\alpha^\beta w(t)\left|\int\limits_t^\beta f(s)ds\right|^\theta dt\right)^{\frac{1}{\theta}}\le H^+(\alpha,\beta)\left(\int\limits_\alpha^\beta v(t)|f(t)|^p dt\right)^{\frac{1}{p}},$$

$$\left(\int\limits_\alpha^\beta w(t)\left|\int\limits_\alpha^t f(s)ds\right|^\theta dt\right)^{\frac{1}{\theta}}\le H^-(\alpha,\beta)\left(\int\limits_\alpha^\beta v(t)|f(t)|^p dt\right)^{\frac{1}{p}},$$

respectively.

From the results of the works [4] and [8] (see also [5]) for Hardy inequalities we have

Lemma A. (i) *If $1\le p\le\theta<\infty$, then*

$$A^\pm(\alpha,\beta)\le H^\pm(\alpha,\beta)\le p^{\frac{1}{\theta}}(p')^{\frac{1}{p'}}A^\pm(\alpha,\beta). \tag{1.2}$$

(ii) *If $0<\theta<p$ and $1<p<\infty$, then*

$$(p')^{\frac{1}{p'}}\theta^{\frac{1}{p}}\left(1-\frac{\theta}{p}\right)B^\pm(\alpha,\beta)\le H^\pm(\alpha,\beta)\le\left(\frac{p}{p-\theta}\right)^{\frac{p-\theta}{p\theta}}p^{\frac{1}{p}}(p')^{\frac{1}{p'}}B^\pm(\alpha,\beta). \tag{1.3}$$

(iii) *If $0<\theta<1=p$, then*

$$\theta(1-\theta)D^\pm(\alpha,\beta)\le H^\pm(\alpha,\beta)\le(1-\theta)^{\frac{1-\theta}{\theta}}D^\pm(\alpha,\beta). \tag{1.4}$$

Since the expressions $A^\pm$, $B^\pm$ and $D^\pm$ are decreasing in α and increasing in β, then from (1.2), (1.3) and (1.4) we have that $H^\pm(\alpha,\beta)$ are equivalent to a decreasing function in α and equivalent to a increasing function in β. It means that for each case (i), (ii) and (iii) there exists a constant $C>0$ depending only on p and θ such that $H^\pm(\alpha,\beta)\le CH^\pm(\alpha_1,\beta_1)$ holds for $\alpha_1\le\alpha<\beta\le\beta_1$. For example, for the case (i) we have $C=p^{\frac{1}{\theta}}(p')^{\frac{1}{p'}}$.

Denote $A^+(0,\beta)\equiv A^+(\beta)$, $B^+(0,\beta)\equiv B^+(\beta)$, $D^+(0,\beta)\equiv D^+(\beta)$, $H^+(0,\beta)\equiv H^+(\beta)$, $A^-(\alpha,\infty)\equiv A^-(\alpha)$, $B^-(\alpha,\infty)\equiv B^-(\alpha)$, $D^-(\alpha,\infty)\equiv D^-(\alpha)$ and $H^-(\alpha,\infty)\equiv H^-(\alpha)$.

In what follows we write $A\ll B$ if $A\le CB$ with some constant $C>0$ that depends only on θ, q and p. The expression $A\approx B$ means $A\ll B$ and $B\ll A$.

2. Main results

Let

$$E^{+}=\left(\int_{0}^{\infty}u(x)\left(\int_{x}^{\infty}u(s)ds\right)^{\frac{q}{p-q}}\left(H^{+}(x)\right)^{\frac{pq}{p-q}}dx\right)^{\frac{p-q}{pq}},$$

$$E^{-}=\left(\int_{0}^{\infty}u(x)\left(\int_{0}^{x}u(s)ds\right)^{\frac{p}{p-q}}\left(H^{-}(x)\right)^{\frac{pq}{p-q}}dx\right)^{\frac{p-q}{pq}}.$$

Theorem 2.1. *Let $0<q<p$, $p\geq 1$ and $0<\theta<\infty$. If $E^{\pm}<\infty$, then inequality (1.1) holds. Moreover, $C^{\pm}\ll E^{\pm}$, where $C^{\pm}>0$ is the best constant in (1.1).*

Proof. Let us prove Theorem 2.1 only for the operator T_θ^{+}. For the operator T_θ^{-} it can be proved similarly.

In the same way as in the proof of the sufficiency part of Theorem 2.1 of [5] we define a sequence of points $\{x_k\}\subset I$ such that

$$I=\bigcup_k[x_k,x_{k+1}),\quad [x_k,x_{k+1})\bigcap[x_i,x_{i+1})=\varnothing,\quad i\neq k, \tag{2.1}$$

$$(T_\theta^{+}f(x_k))^{\theta}\equiv\int_{0}^{x_k}w(t)\left(\int_{t}^{x_k}f(s)ds\right)^{\theta}dt=2^{\theta k}\ \text{ if }\ x_k<\infty, \tag{2.2}$$

$$2^{\theta k}\leq(T_\theta^{+}f(x))^{\theta}\equiv\int_{0}^{x}w(t)\left(\int_{t}^{x}f(s)ds\right)^{\theta}dt<2^{\theta(k+1)}\ \text{ if }\ x_k\leq x<x_{k+1}. \tag{2.3}$$

From (2.2) and (2.3) it follows

$$2^{k-1}\ll\left(\int_{x_{k-1}}^{x_k}w(t)\left(\int_{t}^{x_k}f(s)ds\right)^{\theta}dt\right)^{\frac{1}{\theta}}+\left(\int_{0}^{x_{k-1}}w(t)dt\right)^{\frac{1}{\theta}}\int_{x_{k-1}}^{x_k}f(s)ds. \tag{2.4}$$

Using (2.1), (2.2) and (2.4) as in [5] we have

$$L\equiv\int_{0}^{\infty}u(x)\left(T_\theta^{+}f(x)\right)^{q}dx=\sum_k\int_{x_k}^{x_{k+1}}u(x)\left(T_\theta^{+}f(x)\right)^{q}dx\leq 2^{2q}\sum_k 2^{q(k-1)}\int_{x_k}^{x_{k+1}}u(x)dx$$

$$\ll\sum_k\left(\int_{x_{k-1}}^{x_k}w(t)\left(\int_{t}^{x_k}f(s)ds\right)^{\theta}dt\right)^{\frac{q}{\theta}}\int_{x_k}^{x_{k+1}}u(x)dx$$

$$+\sum_k\left(\int_{0}^{x_{k-1}}w(t)dt\right)^{\frac{q}{\theta}}\left(\int_{x_{k-1}}^{x_k}f(s)ds\right)^{q}\int_{x_k}^{x_{k+1}}u(x)dx=L_1+L_2. \tag{2.5}$$

Let us estimate L_1 and L_2 separately.

To estimate L_1 first we use Hardy inequality, then we apply Hölder's inequality for sequences with the parameters $\frac{p}{q}$ and $\frac{p}{p-q}$ and get

$$L_1 \le \sum_k \int_{x_k}^{x_{k+1}} u(x)dx \left(H^+(x_{k-1},x_k)\right)^q \left(\int_{x_{k-1}}^{x_k} v(t)f^p(t)dt\right)^{\frac{q}{p}}$$

$$\le \left(\sum_k \left(\int_{x_k}^{x_{k+1}} u(x)dx\right)^{\frac{p}{p-q}} \left(H^+(x_{k-1},x_k)\right)^{\frac{qp}{p-q}}\right)^{\frac{p-q}{p}} \left(\sum_k \int_{x_{k-1}}^{x_k} v(t)f^p(t)dt\right)^{\frac{q}{p}}$$

$$\ll \left(\sum_k \int_{x_k}^{x_{k+1}} u(x) \left(\int_x^{x_{k+1}} u(s)ds\right)^{\frac{q}{p-q}} \left(H^+(x_{k-1},x_k)\right)^{\frac{qp}{p-q}} dx\right)^{\frac{p-q}{p}} \left(\int_0^\infty v(t)f^p(t)dt\right)^{\frac{q}{p}}$$

$$\ll \left(\sum_k \int_{x_k}^{x_{k+1}} u(x) \left(\int_x^{\infty} u(s)ds\right)^{\frac{q}{p-q}} \left(H^+(0,x)\right)^{\frac{qp}{p-q}} dx\right)^{\frac{p-q}{p}} \left(\int_0^\infty v(t)f^p(t)dt\right)^{\frac{q}{p}}$$

$$\le (E^+)^q \left(\int_0^\infty v(t)f^p(t)dt\right)^{\frac{q}{p}}. \tag{2.6}$$

Now, we estimate L_2 for each case of Lemma A separately.

Let $1 \le p \le \theta < \infty$. Twice using Hölder's inequality we get

$$L_2 \le \sum_k \left(\int_0^{x_{k-1}} w(t)dt\right)^{\frac{q}{\theta}} \left(\int_{x_{k-1}}^{x_k} v^{1-p'}(s)ds\right)^{\frac{q}{p'}} \int_{x_k}^{x_{k+1}} u(x)dx \left(\int_{x_{k-1}}^{x_k} v(t)f^p(t)dt\right)^{\frac{q}{p}}$$

$$\le \left(\sum_k \left(\int_{x_k}^{x_{k+1}} u(x)dx\right)^{\frac{p}{p-q}} \left(\left(\int_0^{x_{k-1}} w(t)dt\right)^{\frac{1}{\theta}} \left(\int_{x_{k-1}}^{x_k} v^{1-p'}(s)ds\right)^{\frac{1}{p'}}\right)^{\frac{qp}{p-q}}\right)^{\frac{p-q}{p}}$$

$$\times \left(\sum_k \int_{x_{k-1}}^{x_k} v(t)f^p(t)dt\right)^{\frac{q}{p}} \tag{2.7}$$

$$\ll \left(\sum_k \int_{x_k}^{x_{k+1}} u(x) \left(\int_x^{x_{k+1}} u(s)ds\right)^{\frac{q}{p-q}} dx\right.$$

$$\left.\times \left(\sup_{0<z<x_k} \left(\int_0^z w(t)dt\right)^{\frac{1}{\theta}} \left(\int_z^{x_k} v^{1-p'}(s)ds\right)^{\frac{1}{p'}}\right)^{\frac{qp}{p-q}}\right)^{\frac{p-q}{p}} \left(\int_0^\infty v(t)f^p(t)dt\right)^{\frac{q}{p}}$$

$$\leq \left(\sum_k \int_{x_k}^{x_{k+1}} u(x) \left(\int_x^\infty u(s)ds\right)^{\frac{q}{p-q}} \left(A^+(x)\right)^{\frac{qp}{p-q}} dx\right)^{\frac{p-q}{p}} \left(\int_0^\infty v(t)f^p(t)dt\right)^{\frac{q}{p}}$$

(due to (1.2))

$$\ll \left(\int_0^\infty u(x) \left(\int_x^\infty u(s)ds\right)^{\frac{q}{p-q}} \left(H^+(x)\right)^{\frac{qp}{p-q}} dx\right)^{\frac{p-q}{p}} \left(\int_0^\infty v(t)f^p(t)dt\right)^{\frac{q}{p}}$$

$$= (E^+)^q \left(\int_0^\infty v(t)f^p(t)dt\right)^{\frac{q}{p}}. \tag{2.8}$$

Now, let $0 < \theta < p$ and $1 < p < \infty$. Starting from (2.7) and using Lemma A(ii), we get

$$L_2 \leq \left(\sum_k \left(\int_{x_k}^{x_{k+1}} u(x)dx\right)^{\frac{p}{p-q}} \left(\int_0^{x_{k-1}} w(t)dt\right)^{\frac{qp}{\theta(p-q)}}\right.$$

$$\left.\times \left(\int_{x_{k-1}}^{x_k} v^{1-p'}(s)ds\right)^{\frac{q(p-1)}{p-q}}\right)^{\frac{p-q}{p}} \left(\int_0^\infty v(t)f^p(t)dt\right)^{\frac{q}{p}}$$

$$\ll \left(\sum_k \left(\int_{x_k}^{x_{k+1}} u(x)dx\right)^{\frac{p}{p-q}} \left(\int_0^{x_{k-1}} w(t) \left(\int_0^t w(s)ds\right)^{\frac{\theta}{p-\theta}} dt\right.\right.$$

$$\left.\left.\times \left(\int_{x_{k-1}}^{x_k} v^{1-p'}(s)ds\right)^{\frac{\theta(p-1)}{p-\theta}}\right)^{\frac{q(p-\theta)}{\theta(p-q)}}\right)^{\frac{p-q}{p}} \left(\int_0^\infty v(t)f^p(t)dt\right)^{\frac{q}{p}}$$

$$\leq \left(\sum_k \left(\int_{x_k}^{x_{k+1}} u(x)dx \right)^{\frac{p}{p-q}} \left(B^+(x_k)\right)^{\frac{pq}{p-q}} \right)^{\frac{p-q}{p}} \left(\int_0^\infty v(t) f^p(t) dt \right)^{\frac{q}{p}}$$

$$\ll \left(\sum_k \int_{x_k}^{x_{k+1}} u(x) \left(\int_x^\infty u(t)dt \right)^{\frac{q}{p-q}} \left(H^+(x)\right)^{\frac{pq}{p-q}} dx \right)^{\frac{p-q}{p}} \left(\int_0^\infty v(t) f^p(t) dt \right)^{\frac{q}{p}}$$

$$\leq (E^+)^q \left(\int_0^\infty v(t) f^p(t) dt \right)^{\frac{q}{p}}. \tag{2.9}$$

In the case $0 < \theta < 1 = p$ we have, following (2.5),

$$L_2 = \sum_k \left(\int_0^{x_{k-1}} w(t)dt \right)^{\frac{q}{\theta}} \left(\int_{x_{k-1}}^{x_k} \frac{1}{v(t)} v(t) f(t) dt \right)^q \int_{x_k}^{x_{k+1}} u(x)dx$$

$$\leq \sum_k \int_{x_k}^{x_{k+1}} u(x)dx \left(\int_0^{x_{k-1}} w(t)dt \right)^{\frac{q}{\theta}} \left(\underline{v}(x_{k-1}, x_k)\right)^{-q} \left(\int_{x_{k-1}}^{x_k} v(t) f(t) dt \right)^q$$

(since we have that $q < p = 1$, we use Hölder's inequality with the parameters $\frac{1}{q}$ and $\frac{1}{1-q}$)

$$\leq \left(\sum_k \left(\int_{x_k}^{x_{k+1}} u(x)dx \right)^{\frac{1}{1-q}} \left(\int_0^{x_{k-1}} w(t)dt \right)^{\frac{q}{\theta(1-q)}} \left(\underline{v}(x_{k-1}, x_k)\right)^{\frac{q}{q-1}} \right)^{1-q} \times \left(\sum_k \int_{x_{k-1}}^{x_k} v(t) f(t) dt \right)^q$$

$$= \left(\sum_k \left(\int_{x_k}^{x_{k+1}} u(x)dx \right)^{\frac{1}{1-q}} \left(\left(\int_0^{x_{k-1}} w(t)dt \right)^{\frac{1}{1-\theta}} \left(\underline{v}(x_{k-1}, x_k)\right)^{\frac{\theta}{\theta-1}} \right)^{\frac{q(1-\theta)}{\theta(1-q)}} \right)^{1-q} \times \left(\int_0^\infty v(t) f(t) dt \right)^q$$

$$\ll \left(\sum_k \left(\int\limits_{x_k}^{x_{k+1}} u(x)dx \right)^{\frac{1}{1-q}} \left(\int\limits_0^{x_{k-1}} w(t) \left(\int\limits_0^t w(s)ds \right)^{\frac{\theta}{1-\theta}} \right.\right.$$

$$\left.\left. \times \left(\underline{v}(t,x_k)\right)^{\frac{\theta}{\theta-1}} dt \right)^{\frac{q(1-\theta)}{\theta(1-q)}} \right)^{1-q} \left(\int\limits_0^\infty v(t)f(t)dt \right)^q$$

$$\ll \left(\sum_k \int\limits_{x_k}^{x_{k+1}} u(x) \left(\int\limits_x^{x_{k+1}} u(s)ds \right)^{\frac{q}{1-q}} \left(D^+(x)\right)^{\frac{q}{1-q}} dx \right)^{1-q} \left(\int\limits_0^\infty v(t)f(t)dt \right)^q$$

$$\ll \left(\int\limits_0^\infty u(x) \left(\int\limits_x^\infty u(s)ds \right)^{\frac{q}{1-q}} \left(H^+(x)\right)^{\frac{q}{1-q}} dx \right)^{1-q} \left(\int\limits_0^\infty v(t)f(t)dt \right)^q$$

$$= (E^+)^q \left(\int\limits_0^\infty v(t)f(t)dt \right)^{\frac{q}{p}}. \tag{2.10}$$

From (2.5), (2.6), (2.8), (2.9) and (2.10) it follows that (1.1) holds with the estimate $C^+ \ll E^+$ for the best constant $C^+ > 0$ in (1.1). The proof of Theorem 2.1 is complete. □

Let

$$F^+ = \left(\int\limits_0^\infty u(x) \left(\int\limits_x^\infty u(s)ds \right)^{\frac{q}{p-q}} \left(B^+(x)\right)^{\frac{pq}{p-q}} dx \right)^{\frac{p-q}{pq}},$$

$$F^- = \left(\int\limits_0^\infty u(x) \left(\int\limits_0^x u(s)ds \right)^{\frac{p}{p-q}} \left(B^-(x)\right)^{\frac{pq}{p-q}} dx \right)^{\frac{p-q}{pq}}.$$

Theorem 2.2. *Let $0 < \theta, q < \infty$, $\max\{\theta, q\} < p$ and $p > 1$. Then inequality (1.1) holds if and only if $E^\pm < \infty$. Moreover, $E^\pm \approx C^\pm$, where $C^\pm > 0$ is the best constant in (1.1).*

Proof. The sufficiency follows from Theorem 2.1.

We prove the necessity for the operator T_θ^+. For the operator T_θ^- it can be proved analogously. Suppose that inequality (1.1) holds for T_θ^+ with the best constant $C^+ > 0$. It suffices to prove that $F^+ \ll C^+$ since in the case $\max\{\theta, q\} < p$ and $p > 1$ we have that $F^\pm \approx E^\pm$, by Lemma A. We consider two cases $q \leq \theta$ and $q > \theta$.

First we consider the case $q \leq \theta$. Let $0 < y < z < \infty$. Due to local integrability of the functions w and $v^{1-p'}$ on I the following function

$$F(x) \equiv F_y(x) = \int_y^x w(t) \left(\int_y^t w(s)ds \right)^{\frac{\theta}{p-\theta}} \left(\int_t^x v^{1-p'}(s)ds \right)^{\frac{\theta(p-1)}{p-\theta}} dt$$

$$= \left(B^+(y,x) \right)^{\frac{p\theta}{p-\theta}}$$

is defined for all $x > y$.

The function $F(x)$ for any $\tau > y$ is absolutely continuous on the interval $[y, \tau]$. Therefore, its derivative

$$F'(x) = \frac{\theta(p-1)}{p-\theta} \int_y^x w(t) \left(\int_y^t w(s)ds \right)^{\frac{\theta}{p-\theta}} \left(\int_t^x v^{1-p'}(s)ds \right)^{\frac{p(\theta-1)}{p-\theta}} dt \; v^{1-p'}(x)$$

$$\equiv \frac{\theta(p-1)}{p-\theta} g(x) v^{1-p'}(x)$$

is integrable on the interval $[y, \tau]$ for any $\tau > y$. Here

$$g(x) = \int_y^x w(t) \left(\int_y^t w(s)ds \right)^{\frac{\theta}{p-\theta}} \left(\int_t^x v^{1-p'}(s)ds \right)^{\frac{p(\theta-1)}{p-\theta}} dt.$$

We introduce the function

$$f_{y,z}(t) = \chi_{(y,z)}(t) \left(\int_t^z u(x)dx \right)^{\frac{1}{p-q}} (F(t))^{\frac{q-\theta}{\theta(p-q)}} (g(t))^{\frac{1}{p}} v^{1-p'}(t),$$

where $\chi_{(y,z)}(\cdot)$ is the characteristic function of the interval (y, z). Then due to local integrability of the functions u, w, $v^{1-p'}$ and $gv^{1-p'}$ we have

$$\left(\int_0^\infty v(t) f_{y,z}^p(t)dt \right)^{\frac{1}{p}} = \left(\int_y^z \left(\int_t^z u(x)dx \right)^{\frac{p}{p-q}} (F(t))^{\frac{p(q-\theta)}{\theta(p-q)}} g(t) v^{1-p'}(t)dt \right)^{\frac{1}{p}} < \infty.$$

From the last expression by integration by parts we get

$$\left(\int_0^\infty v(t) f_{y,z}^p(t)dt \right)^{\frac{1}{p}} \approx \left(\int_y^z u(t) \left(\int_t^z u(x)dx \right)^{\frac{q}{p-q}} (F(t))^{\frac{q(p-\theta)}{\theta(p-q)}} dt \right)^{\frac{1}{p}}. \quad (2.11)$$

We estimate the left side of (1.1) for $f = f_{y,z}$ from below. For this purpose first we estimate the expression $T_\theta^+ f_{y,z}(x)$ for a fixed $x \in (y, z)$ from below. Using monotonicity of the functions $(F(t))^{\frac{q-\theta}{\theta(p-q)}}$ and $\left(\int_t^z u(x)dx \right)^{\frac{1}{p-q}}$ for $t \in (y, z)$ we have

$$\left(T_\theta^+ f_{y,z}(x)\right)^\theta = \int_y^x w(t)\left(\int_t^x f_{y,z}(s)ds\right)^\theta dt$$

$$= \int_y^x w(t)\left(\int_t^x \left(\int_s^z u(t)dt\right)^{\frac{1}{p-q}} (F(s))^{\frac{q-\theta}{\theta(p-q)}} g^{\frac{1}{p}}(s)v^{1-p'}(s)ds\right)^\theta dt$$

$$\geq \left(\int_x^z u(t)dt\right)^{\frac{\theta}{p-q}} (F(x))^{\frac{q-\theta}{p-q}} \int_y^x w(t)\left(\int_t^x g^{\frac{1}{p}}(s)v^{1-p'}(s)ds\right)^\theta dt. \tag{2.12}$$

We estimate the integral $\int_t^x g^{\frac{1}{p}}(s)v^{1-p'}(s)ds$ separately:

$$\int_t^x g^{\frac{1}{p}}(s)v^{1-p'}(s)ds$$

$$= \int_t^x \left(\int_y^s w(\varsigma)\left(\int_y^\varsigma w(\tau)d\tau\right)^{\frac{\theta}{p-\theta}} \left(\int_\varsigma^s v^{1-p'}(\tau)d\tau\right)^{\frac{p(\theta-1)}{p-\theta}} d\varsigma\right)^{\frac{1}{p}} v^{1-p'}(s)ds$$

$$\geq \int_t^x \left(\int_y^t w(\varsigma)\left(\int_y^\varsigma w(\tau)d\tau\right)^{\frac{\theta}{p-\theta}} \left(\int_t^s v^{1-p'}(\tau)d\tau\right)^{\frac{p(\theta-1)}{p-\theta}} d\varsigma\right)^{\frac{1}{p}} v^{1-p'}(s)ds$$

$$\approx \left(\int_y^t w(\tau)d\tau\right)^{\frac{1}{p-\theta}} \int_t^x \left(\int_t^s v^{1-p'}(\tau)d\tau\right)^{\frac{\theta-1}{p-\theta}} v^{1-p'}(s)ds$$

$$\approx \left(\int_y^t w(\tau)d\tau\right)^{\frac{1}{p-\theta}} \left(\int_t^x v^{1-p'}(\tau)d\tau\right)^{\frac{p-1}{p-\theta}}. \tag{2.13}$$

From (2.12) and (2.13) for $x \in (y,z)$ we have

$$T_\theta^+ f_{y,z}(x) \gg \left(\int_x^z u(t)dt\right)^{\frac{1}{p-q}} (F(x))^{\frac{q-\theta}{\theta(p-q)}}$$

$$\left(\int_y^x w(t)\left(\int_y^t w(\tau)d\tau\right)^{\frac{\theta}{p-\theta}} \left(\int_t^x v^{1-p'}(\tau)d\tau\right)^{\frac{\theta(p-1)}{p-\theta}} dt\right)^{\frac{1}{\theta}}$$

$$= \left(\int_x^z u(t)dt \right)^{\frac{1}{p-q}} (F(x))^{\frac{p-\theta}{\theta(p-q)}}.$$

Then

$$\left(\int_0^\infty u(x) \left(T_\theta^+ f_{y,z}(x)\right)^q dx \right)^{\frac{1}{q}} \geq \left(\int_y^z u(x) \left(T_\theta^+ f_{y,z}(x)\right)^q dx \right)^{\frac{1}{q}}$$

$$\gg \left(\int_y^z u(x) \left(\int_x^z u(t)dt \right)^{\frac{q}{p-q}} (F(x))^{\frac{q(p-\theta)}{\theta(p-q)}} dx \right)^{\frac{1}{q}}. \tag{2.14}$$

From (1.1), (2.11) and (2.14) we get

$$\left(\int_y^z u(x) \left(\int_x^z u(t)dt \right)^{\frac{q}{p-q}} (F(x))^{\frac{q(p-\theta)}{\theta(p-q)}} dx \right)^{\frac{p-q}{pq}} \ll C^+$$

for all $(y,z) \subset I$.

Proceeding to the limits $y \to 0$ and $z \to \infty$ and taking into account that $\lim\limits_{\substack{y\to 0\\ z\to\infty}} F_{y,z}(x) = (B^+(x))^{\frac{p\theta}{p-\theta}}$ we have

$$F^+ \ll C^+. \tag{2.15}$$

Thus, the proof of the necessity for the case $q \leq \theta$ is complete.

Now, let $q > \theta$. Then $\gamma = \frac{q}{\theta} > 1$. Let f and φ be non-negative functions such that $\int\limits_0^\infty v(t)f^p(t)dt < \infty$ and $\int\limits_0^\infty u^{1-\gamma'}(s)\varphi^{\gamma'}(s)ds < \infty$. Inequality (1.1) is rewritten in the form:

$$\left(\int_0^\infty u(x) \left(\int_0^x w(t) \left(\int_t^x f(s)ds \right)^\theta dt \right)^\gamma dx \right)^{\frac{1}{\gamma}} \leq (C^+)^\theta \left(\int_0^\infty v(t)f^p(t)dt \right)^{\frac{\theta}{p}}.$$

By Hölder's inequality, this implies

$$\int_0^\infty \varphi(x) \int_0^x w(t) \left(\int_t^x f(s)ds \right)^\theta dtdx$$

$$\leq (C^+)^\theta \left(\int_0^\infty u^{1-\gamma'}(s)\varphi^{\gamma'}(s)ds \right)^{\frac{1}{\gamma'}} \left(\int_0^\infty v(t)f^p(t)dt \right)^{\frac{\theta}{p}}.$$

Since f was chosen arbitrarily, we get

$$G \equiv \sup_{f\geq 0} \frac{\left(\int\limits_0^\infty \varphi(x) \int\limits_0^x w(t) \left(\int\limits_t^x f(s)ds\right)^\theta dtdx\right)^{\frac{1}{\theta}}}{\left(\int\limits_0^\infty v(t) f^p(t)dt\right)^{\frac{1}{p}}} \leq C^+ \left(\int\limits_0^\infty u^{1-\gamma'}(s)\varphi^{\gamma'}(s)ds\right)^{\frac{1}{\theta\gamma'}}. \tag{2.16}$$

For the fixed φ, the quantity G in (2.16) is equal to the least constant C^+ of inequality (1.1) in which $u(x) \equiv \varphi(x)$ and $q = \theta$.

Therefore, using the first part of the proof, we have

$$G \gg \left(\int\limits_0^\infty \varphi(x) \left(\int\limits_x^\infty \varphi(t)dt\right)^{\frac{\theta}{p-\theta}} \widetilde{F}(x)dx\right)^{\frac{p-\theta}{\theta p}},$$

where $\widetilde{F}(x) = (B^+(x))^{\frac{p\theta}{p-\theta}}$.

Integration by parts of the last expression gives

$$G \gg \left(\int\limits_0^\infty \left(\int\limits_x^\infty \varphi(t)dt\right)^{\frac{p}{p-\theta}} \widetilde{F}'(x)dx\right)^{\frac{p-\theta}{\theta p}}.$$

Then from (2.16) we have the following Hardy inequality:

$$\left(\int\limits_0^\infty \left(\int\limits_x^\infty \varphi(t)dt\right)^{\mu} \widetilde{F}'(x)dx\right)^{\frac{1}{\mu}} \ll (C^+)^\theta \left(\int\limits_0^\infty u^{1-\gamma'}(s)\varphi^{\gamma'}(s)ds\right)^{\frac{1}{\gamma'}}, \tag{2.17}$$

where $\mu = \frac{p}{p-\theta}$.

Since $\gamma' = \frac{q}{q-\theta}$, it holds $\gamma' > \mu$. Since φ was arbitrary, (2.17) holds for all φ such that $\int\limits_0^\infty u^{1-\gamma'}(s)\varphi^{\gamma'}(s)ds < \infty$. Hence, by Lemma A we have

$$\left(\int\limits_0^\infty \widetilde{F}'(x) \left(\int\limits_0^x \widetilde{F}'(t)dt\right)^{\frac{\mu}{\gamma'-\mu}} \left(\int\limits_x^\infty u(s)ds\right)^{\frac{\mu(\gamma'-1)}{\gamma'-\mu}} dx\right)^{\frac{\gamma'-\mu}{\mu\gamma'}} \ll (C^+)^\theta.$$

Integration by parts yields

$$\left(\int\limits_0^\infty u(x) \left(\int\limits_x^\infty u(s)ds\right)^{\frac{\gamma'(\mu-1)}{\gamma'-\mu}} \left(\widetilde{F}(x)\right)^{\frac{\gamma'}{\gamma'-\mu}} dx\right)^{\frac{\gamma'-\mu}{\theta\mu\gamma'}} \ll C^+.$$

Since $\frac{\gamma'(\mu-1)}{\gamma'-\mu} = \frac{q}{p-q}$, $\frac{\gamma'}{\gamma'-\mu} = \frac{q(p-\theta)}{\theta(p-q)}$, $\frac{\gamma'-\mu}{\theta\mu\gamma'} = \frac{p-q}{pq}$ and $\left(\widetilde{F}(x)\right)^{\frac{p-\theta}{p\theta}} = B^+(x)$, we have

$$F^+ \ll C^+. \tag{2.18}$$

Relations (2.15) and (2.18), together with the relation $C^+ \ll E^+$ obtained in Theorem 2.1, give $E^+ \approx C^+$. The proof of Theorem 2.2 is complete. □

Let

$$F_1^+ = \left(\int_0^\infty u(x)\left(\int_x^\infty u(s)ds\right)^{\frac{q}{1-q}}\left(\int_0^x w(s)ds\right)^{\frac{q}{\theta(1-q)}} dx\right)^{\frac{1-q}{q}},$$

$$F_1^- = \left(\int_0^\infty u(x)\left(\int_0^x u(s)ds\right)^{\frac{q}{1-q}}\left(\int_x^\infty w(s)ds\right)^{\frac{q}{\theta(1-q)}} dx\right)^{\frac{1-q}{q}}.$$

Theorem 2.3. *Let* $\max\{\theta, q\} < 1 = p$ *and* $v(x) \equiv 1$. *Then inequality* (1.1) *holds if and only if* $E^\pm < \infty$. *Moreover,* $E^\pm \approx C^\pm$, *where* $C^\pm > 0$ *is the best constant in* (1.1).

Proof. The sufficiency follows from Theorem 2.1.

Let us prove the necessity only for T_θ^+ since for T_θ^- it can be proved similarly. Suppose that inequality (1.1) holds for T_θ^+ with the best constant $C^+ > 0$. Since $v(x) \equiv 1$, we have $D^+(x) \approx \left(\int_0^x w(s)ds\right)^{\frac{1}{\theta}}$. Here and below the equivalence constants do not depend on $x \in I$. Due to the relations $D^+(x) \approx H^+(x)$ the values F_1^+ are equivalent to the values E^+, respectively. Therefore, it suffices to prove the estimates $F_1^+ \ll C^+$.

Let $0 < y < z < \infty$. Assume

$$f_{y,z}(t) = \chi_{(y,z)}(t)\left(\int_t^z u(s)ds\right)^{\frac{1}{1-q}}\left(\int_y^t w(s)ds\right)^{\frac{q}{\theta(1-q)}-1} w(t).$$

Then

$$\int_0^\infty f_{y,z}(t)dt = \int_y^z \left(\int_t^z u(s)ds\right)^{\frac{1}{1-q}}\left(\int_y^t w(s)ds\right)^{\frac{q}{\theta(1-q)}-1} w(t)dt$$

$$\approx \int_y^z u(t)\left(\int_t^z u(s)ds\right)^{\frac{q}{1-q}}\left(\int_y^t w(s)ds\right)^{\frac{q}{\theta(1-q)}} dt. \tag{2.19}$$

Now, we estimate the left side of (1.1) for $f = f_{y,z}$ from below. Let the function $\sigma(x) \equiv \sigma_y(x)$ be such that $\sigma(x) < x$ and

$$\int_y^x w(t)dt = 2\int_y^{\sigma(x)} w(t)dt \quad \text{for all} \quad x \in (y, \infty).$$

Then

$$\left(\int_y^x w(t)\left(\left(\int_y^x w(s)ds\right)^{\frac{q}{\theta(1-q)}} - \left(\int_y^t w(s)ds\right)^{\frac{q}{\theta(1-q)}}\right)^{\theta} dt\right)^{\frac{q}{\theta}}$$

$$\geq \left(\int_y^{\sigma(x)} w(t)\left(\left(\int_y^x w(s)ds\right)^{\frac{q}{\theta(1-q)}} - \left(\int_y^t w(s)ds\right)^{\frac{q}{\theta(1-q)}}\right)^{\theta} dt\right)^{\frac{q}{\theta}}$$

$$\geq \left(\int_y^{\sigma(x)} w(t)dt\right)^{\frac{q}{\theta}} \left(\left(\int_y^x w(s)ds\right)^{\frac{q}{\theta(1-q)}} - \left(\int_y^{\sigma(x)} w(s)ds\right)^{\frac{q}{\theta(1-q)}}\right)^{q}$$

$$= \left(\frac{1}{2}\right)^{\frac{q}{\theta}} \left(1 - \left(\frac{1}{2}\right)^{\frac{q}{\theta(1-q)}}\right) \left(\int_y^x w(s)ds\right)^{\frac{q}{\theta(1-q)}}. \tag{2.20}$$

Using estimate (2.20) for $x \in (y, z)$ we get

$$\left(T_\theta^+ f_{y,z}(x)\right)^q = \left(\int_y^x w(t)\left(\int_t^x f_{y,z}(s)ds\right)^{\theta} dt\right)^{\frac{q}{\theta}}$$

$$\geq \left(\int_x^z u(s)ds\right)^{\frac{q}{1-q}} \left(\int_y^x w(t)\left(\int_t^x \left(\int_y^s w(\tau)d\tau\right)^{\frac{q}{\theta(1-q)}-1} w(s)ds\right)^{\theta} dt\right)^{\frac{q}{\theta}}$$

$$\gg \left(\int_x^z u(s)ds\right)^{\frac{q}{1-q}} \left(\int_y^x w(t)dt\right)^{\frac{q}{\theta(1-q)}}. \tag{2.21}$$

Then

$$\int_0^\infty u(x)\left(T_\theta^+ f_{y,z}(x)\right)^q dx \geq \int_y^z u(x)\left(T_\theta^+ f_{y,z}(x)\right)^q dx$$

$$\gg \int_y^z u(x)\left(\int_x^z u(s)ds\right)^{\frac{q}{1-q}} \left(\int_y^x w(t)dt\right)^{\frac{q}{\theta(1-q)}} dx. \tag{2.22}$$

From (1.1), (2.19) and (2.22) we have

$$\left(\int_y^z u(x)\left(\int_x^z u(s)ds\right)^{\frac{q}{1-q}}\left(\int_y^x w(t)dt\right)^{\frac{q}{\theta(1-q)}}dx\right)^{\frac{1-q}{q}} \ll C^+$$

for all $(y,z)\subset I$.

Taking the limits $y\to 0$ and $z\to\infty$ we get the estimate $F_1^+ \ll C^+$ which, together with the estimate $E^+ \ll C^+$ from the sufficiency part, gives $E^+ \approx C^+$. The proof of Theorem 2.3 is complete. □

3. Applications

In the paper [3] the following inequalities

$$\|\varphi\widetilde{H}_n f\|_{LM_{\theta q,\tau}} \le C\|f\|_{L_{p,V}} \tag{3.1}$$

and

$$\|\varphi H_n f\|_{{}^{c}LM_{\theta q,\tau}} \le C\|f\|_{L_{p,V}} \tag{3.2}$$

were studied, where $LM_{\theta q,\tau}$ is the local Morrey-type space with the norm

$$\|f\|_{LM_{\theta q,\tau}} = \|\tau(r)\|f\|_{L_\theta(B_r)}\|_{L_q(0,\infty)},$$

and ${}^{C}LM_{\theta q,\tau}$ is the complementary local Morrey-type space with the norm

$$\|f\|_{{}^{c}LM_{\theta q,\tau}} = \|\tau(r)\|f\|_{L_\theta(CB_r)}\|_{L_q(0,\infty)},$$

B_r is the open ball in R^n centered at 0 with radius r and CB_r is the complement of the ball B_r in R^n,

$$H_n f(x) = \int_{B_{|x|}} f(s)ds \quad\text{and}\quad \widetilde{H}_n f(x) = \int_{CB_{|x|}} f(s)ds$$

are multidimensional Hardy operators.

In [3] assuming that $\varphi(x)\equiv\varphi(|x|)$ and $V(x)\equiv V(|x|)$ it was proved that the validity of inequalities (3.1) and (3.2) are equivalent to the validity of the inequalities

$$\left(\int_0^\infty u(x)\left(\int_0^x w(t)\left(\int_t^\infty f(s)ds\right)^\theta dt\right)^{\frac{q}{\theta}}dx\right)^{\frac{1}{q}} \le C\left(\int_0^\infty v(t)f^p(t)dt\right)^{\frac{1}{p}} \tag{3.3}$$

and

$$\left(\int_0^\infty u(x)\left(\int_x^\infty w(t)\left(\int_0^t f(s)ds\right)^\theta dt\right)^{\frac{q}{\theta}}dx\right)^{\frac{1}{q}} \le C\left(\int_0^\infty v(t)f^p(t)dt\right)^{\frac{1}{p}}, \tag{3.4}$$

respectively, where $u(x)=\tau^q(x)$, $w(t)=\varphi(t)t^{\frac{n-1}{\theta}}$ and $v(t)=V(t)t^{-\frac{n-1}{p'}}$.

In the papers [1], [2] and [6] by different approaches necessary and sufficient conditions for the validity of inequalities (3.3) and (3.4) are obtained for different relations between the parameters $0 < p, q, \theta \leq \infty$. Moreover, in [6] other inequalities of the type (3.3) and (3.4) are considered. In [3] characterizations of (3.3) and (3.4) are found only for the case $1 \leq p \leq q < \infty$ and $0 < \theta < \infty$ but by a method different from those in [1], [2] and [6].

Investigation of inequality (1.1) gives this alternative method to characterize inequality (3.3) since the validity of inequality (3.3) is equivalent to the validity of inequality (1.1) for T_θ^+ and the Hardy inequality

$$\left(\int_0^\infty u(x)\left(\int_0^x w(t)dt\right)^{\frac{q}{\theta}}\left(\int_x^\infty f(s)ds\right)^q dx\right)^{\frac{1}{q}} \leq C_1\left(\int_0^\infty v(t)f^p(t)dt\right)^{\frac{1}{p}}.$$

The similar splitting can be done for inequality (3.4).

Therefore, for example, from Theorem 2.2 and Lemma A we have

Theorem 3.1. *Let $0 < \theta, q < \infty$, $p > 1$ and $\max\{\theta, q\} < p$. Let $\varphi(x) = \varphi(|x|)$, $V(x) = V(|x|)$, $u(x) = \tau^q(x)$, $w(t) = \varphi(t)t^{\frac{n-1}{\theta}}$ and $v(t) = V(t)t^{-\frac{n-1}{p'}}$. Then inequality (3.1) ((3.3)) holds if and only if $E^+ < \infty$ and*

$$G^+ = \left(\int_0^\infty u(x)\left(\int_0^x w\right)^{\frac{q}{\theta}}\left(\int_0^x u(t)\left(\int_0^t w\right)^{\frac{q}{\theta}}dt\right)^{\frac{q}{p-q}}\right.$$
$$\left.\times\left(\int_x^\infty v^{1-p'}\right)^{\frac{q(p-1)}{p-q}}dx\right)^{\frac{p-q}{pq}} < \infty.$$

Moreover, $\max\{E^+, G^+\} \approx C$, where $C > 0$ is the best constant in (3.1) ((3.3)).

The similar statement follows from Theorem 2.2 and Lemma A for inequality (3.2) ((3.4)).

The characterizations of inequality (3.3) in Theorem 3.1 are respectively equivalent to those obtained earlier in [1](Theorem 3.1, (iv)) and in [6](Theorem 5, $\max\{\theta, q\} < p$).

Let us also note that inequalities of the type (3.3) and (3.4) with kernels are considered in [7].

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