Characterizations of the topology of uniform convergence on order-intervals

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1. Introduction

Nakano's theorem [6], which states that a boundedly and locally order complete topological vector lattice $(E, C, \mathscr{T})$ is complete, is one of the deepest results in the theory of topological vector lattices. It is known from [16, (13.9)] that the converse of Nakano's theorem holds for the topology $o(E, E')$ of uniform convergence on all order-intervals in E'. Therefore it is interesting to find some characterizations for the topology $\mathscr{T}$ to be $o(E, E')$ where $(E, C, \mathscr{T})$ is only assumed to be a locally solid space. The purpose of this paper is to give such characterizations in terms of some special continuous linear mappings.

Definitions and some remarkable properties of ordered sequence vector spaces, which we shall need in what follows, are explained in section 2.

Cone-absolutely summing mappings were first considered by Schaefer [12] and Schlotterbeck in the Banach lattice case, and were extended by Walsh [13] to the case of locally solid spaces. Using a characterization of cone-absolutely summing mappings defined on a locally solid space $(E, C, \mathscr{T})$, we obtain a necessary and sufficient condition for $\mathscr{T}$ to be $o(E, E')$. A connection between the nuclearity and the topology $\sigma(E, E')$ is given by Theorem 3.7.

In section 4 we define L-prenuclear seminorms in terms of the notion of cone-absolutely summing mappings, and then it is shown that $\mathscr{T}=o(E, E')$ if and only if each continuous seminorm is L-prenuclear. On the other hand, it is known from Schaefer [12, p. 178] that the notion of prenuclear seminorm is useful for the investigation of nuclearity. A connection between prenuclear seminorms and absolutely summing mappings is given in this section.

L-nuclear mappings are defined by means of L-prenuclear seminorms. Another characterization of $\mathscr{T}$ to be $o(E, E')$ is given in terms of L-nuclear mappings. In particular, the identity map is L-nuclear if and only if $\mathscr{T}=o(E, E')$ and E' has an order unit. It is amusement to compare this result with the Dvoretzky and Rogers theorem. Similarly prenuclear linear mappings are defined in terms of prenuclear seminorms. A characterization of nuclearity is given by means of prenuclear linear mappings. An interesting property of prenuclear linear mappings is that the identity map on X (a locally convex space) is prenuclear if and only if X is normable and finite dimensional. As a consequence of this result, we obtain the theorem of Dvoretzky and Rogers.

Section 6 is devoted to a studying of lattice properties of L-nuclear mappings and of cone-absolutely summing mappings, of course the domain and range spaces are assumed to be locally convex Riesz sapces.

In the final section, we are studying the factorization of continuous linear mappings.

Throughout this paper, $(E, C, \mathscr{T})$ and $(F, K, \mathscr{P})$ are always assumed to be ordered convex spaces, while X and Y are locally convex spaces. E^* denotes the algebraic dual, E' denoes the topological dual of $(E, \mathscr{T})$, C^* is the set of all positive linear functionals on E, while C' is the dual cone of C, that is $C'=C^* \cap E'$. Also $L(X, Y)$ denotes the vector space of all linear mappings of X into Y, the subspace consisting of continuous linear mappings is $\mathscr{L}(X, Y)$.

Terminology and notation concerning ordered vector spaces will follow [16], while [11] will serve as our reference for material on topological vector spaces, the background material concerning absolutely summing mappings can be found in [9].

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2. Prelimsnary results of ordered vector spaces

Let (E, C) be an ordered vector space with a positive cone C, and let p be a seminorm on E. p is said to be *strongly monotone* if

$$y \leqslant x \leqslant z \Rightarrow p(x) \leqslant \max \{p(y), p(z)\};$$

and p is called a *Riesz seminorm* if it satisfies the following conditions:

(i) $-u \leqslant x \leqslant u \Rightarrow p(x) \leqslant p(u)$;

(ii) for any $x \in E$ and $\varepsilon > 0$ there exists a $u \in C$ with $-u \leqslant x \leqslant u$ such that $p(u) < p(x) + \varepsilon$.

It is easy to see that our definition of Riesz seminorms coincides with the usual definition in the Riesz space (i.e. vector lattice) case.

Let V be a subset of E. Define

$$F(V) = (V + C) \cap (V - C);$$

$$S(V) = \cup\{[-u, u] : u \in V \cap C\};$$

$$D(V) = \{x \in V : x = \lambda x_1 - (1-\lambda) x_2,\ \lambda \in [0, 1],\ x_1, x_2 \in V \cap C\}.$$

V is said to be *order-convex* (resp. *solid, decomposable*) if $V = F(V)$ (resp. $V = S(V)$, $V = D(V)$). It is clear that a seminorm p is strongly monotone (resp. a Riesz seminorm) if and only if its open unit ball is order-convex (resp. solid). A topology $\mathscr{T}$ on E is said to be *locally solid* (resp. *locally decomposable*) if it admits a neighbourhood base at o consisting of convex and solid (resp. convex and decomposable) sets; and $(E, C, \mathscr{T})$ is called a *locally solid* (resp. *locally decomposable*) *space* if $\mathscr{T}$ is locally solid (resp. locally decomposable). It should be noted that for a locally solid space $(E, C, \mathscr{T})$, (E, C) need not be a vector lattice; but locally solid spaces share a number of important properties with locally convex Riesz spaces (i.e., locally convex vector lattices) (see [13] and [16]).

Let $(E, C, \mathscr{T})$ be a locally o-convex space (i.e., C is a normal cone in $(E, \mathscr{T})$), $E = C - C$ and suppose that $\mathscr{U}$ is a neighbourhood base at o consisting of o-convex and circled sets. Define

$$S(\mathscr{U}) = \{S(V) : V \in \mathscr{U}\};$$

$$D(\mathscr{U}) = \{D(V) : V \in \mathscr{U}\}.$$

The topology determined by $S(\mathscr{U})$ (resp. $D(\mathscr{U})$), denoted by $\mathscr{T}_S$ (resp. $\mathscr{T}_D$), is called the *locally solid* (resp. *locally decomposable*) *topology on E associated with* $\mathscr{T}$. $\mathscr{T}_S$ (resp. $\mathscr{T}_D$) is the smallest locally solid (resp. locally decomposable) topology on E which is finer than $\mathscr{T}$. Since $\mathscr{T}$ is locally o-convex, it follows that $\mathscr{T}_S = \mathscr{T}_D$. Let V be in $\mathscr{U}$ and p_V the gauge of V. We define, for each $x \in E$, that

$$p_{V,D}(x) = \inf\{p_V(x_1) + p_V(x_2) : x_1, x_2 \in C \text{ with } x = x_1 - x_2\};$$

$$p_{V,S}(x) = \inf\{p_V(u) : u \in C \text{ with } -u \leqslant x \leqslant u\}.$$

Then

$$p_{V,D}(u) = p_{V,S}(u) = p_V(u) \text{ for all } u \in C,$$

and $p_{V,D}$ (resp. $p_{V,S}$) is the gauge of $D(V)$ (resp. $S(V)$); therefore $\mathscr{T}_S$ is determined by $\{p_{V,S} : V \in \mathscr{U}\}$, or by $\{p_{V,D} : V \in \mathscr{U}\}$ (see Walsh [13]). According to the above remark, each $p_{V,S}$ is a Riesz seminorm. For further information about the locally solid spaces and the topologies $\mathscr{T}_S$ and $\mathscr{T}_D$, we refer the reader to [16] or [13].

Suppose now that $(E, C, \mathscr{P})$ is an ordered convex space whose dual is E', and that $E=C-C$ and $E'=C'-C'$, where C' is the dual cone of C. Then, by a result of Namioka (see [16]), $\sigma(E, E')$ is a locally o-convex topology and is determined by the family $\{p_f : f \in C'\}$ of monotone seminorms, where each p_f is defined by

$$p_f(x) = \left| f(x) \right| \qquad (x \in E).$$

By the above remark, $\sigma_S(E, E')$ is determined by the family $\{p_{f,S} : f \in C'\}$ as well as by $\{p_{f,D} : f \in C'\}$ of seminorms, where $p_{f,S}$ and $p_{f,D}$ are given by

$$p_{f,S}(x) = \inf \left\{ f(u) : u \in C,\ x \in [-u, u] \right\}$$

$$p_{f,D}(x) = \inf \left\{ f(u+w) : u,\ w \in C,\ x = u - w \right\} \qquad (x \in E).$$

On the other hand, for any $f \in C'$, let

$$q_f(x) = \sup \left\{ g(x) : g \in E',\ -f \leqslant g \leqslant f \right\} \qquad (x \in E).$$

Then the family $\{q_f : f \in C'\}$ of seminorms determines the topology $o(E, E')$ of uniform convergence on all order-intervals in E' (see Peressini [7]).

A trivial modification of [14] yields the following more general result, but for completeness we shall give the entire proof.

LEMMA 2.1 *Let $(E, C, \mathscr{T})$ be an ordered convex space such that $E=C-C$ and $E'=C'-C'$. For each $f \in C'$, let q_f, $p_{f,S}$ be as defined above, and let*

$$W_f = \left\{ x \in E : p_f(x) = \left| f(x) \right| \leqslant 1 \right\} \text{ and } V_f = \left\{ x \in E : q_f(x) \leqslant 1 \right\}.$$

Then the following assertion holds:

(a) *$\overline{D(W_f)} = \overline{S(V_f)} = V_f$, hence $q_f \leqslant p_{f,S}$ and $o(E, E') \leqslant \sigma_S(E, E')$;*

(b) *if, in addition, E' is order-convex in (E^*, C^*), then $p_{f,S} = p_{s,D} = q_f$ and thus $\sigma_S(E, E') = o(E, E')$.*

Proof. (a) We first note that $D(W_f) \subset S(W_f)$ and that $p_{f,S}$ and $p_{f,D}$ are the gauges of $S(W_f)$ and $D(W_f)$ respectively. Also it is not hard to see that $q_f \leqslant p_{f,S}$; consequently $S(W_f) \subset V_f$. As V_f is the polar of $\{g \in E' : -f \leqslant g \leqslant f\} = [-f, f]$, V_f is $\mathscr{T}$-closed, and thus $\overline{D(W_f)} \subseteq \overline{S(W_f)} \subseteq V_f$.

In order to verify that $V_f \subseteq \overline{D(W_f)}$, it is sufficient to show, by the bipolar theorem, that $(D(W_f))^o \subseteq V_f^o = [-f, f]$, where $(D(W_f))^o$ is the polar of $D(W_f)$, taken in E' In fact, if $g \in (D(W_f))^o$, then $|g(x)| \leqslant p_{f,D}(x)$ for all $x \in E$; in particular,

$$\left|g(u)\right| \leqslant p_{f,D}(u) = f(u) \qquad \text{for all } u \in C$$

which implies that $g \in [-f, f] = V_f^o$. Therefore we have that $V_f = \overline{D(W_f)} = \overline{S(W_f)}$.

(b) If E' is order-convex in (E^*, C^*), then $\sigma_S(E, E')$ is consistent with the dual pair $\langle E, E' \rangle$, and thus $\overline{S(W_f)}$ is the $\sigma_S(E, E')$-closure of $S(W_f)$. As $S(W_f)$ is a convex, circled $\sigma_S(E, E')$-neighbourhood of o, $p_{f,S}$ is the gauge of $\overline{S(W_f)}$. We conclude from $V_f = \overline{S(W_f)}$ that $p_{f,S} = q_f$. Similarly there is $p_{f,D} = q_f$. Therefore, $\sigma_S(E, E') = o(E, E')$.

Let (E, C) be a Riesc space (i.e., vector lattice) and let $\mathscr{T}$ be a locally convex topology on E such that the lattice operations are $\mathscr{T}$-continuous. Then it is easily seen that $\mathscr{T}$ is a locally decomposable topology. Therefore the following two results are generalizations of Peressini [7, (2.8), (2.10), (2.13) and (2.14) of Chap. 3].

Corollary 2.2 *Let (E, C) and (G, K) be ordered vector spaces which form a dual pair, let $K = -C^o$ and suppose that $E = C - C$ and $G = K - K$. Then the following assertions hold*:

(a) *If $\mathscr{P}$ is a locally decomposable topology on E finer than $\sigma(E, G)$, then $\sigma_S(E, G) \leqslant \mathscr{P}$ and hence $o(E, G) \leqslant \mathscr{P}$.*

(b) *If there is an order-interval in G that is not $\sigma(G, E)$-compact, then there does not exist a locally decomposable topology on E which is consistent with $\langle E, G \rangle$.*

(c) *If $\sigma(E, G)$ is locally decomposable, then each order-interval in (G, K) is contained in a finite dimensional subspace of G; if, in addition, G contains an order unit, then G and E are finite dimensional.*

Proof. (a) Since $\sigma_S(E, G)$ is the smallest locally decomposable topology on E finer than $\sigma(E, G)$, it follows that $\sigma_S(E, G) \leqslant \mathscr{P}$, and hence from Lemma 2,1 (a) that $o(E, G) \leqslant \mathscr{P}$,

(b) If there exists a locally decomposable topology $\mathscr{P}$ on E consistent with $\langle E, G \rangle$, then $o(E, G) \leqslant \mathscr{P}$ and thus each order-interval in G is $\sigma(G, E)$-compact.

(c) If $\sigma(E, G)$ is locally decomposable, then $\sigma(E, G) = o(E, G) = \sigma_S(E, G)$ by Lemma 2.1, hence each order-interval in (G, K) is $\sigma(E, G)$-equicontinuous and surely must be contained in a finite dimensional subspace of G.

The proof of the following corollary is similar to that given in Peressini [7, (2.14) p. 134], and hence will be omitted.

Corollary 2.3. *Let (E, C) and (G, K) be as in the preceding corollary and let $\sigma(E, G)$ be locally decomposable. If there exists a metrizable vector* *topology $\mathscr{L}$ on G such that K is $\mathscr{L}$-complete, then E is finite dimensional.*

Let $(E, C, \mathscr{T})$ be an ordered convex space. For a non-empty set A, denote by E^A the algebraic product of E ordered by the product cone C^A, and by $E^{(A)}$ the algebraic direct sum of E ordered by the relative cone $C^{(A)} = E^{(A)} \cap C^A$. Elements in E^A will be written as families (x_i, A). Let $\mathscr{T}^A$ be the product topology and $\mathscr{T}^{(A)}$ the locally convex direct sum topology. It is known that if $\mathscr{T}$ is locally o-convex (resp. locally solid, locally decomposable) then so are $\mathscr{T}^A$ and $\mathscr{T}^{(A)}$. Further, if C is generating then so is C^A, therefore $E^{(A)}$ is a solid subspace of (E^A, C^A).

$\mathscr{F}(A)$ denotes the directed set consisting of all finite subsets of A ordered by the set theoretic inclusion $\supseteq$. Following Pietsch [9], an element (x_i, A) in E^A is said to be *summable* if the net $\left\{\sum_{i\in\alpha} x_i : \alpha\in\mathscr{F}(A)\right\}$ is Cauchy; and (x_i, A) is said to be *absolutely summable* if for any continuous seminorm p on E, $(p(x_i), A)$ is a summable family in $I\!R$. The set consisting of all summable families (resp. absolutely summable families) in E, denoted by $l^1(A, E)$ (resp. $l^1[A, E]$), is a vector subspace of E^A. It should be noted that our terminology for a summable (resp. absolutely summable) family (x_i, A) differs slightly from that of Schaefer [11] in that we require $\left\{\sum_{i\in\alpha} x_i : \alpha\in\mathscr{F}(A)\right\}$ to be convergent $\left(\text{resp. } \left\{\sum_{i\in\alpha} x_i : \alpha\in\mathscr{F}(A)\right\} \text{ to be convergent}\right.$ and $(p(x_i), A)$ to be summable for any continuous seminorm $\left.p\right)$. Let p be a continuous seminorm on, E we define

$$\varepsilon_p(x_i, A) = \sup\left\{\textstyle\sum_A \left|\langle x_i, f\rangle\right| : f\in V_p^\circ\right\} \qquad \left((x_i, A)\in l^1(A, E)\right)$$

$$\pi_p(x_i, A) = \textstyle\sum_A p(x_i) \qquad \left((x_i, A)\in l^1[A, E]\right)$$

where $V_p = \{x\in E : p(x)\leqslant 1\}$. Then ε_p and π_p are seminorms on $l^1(A, E)$ and $l^1[A, E]$ resp., and

$$\varepsilon_p(x_i, A) \leqslant \pi_p(x_i, A) \text{ for all } (x_i, A)\in l^1[A, E].$$

Let $\mathscr{L}_\varepsilon$ (resp. $\mathscr{L}_\pi$) denote the topology on $l^1(A, E)$ (resp. on $l^1[A, E]$) generated by $\{\varepsilon_p : p\in P\}$ (resp. by $\{\pi_p : p\in P\}$), where P is a family consisting of continuous seminorms which generates $\mathscr{T}$. Also we define

$$C_\varepsilon(A, E) = C^A \cap l^1(A, E) \text{ and } C_\pi(A, E) = C^A \cap l^1[A, E].$$

Then $(l^1(A, E), C_\varepsilon(A, E), \mathscr{L}_\varepsilon)$ and $(l^1[A, E], C_\pi(A, E), \mathscr{L}_\pi)$ are ordered convex spaces. If $\mathscr{T}$ is a locally o-convex topology then so is $\mathscr{L}_\pi$. Further, we have:

LEMMA 2.4. *Let $(E, C, \mathscr{T})$ be an ordered convex space, V an o-convex, circled $\mathscr{T}$-neighbourhood of o and suppose that p is the gauge of V. Then, for any $(u_i, A)\in C_\epsilon(A, E)$, we have*

$$\varepsilon_p(u_i, A) = \sup\left\{\sum_A \langle u_i, x'\rangle : x'\in V^o\cap C'\right\}$$
$$= \sup\left\{p\left(\sum_{i\in\alpha} u_i\right) : \alpha\in\mathscr{F}(A)\right\},$$

where V^o is the polar of V taken in E'. Consequently, if $\mathscr{T}$ is locally o-convex then so is $\mathscr{L}_\epsilon$.

Proof. It is trivial that

$$\sup\left\{\sum_A\langle u_i, x'\rangle : x'\in V^o\cap C'\right\}\leqslant \varepsilon_p(u_i, A).$$

Since V is o-convex, it follows from [16, Theorem (2.11)] that V^o is decomposable, and hence they must be equal.

It is also clear that

$$\varepsilon_p(u_i, A)\leqslant \sup\left\{p\left(\sum_{i\in\alpha} u_i\right) : \alpha\in\mathscr{F}(A)\right\}. \tag{2.1}$$

On the other hand, for any $\alpha\in\mathscr{F}(A)$, the $\sigma(E', E)$-compactness of $V^o\cap C'$ insures that there exists $g_\alpha\in V^o\cap C'$ such that

$$p\left(\sum_{i\in\alpha} u_i\right) = \left\langle \sum_{i\in\alpha} u_i, g_\alpha\right\rangle.$$

Since $u_i\in C$ and $g_\alpha\in V^o\cap C'$, it follows that

$$\left\langle \sum_{i\in\alpha} u_i, g_\alpha\right\rangle \leqslant \sum_A\langle u_i, g_\alpha\rangle\leqslant \varepsilon_p(u_i, A).$$

As α was arbitrary, we conclude that

$$\sup\left\{p\left(\sum_{i\in\alpha} u_i\right) : \alpha\in\mathscr{F}(A)\right\}\leqslant \varepsilon_p(u_i, A).$$

Combining this with (2.1), we get the required equality.

Let $(E, C, \mathscr{T})$ be a locally o-convex space and let $\mathscr{T}$ be determined by a family P of strongly monotone seminorms. Define

$$l^1\langle A, E\rangle = C_\epsilon(A, E) - C_\epsilon(A, E)$$

and denote by $\mathscr{L}_{\epsilon,D}$ the locally decomposable topology on $l^1\langle A, E\rangle$ associated with the relative topology on $l^1\langle A, E\rangle$ induced by $\mathscr{L}_\epsilon$. Then $(l^1\langle A, E\rangle, C_\epsilon(A, E), \mathscr{L}_{\epsilon,D})$ is a locally solid space, and $\mathscr{L}_{\epsilon,D}$ is generated by the family $\{\varepsilon_{p,D} : p\in P\}$ of seminorms. For simplicity of notation, we write $p_1 = \varepsilon_{p,D}$, that is

$$p_1(x_i, A) = \inf\Big\{\varepsilon_p(u_i, A) + \varepsilon_p(w_i, A) : (u_i, A),\ (w_i, A) \in C_\iota(A, E)$$
$$\text{with } (u_i, A) - (w_i, A) = (x_i, A)\Big\}.$$

According to Lemma 2.4, the locally solid space $(l^1\langle A, E\rangle,\ C_\iota(A, E),\ \mathscr{L}_{\iota,D})$ coincides with $a_0(A, E)$, defined by Walsh [13, (2.3.3) and (2.3.10)]. (Here $C_\iota(A, E)$ is not necessarily closed.)

It is clear that if $(E, C, \mathscr{T})$ is a locally convex Riesz space then $l^1[N, E]$ is a solid subspace of (E^N, C^N) and $(l^1[N, E],\ C_\pi(N, E),\ \mathscr{L}_\pi)$ is a locally convex Riesz space, where N is the set of oll natural numbers. For locally solid spaces, we have the following result, but the proof is straighforward and hence will be omitted.

LEMMA 2.5. *If $(E, C, \mathscr{T})$ is a metrizable locally solid space then so is $(l^1[N, E],\ C_\pi(N, E),\ \mathscr{L}_\pi)$, where N stands for the set of all natural numbers. Consequently $l^1[N, E]$ is a solid subspace of E^A.*

Define

$$m_\infty^{+}(A, E) = \Big\{(x_i, A) \in C^A : \exists u \in C \text{ with } x_i \leqslant u \text{ for all } i \in A\Big\}$$

$$m_\infty(A, E) = m_\infty^{+}(A, E) - m_\infty^{+}(A, E).$$

Then we have:

THEOREM 2.6. (Walsh [13, (2.4.9)]. *Let $(E, C, \mathscr{T})$ be a locally solid space with the topological dual E'. A linear functional f on $l^1\langle A, E\rangle$ is $\mathscr{L}_{\iota,D}$-continuous if and only if there exists a unique $(x_i', A) \in m_\infty(A, E')$ such that*

$$\Big\langle (x_i, A), f\Big\rangle = \textstyle\sum_A \langle x_i, x_i'\rangle$$

for all $(x_i, A) \in (l^1\langle A, E\rangle$.

Recall that a subset M of E' is prenuclear [11] if there exists a $\sigma(E', E)$-closed equicontinuous subset B of E' and a positive Radon measure μ on B such that

$$\Big|\langle x, f\rangle\Big| \leqslant \int_B \Big|\langle x, x'\rangle\Big|\, \mathrm{d}\mu(x') \text{ for all } x \in E \text{ and } f \in M;$$

and a family $\{x_i' : i \in A\}$ is prenuclear if its range is a prenuclear subset of E'.

LEMMA 2.7. *Let X be a locally convex space with the topological dual X'. The following statements hold:*

(a) *A linear functional f on $l^1[A, X]$ is $\mathscr{L}_\pi$-continuous if and only if there exists a unique equicontinuous family (x_i', A) in X' such that*

$$\langle (x_i, A), f \rangle = \textstyle\sum_A \langle x_i, x_i' \rangle \text{ for all } (x_i, A) \in l^1[A, X].$$

(b) *A linear functional f on $l^1(A, X)$ is $\mathscr{L}_\varepsilon$-continuous if and only if there exists a unique prenuclear family (x_i', A) in X' such that*

$$\langle (x_i, A), f \rangle = \textstyle\sum_A \langle x_i, x_i' \rangle \text{ for all } (x_i, A) \in l^1(A, X).$$

Proof. It should be noted that the definitions of $\mathscr{L}_\pi$ and $\mathscr{L}_\varepsilon$ are the same as those defined by ([11, p. 180]). On the other hand, since $X^{(A)}$ is $\mathscr{L}_\pi$-dense in $l^1[A, X]$ as well as $\mathscr{L}_\varepsilon$-dense in $l^1(A, X)$, and since each element (x_i, A) in $X^{(A)}$ is absolutely summable in the sense of Schaefer [11], the result now follows from Schaefer [11, (10.3) and (10.4), p. 180–181].

Let Y be a locally convex space and let T be a linear mapping from E into Y. For a non-empty index set A, we define a linear map T_A, say, from E^A into Y^A by setting

$$T_A(x_i, A) = (Tx_i, A) \qquad \big((x_i, A) \in E^A\big).$$

3. Cone-absolutely summing mappings

Les $(E, C, \mathscr{T})$ be an ordered convex space. A linear map T from E into Y is said to be *cone-absolutely summing* if for any continuous seminorm q on Y there is a continuous seminorm p on E such that the inequality

$$\sum_{i=1}^{n} q(Tu_i) \leqslant p\Big(\sum_{i=1}^{n} u_i\Big)$$

holds for any finite subset $\{u_1, \cdots, u_n\}$ of C.

Clearly cone-absolutely summing maps are continuous mappings from C into Y. Therefore, if E is locally decomposable, cone-absolutely summing mappings are continuous. The set consisting of all cone-absolutely summing mappings, denoted by $\mathscr{L}^l(E, Y)$, is a vector subspace of $L(E, Y)$.

Schaefer [12] and Schlotterbeck seem to be the first to investigate cone-absolutely summing mappings defined on Banach lattices. In [15], we use the notion of cone-absolutely summing mappings, defined on locally convex Riesz spaces, to study the Dieudonné topology $\sigma_S(E, E')$. Recently Walsh [13] also studies cone-absolutely summing mappings, defined on locally o-convex spaces with closed and generating cones. We shall see from Theorem 3.2 that our definition of cone-absolutely summing mappings is the same as that defined by Walsh [13, (3.2.1)].

LEMMA 3.1. *Let $(E, C, \mathscr{T})$ and $(F, K, \mathscr{P})$ be ordered convex spaces. The following assertions hold:*

(a) *If* $T\in\mathscr{L}^l(E, X)$ *and* $S\in\mathscr{L}(X, Y)$ *then* $S\circ T\in\mathscr{L}^l(E, Y)$.

(b) *If* $S\in\mathscr{L}(E, F)$ *is positive and if* $T\in\mathscr{L}^l(F, Y)$ *then* $T\circ S\in\mathscr{L}^l(E,Y)$.

The proof is straightforward and hence will be omitted.

THEOREM 3.2. *Let* $(E, C, \mathscr{T})$ *be a locally solid space and suppose that* T *is a linear map from* E *into* Y. *Then the following statements are equivalent.*

(a) $T\in\mathscr{L}^l(E, Y)$.

(b) *For any index set* A, $T_A\in\mathscr{L}(l^1\langle A, E\rangle, l^1[A, Y])$.

(c) *The adjoint map* T' *of* T *sends equicontinuous subsets of* Y' *into order-bounded subsets of* E'.

(d) *For any continuous seminorm* q *on* Y *there is* $f\in C'$ *such that*

$$q(Tx)\leqslant \sup\left\{g(x): g\in[-f, f]\right\} \qquad (x\in E).$$

(e) $T\in\mathscr{L}(E(\sigma_S), Y)$, *where* $E(\sigma_S)=(E, C, \sigma_S(E, E'))$.

If E *and* Y *are Frenchet spaces then* (*a*) *is equivalent to the following*

(f) $T\in\mathscr{L}(E, Y)$ *and* $T_N\in L(l^1\langle N, E\rangle, l^1[N, Y])$.

Proof. The equivalence of (a) and (b) is a restatement of the definition of cone-absolutely summing mappings, the equivalence of (d) and (e) is obvious, and the implication (b)$\Rightarrow$(c) follows from Theorem 2.6. On the other hand, the proof of the equivalence of (b) and (f) was given by Walsh [13, (3.2.5)], Therefore to complete the proof we have only to show the implications (c)$\Rightarrow$(d)$\Rightarrow$(a).

(c)$\Rightarrow$(d): Let q be a continuous seminorm on Y and suppose that $U=\{y\in Y: q(y)\leqslant 1\}$, Then there exists $f\in C'$ such that $-f\leqslant T'y'\leqslant f$ for all $y'\in U^\circ$. Thus for any $x\in E$, we have

$$q(Tx)=\sup\left\{\langle Tx, y'\rangle : y'\in U^\circ\right\}\leqslant \sup\left\{g(x): g\in[-f, f]\right\}.$$

(d)$\Rightarrow$(a): Let U be a convex, circled o-neighbourhood in Y and let q be the gauge of U. By the assumption, there exists $f\in C'$ such that

$$q(Tx)\leqslant \sup\left\{g(x): g\in[-f, f]\right\},$$

and hence by Lemma 2.1, we have

$$q(Tx)\leqslant \inf\left\{f(u): u\in C,\ x\in[-u, u]\right\} \qquad (x\in E);$$

in particular, for any $w\in C$,

$$q(Tw)\leqslant \inf\left\{f(u): u\in C,\ w\in[-u, u]\right\}=f(w).$$

Let p be defined by

$$p(x)=\inf\left\{f(u): u\in C,\ x\in[-u, u]\right\} \qquad (x\in E).$$

Then p is a continuous seminorm on E such that

$$\sum_{i=1}^{n} q(Tu_i)\leqslant \sum_{i=1}^{n} f(u_i)=p\left(\sum_{i=1}^{n} u_i\right)$$

holds for any finite subset $\{u_1, \cdots, u_n\}$ of C. Therefore $T\in\mathscr{L}^l(E, Y)$.

The equivalence of (b), (c) and (f) in the preceding theorem are due to Walsh [13, (3.2.5)].

Corollary 3.3. *For a locally solid space* $(E, C, \mathscr{T})$, *the following statements are equivalent.*

(a) *The identity map from E onto E is cone-absolutely summing.*

(b) *Equicontinuous subset of E′ are order-bounded.*

(c) $\mathscr{T}=\sigma_S(E, E')$.

(d) $\mathscr{L}(E, Y)=\mathscr{L}^l(E, Y)$ *for any locally convex space Y.*

If E is either metrizable or locally convex Riesz space, then (a) *is equivalent to the following*

(e) *The embedding map from* $l^1[N, E]$ *into* $l^1\langle N, E\rangle$ *is a topological isomorphism of the firsts space onto the second space.*

If E is an F-space then (a) *is equivalent to the following*

(f) *Each positive summable sequence in E is absolutely summable.*

Proof. The equivalence of (a) and (e) follows from Theorem 3.2, Lemma 2.5 and Walsh [13, (3.2.2)], while other equivalence follows from Theorem 3.2 and Lemma 3.1.

Let us say that a locally solid space $(E, C, \mathscr{T})$ satisfying one of the equivalent properties in Corollary 3.3 is an *L-nuclear space* (or a *Dieudonné space*). According to the definition $\varepsilon_{q,D}$, we have that if q is a continuous monotone seminorm on an L-nuclear space E, then

$$\pi_q(u_n, N)=\varepsilon_q(u_n, N) \text{ whenever } (u_n, N)\in C_\pi(N, E).$$

A linear map $T: X\to Y$ is said to be *absolutely summing* if for any continuous seminorm q on Y there exists a convex, circled, o-neighbourhood V in X such that the inequality

$$\sum_{i=1}^{n} q(Tx_i)\leqslant \sup\left\{\sum_{i=1}^{n}\left|\langle x_i, x'\rangle\right| : x'\in V^o\right\}$$

holds for any finite subset $\{x_1, \cdots, x_n\}$ of X.

Clearly absolutely summing mappings are continuous. Therefore the set consisting of all absolutely summing mappings from X into Y, denoted by $\mathscr{L}^s(X, Y)$, is a vector subspace of $\mathscr{L}(X, Y)$, where $\mathscr{L}(X, Y)$ (resp. $L(X, Y)$) denotes the space consisting of all continuous linear (resp. linear) maps from X into Y. By Lemma 2.4, we have $\mathscr{L}^s(E, Y) \subset \mathscr{L}^l(E, Y)$ provided that E is a locally solid space.

LEMMA 3.4. *Let Z be a locally convex space and suppose that $T \in \mathscr{L}(X, Y)$, $S \in \mathscr{L}(Y, Z)$. If one of them is absolutely summing then $S \circ T \in \mathscr{L}^s(X, Z)$.*

The proof is straightforward and henee will be omitted.

THEOREM 3.5. *For a linear map T from X into Y, the following statements are equivalent.*

(a) *$T \in \mathscr{L}^s(X, Y)$.*

(b) *For any index set A, $T_A \in \mathscr{L}(l^1(A, X), l^1[A, Y])$.*

(c) *The adjoint map T' of T sends equicontinuous subsets of Y' into prenuclear subsets of X'.*

(d) *For any continuous seminorm q on Y there exists a $\sigma(X', X)$-closed equicontinuous subset B of X' and a positive Radon measure μ on B such that*

$$q(Tx) \leqslant \int_B \left|\langle x, x'\rangle\right| d\mu(x') \qquad (x \in X).$$

If X is metrizable then (a) *is equivalent to the following*

(e) *$T \in \mathscr{L}(X, Y)$ and $T_A \in L(l^1(A, X), l^1[A, Y])$ for any index set A.*

Proof. The equivalence of (a) and (b) is just a restatement of the definition of absolutely summing mappings; while the implication (b)$\Rightarrow$(c) follows from Lemma 2.7.

(c)$\Rightarrow$(d): Let q be a continuous seminorm on Y and suppose that $U = \{y \in Y : q(y) \leqslant 1\}$. Then $T'(U^\circ)$ is a prenuclear set in X', hence there is a $\sigma(X', X)$-closed equicontinuous subset B of X' and a positive Radon measure μ on B such that

$$\left|\left\langle x, T'(y')\right\rangle\right| \leqslant \int_B \left|\langle x, x'\rangle\right| d\mu(x') \qquad (x \in X,\ y' \in U^\circ).$$

It then follows that

$$q(Tx) \leqslant \int_B \left|\langle x, x'\rangle\right| d\mu(x') \qquad (x \in X)$$

as required in (d).

(d)$\Rightarrow$(a): Let $V=B^o$. Then V is a convex, o-neighbourhood in X such that V^o is the $\sigma(X', X)$-closed convex hull of $B\cup\{0\}$. For any finite subset $\{x_1, \cdots, x_n\}$ of X, we have

$$\sum_{i=1}^{n} q(Tx_i) \leqslant \int_B \sum_{i=1}^{n} \left|\langle x_i, x'\rangle\right| d\mu(x')$$
$$\leqslant \mu(B) \sup\left\{\sum_{i=1}^{n}\left|\langle x_i, x'\rangle\right| : x' \in V^o\right\},$$

which shows that T is absolutely summing.

The implication (a)$\Rightarrow$(e) is obvious. Conversely, if X is assumed to be metrizable and if the statement (e) holds, then T sends each summable family in X into an absolutely summable family in Y. In view of Pietsch [9, (2.1.3)], $T_A \in \mathscr{L}(l^1(A, X), l^1[A, Y])$; therefore (e) implies (b).

The preceding theorem shows that our definition of absolutely summing mappings coincides with the usual definition [9, p. 34] in the normed space case.

Recall that a locally convex space $(X, \mathscr{P})$ is nuclear if and only if each $\mathscr{P}$-equicontinuous subset of X' is prenuclear (see Schaefer [11, p. 178]).

Corollary 3.6. *For a locally convex space X, the following statements are equivalent.*

(a) *The identity map of X onto X is absolutely summing.*

(b) *X is nuclear.*

(c) *The embedding map of $l^1[N, X]$ into $l^1(N, X)$ is a topological isomorphism of the first space onto the second space.*

(d) *$\mathscr{L}(X, Y)=\mathscr{L}^s(X, Y)$ for any locally convex space Y.*

If X is metrizable then (a) *is equivalent to the following*

(e) *Each summable sequence in X is absolutely summable.*

The equivalence of (b) and (c) in the preceding result is due to Pietsch (see Schaefer [11, p. 184]).

The following result gives a connection between nuclear spaces and the topology of uniform convergence on order-intervals for locally solid spaces.

THEOREM 3.7. *A locally solid space $(E, C, \mathscr{T})$ is nuclear if and only if it satisfies the following two conditions:*

(i) *$\mathscr{T}=o(E, E')$ (i.e., E is L-nuclear);*

(ii) *order-bounded subsets of E' are prenuclear.*

Proof. The necessity follows from Corollaries 3.3 and 3.6, and from the fact that absolutely summing mappings are cone-absolutely summing.

Conversely, if E satisfies (i) and (ii), then $\{[-f, f]: f \in C'\}$ is a fundamental system of equicontinuous sets in E', and thus equicontinuous subsets of E' are prenuclear.

A part of the preceding theorem, namely nuclearity implies L-nuclearity, due to [14], is a generalization of Kōmura-Koshi's result [4].

For the metrizable case, the condition (ii) can be replaced by the following condition

(ii)* *the cone* $C_s(N, E)$ *in* $l^1(N, E)$ *is generating.*

(see [14]) Therefore this natually suggests the following:

PROBLEM. *Is a (non-metrizable) locally convex Riesz space that satisfies conditions* (i) *and* (ii)* *nuclear?*

Since base normed spaces are L-nuclear spaces and since a normed space is nuclear if and only if it is finite dimensional, it follows that the class of nuclear locally solid spaces is a proper subset of the class consisting of all L-nuclear locally solid spaces. On the other hand, normed, L-nuclear spaces may not be finite dimensional.

It is well-known thate every bounded set in a nuclear space is precompact; but this property is no longer true for L-nuclear spaces; for instance, bounded subsets of l_1 are not precompact, where l_1 is the Banach space consisting of all summable sequences of real numbers. Further, by Corollaries 3.3 and 3.6, we have $\mathscr{L}^s(l_1, l_1) \neq \mathscr{L}^l(l_1, l_1)$.

4. L-prenuclear seminorms

For an ordered convex spoce $(E, C, \mathscr{T})$, if V is a convex, circled and absorbing subset of E, we denote by Q_V the quotient map from E onto $E/p_V^{-1}(0)$, and define

$$\left\|Q_V(x)\right\|_V = p_V(x) \qquad (x \in E),$$

where p_V is the gauge of V. Then $(E/p_V^{-1}(0), \|.\|_V)$ is a normed space. (It should be noted that Q_V is continuous if and only if V is a $\mathscr{T}$-neighbourhood of 0). Further, if V is order-convex, then $p_V^{-1}(0)$ is an order-convex subspace of E, hence $(E/p_V^{-1}(0), Q_V(C), \|.\|_V)$ is an ordered normed space. We denote by E_V the normed space or the ordered normed space just introduced. If p is a seminorm on E and if $V = \{x \in E: p(x) \leqslant 1\}$, we let $E_p = E_V$, $Q_p = Q_V$ and $\|.\|_p = \|.\|_V$. If U and V are convex, circled and absorbing subsets of E and if $V \subseteq U$, then the canonical map from E_V onto E_U is denoted hy $Q_{U,V}$. Thus we have the relation $Q_U = Q_{U,V} \circ Q_V$.

Dually, if B is a convex, circled and bounded subset of X, we let $X(B)=\cup\{nB: n\geqslant 1\}$ and equip $X(B)$ with the norm $\|.\|_B$ defined by

$$\|x\|_B=\inf\{\lambda\geqslant 0: x\in\lambda B\} \qquad \big(x\in X(B)\big).$$

We also let $J_B: X(B)\to X$ to denote the embedding. Of course, J_B is continuous.

Let $(E, C, \mathscr{T})$ be an ordered convex space. A seminorm p on E is said to be *L-prenuclear* if the quotient map $Q_p: E\to E_p$ is cone-absolutely summing. Clearly L-prenuclear seminorms on a locally decomposable space are continuous.

THEOREM. 4.1 *Let $(E, C, \mathscr{T})$ be a locally solid space, p a seminorm on E and suppose that $V=\{x\in E: p(x)\leqslant 1\}$. Then the following statements are equivalent.*

(a) *p is an L-prenuclear seminorm.*

(b) *V^o is an order-bounded subset of E'.*

(c) *There exists $f\in C'$ such that*

$$p(x)\leqslant \sup\big\{g(x): g\in[-f, f]\big\} \qquad (x\in E).$$

(d) *There exists a continuous montone seminorm r on E with $p\leqslant r$ such that $Q_{p,r}: E_r\to E_p$ is cone-absolutely summing.*

Proof. By Theorem 3.2, the implications (a)$\Rightarrow$(b)$\Rightarrow$(c) are valid; while the implication (d)$\Rightarrow$(a) is an immediate consequence of Lemma 3.1. Therefore to complete the proof we have only to show that (c) implies (d).

Define, for each $x\in E$, that

$$r(x)=\sup\big\{g(x): g\in[-f, f]\big\} \qquad (x\in E).$$

By Lemma 2.1, we have

$$r(x)=\inf\big\{f(y): y\in C \text{ with } -y\leqslant x\leqslant y\big\}.$$

r is a continuous seminorm on E such that $p\leqslant r$, and

$$r(u)=h(u) \qquad (u\in C);$$

hence E_r is an ordered normed space. If $Q_r(x)\in Q_r(C)$, then there is $u\in C$ such that $Q_r(x)=Q_r(u)$, and thus $r(x)=r(u)$; therefore we can assume without loss of generrlity that $x\in C$ whenever $Q_r(x)\in Q_r(C)$. Now for any subset $\{Q_r(x_1\}, \cdots, Q_r(x_n)\}$ of $Q_r(C)$, we have

$$\sum_{i=1}^n \Big\|Q_{p,r}\big(Q_r(x_i)\big)\Big\|_p \leqslant \sum_{i=1}^n r(x_i)=r\Big(\sum_{i=1}^n x_i\Big)=\Big\|\sum_{i=1}^n Q_r(x_i)\Big\|_r.$$

Therefore $Q_{p,r}$ is conce-absolutely.

It is remarkable that the preceding theorem shows that a seminorm p on E is L-prenuclear if and only if it is dominated by some $\sigma_S(E, E')$-continuous seminorm. Also the preceding theorem has many important applications, we mention a few below.

Corollary 4.2 *For a locally solid space* $(E, C, \mathscr{T})$ *the following statements arh equivalent.*

(a) $\mathscr{T}=o(E, E')$.

(b) *Each continuous seminorm on E is L-prenuclear.*

(c) *For any continuous seminorm p on E there exists a continuous seminorm r on E with* $p\leqslant r$ *such that* $Q_{p,r}\in\mathscr{L}^l(E_r, E_p)$.

(d) $\mathscr{L}^l(E, Y)=\mathscr{L}(E, Y)$ *for any normed space Y.*

Proof. This follows from Theorem 4.1 and Corollary 3.3.

Corollary 4.3 *If p and q are L-prenuclear seminorms on E then so are* $p+q$ *and* αp *for all* $\alpha\geqslant 0$.

Corollary 4.4 *Let* $(E, C, \mathscr{T})$ *and* $(F, K, \mathscr{P})$ *be locally solid spaces. Then the following statements hold:*

(a) *If* $T\in\mathscr{L}(E, Y)$, *then* $T\in\mathscr{L}^l(E, Y)$ *if and only if for any continuous seminorm q on Y,* $q\circ T$ *is an L-prenuclear seminorm on E.*

(b) *If* $T\in\mathscr{L}(E, F)$ *is positive and if q is an L-prenuclear seminorm on F then* $q\circ T$ *is an L-prenuclear seminorm on E.*

Recall that a seminorm p on X is prenuclear [11] if there is *a* $\sigma(X', X)$-closed equicontinuous subset B of X' and a positive Radon measure μ on B such that

$$p(x)\leqslant\int_B\left|\langle x, x'\rangle\right|d\mu(x') \qquad (x\in X).$$

The following result establishes some relationship between prenuclear seminorms and absoletdly summing mappings.

THEOREM 4.5 *Let p be a seminorm on X and suppose* $V=\{x\in X: p(x)\leqslant 1\}$. *Then the following statements are equivalent.*

(a) *p is a prenuclear seminorm.*

(b) *The quotient map* Q_p *from X onto* X_p *is absolutely summing.*

(c) V^o *is a preuuclear subset of* X'.

(d) *There exists a convex, circled, o-neighbourhood W in X such that the inequality*

$$\sum_{i=1}^{n} p(x_i) \leqslant \sup\left\{\sum_{i=1}^{n}\left|\langle x_i, x'\rangle\right| : x' \in W^o\right\}$$

holds for any finite subset $\{x_1, \cdots, x_n\}$ *of* X.

(e) *There exists a continuous seminorm* r *on* X *with* $p \leqslant r$ *such that* $Q_{p,r} : X_r \to X_p$ *is absolutely summing.*

Proof. Since Q'_p is an isometry from X'_p onto $X'(V^o)$, it follows from Theorem 3.5 that (b) and (c) are equivalent. According to the definition of absolutely summing mappings, it is trivial that (b) and (d) are equivalent. Note that $p(x) = \|Q_p(x)\|_p$; it follows from Theorem 3.5 that (a) and (b) are equivalent. On the other hand, the implication (e)$\Rightarrow$(b) is a consequence of Lemma 3.4. Therefore to complete the proof we have only to verify that (d) implies (e).

Let r be the gauge of W. Then r is a continuous seminorm on X such that $p \leqslant r$. Let Σ denote the closed unit ball in X_r. It is well-known that Q'_r is an isometry from X'_r onto $X'(W^o)$; it then follows that $Q'_r(\Sigma^o) = W^o$. Now for any finite subset $\{Q_r(x_1), \cdots, Q_r(x_n)\}$ of X_r, we have

$$\sup\left\{\sum_{i=1}^{n}\left|\left\langle Q_r(x_i), f\right\rangle\right| : f \in \Sigma^o\right\} = \sup\left\{\sum_{i=1}^{n}\left|\langle x_i, x'\rangle\right| : x' \in W^o\right\}. \quad (1)$$

In view of the hypothesis and the equality (1), we obtain

$$\sum_{i=1}^{n}\left\|Q_{p,r}\left(Q_r(x_i)\right)\right\|_p = \sum_{i=1}^{n} p(x_i) \leqslant \sup\left\{\sum_{i=1}^{n}\left|\left\langle Q_r(x_i), f\right\rangle\right| : f \in \Sigma^o\right\}$$

which shows that (d) simplies (e).

The preceding theorem has many important applications; we mention a few below:

Corollary 4.6 *For a locally convex space* X, *the following statements are equivalent.*

(a) X *is nuclear.*

(b) *For each continuous seminorm* p *on* X, $Q_p \in \mathscr{L}^s(X, X_p)$.

(c) *For each continuous seminorm* p *on* X *there exists a continuous seminorm* r *on* X *with* $p \leqslant r$ *such that* $Q_{p,r} \in \mathscr{L}^s(X_r, X_p)$.

(d) $\mathscr{L}(X, Y) = \mathscr{L}^s(X, Y)$ *for any normed space* Y.

Corollary 4.7 *If* p *and* q *are prenuclear seminorms on* X *then so are* $p+q$ *and* αp *for all* $\alpha \geqslant 0$.

Proof. It is trivial that αp is prenuclear for any $\alpha \geqslant 0$. To see the prenuclearity of $p+q$, we define $T : X \to X_p \times X_q$ by setting

$$T(x)=\big(Q_p(x),\, Q_q(x)\big) \qquad (x\in X).$$

Then $T^{-1}(0)=(p+q)^{-1}(0)$, thus there is a unique algebraic isomorphism $\hat{T}$ from X_{p+q} into $X_p\times X_q$ such that $T=\hat{T}\circ Q_{p,q}$. Since $X_p\times X_q$ is a normed space under the following norm

$$\big\|\big(Q_p(x),\, Q_q(y)\big)\big\|=p(x)+q(y) \qquad (x,\, y\in X)$$

and since

$$\big\|\hat{T}\big(Q_{p+q}(x)\big)\big\|=\|Tx\|=p(x)+q(x)=\big\|Q_{p+q}(x)\big\|_{p+q},$$

it follows that $\hat{T}$ is an isometry from X_{p+q} into $X_p\times X_q$, and hence that $Q_{p+q}=\hat{T}^{-1}\circ T$. We complete the proof by showing that T is absolutely summing.

By Theorem 4.5, there exist convex, circled, o-neighbourhoods V and W in X resp. such that the inequalities

$$\sum_{i=1}^{n} p(x_i)\leqslant \sup\left\{\sum_{i=1}^{n}\Big|\langle x_i,\, x'\rangle\Big| : x'\in V^o\right\}$$

$$\sum_{j=1}^{m} q(y_j)\leqslant \sup\left\{\sum_{j=1}^{m}\Big|\langle y_j,\, y'\rangle\Big| : y'\in W^o\right\}$$

hold for any finite subsets $\{x_1, \cdots, x_n\}$ and $\{y_1, \cdots, y_m\}$ of X. Let $U=2^{-1}(V\cap W)$. Then U is a convex, circled, o-neighbourhood in X such that the inequality

$$\sum_{i=1}^{n}\|Tx_i\|\leqslant \sup\left\{\sum_{i=1}^{n}\Big|\langle x_i,\, f\rangle\Big| : f\in U^o\right\}$$

holds for any finite subset $\{x_1, \cdots, x_n\}$ of X. Therefore T is absolutely summing.

If $T\in L(X, Y)$ and if q is a continuous seminorm on Y, then the functional $q\circ T$, defined by

$$(q\circ T)(x)=q(Tx) \qquad (x\in X)$$

is a seminorm on X. If, in addition, T is absolutely summing then $q\circ T$ is prenuclear, as the following result shows.

Corollary 4.8 *For $T\in\mathscr{L}(X, Y)$, the following statements hold*:

(a) *$T\in\mathscr{L}^s(X, Y)$ if and only if for any continuous seminorm q on Y, $q\circ T$ is a prenucleare seminorm on X.*

(b) *If q is a prenuclear seminorm on Y then $q\circ T$ is a prenuclear seminorm on X.*

Let s denote the F-space of rapidly decreasing sequences, namely the vector space consisting of all number sequences $\lambda=(\lambda_n)$ such that for any integer k,

$$q_k(\lambda)=\sum_n n^k|\lambda_n|<\infty$$

and equipped with the topology determined by the family $\{q_k\}$ of seminorms. Let q be a seminorm on X. Randtke [10] calls q *strongly nuclear* (resp. *quasi-nuclear*) if there exists $\lambda=(\lambda_n)$ in s (resp. in l^1) and an equicontinuous sequence (f_n) in X' such that

$$q(x)\leqslant \sum_n \left|\lambda_n\langle x, f_n\rangle\right| \qquad (x\in X).$$

It is clear that strongly nuclear seminorms are quasi-nuclear, and that each $\sigma(X, X')$-continuous seminorm on X is quasi-nuclear. Further, quasi-nuclear seminorms are prenuclear, as the following result shows.

Corollary 4.9 *Quasi-nuclear seminorms on X are prenuclear.*

Proof. Let q be a quasi-nuclear seminorm on X, let $(\lambda_n)\in l^1$, and let (f_n) be an equicontinuous sequence in X' such that

$$q(x)\leqslant \sum_n \left|\lambda_n\langle x, f_n\rangle\right| \qquad (x\in X).$$

Suppose that V is a convex, circled, o-neighbourhood in X such that $f_n\in V^o$ for all n. Then for any finite subset $\{x_1, \cdots, x_m\}$ of X, we have

$$\begin{aligned}\sum_{j=1}^{m} q(x_j) &\leqslant \sum_n |\lambda_n|\left(\sum_{j=1}^{m}\left|\langle x_j, f_n\rangle\right|\right)\\ &\leqslant \left(\sum_n |\lambda_n|\right)\sup\left\{\sum_{j=1}^{m}\left|\langle x_j, x'\rangle\right| : x'\in V^o\right\}.\end{aligned}$$

Therefore q is prenuclear.

The following result establishes some connection between L-prenuclear seminorms and prenuclear seminorms.

Proposition 4.10 *For a locally solid space $(E, C, \mathscr{T})$, prenuclear seminorms on E are L-prenuclear.*

Proof. This follows from Theorems 4.1 and 4.5.

5. L-nuclear linear mappings

Recall that a linear map $T: X\rightarrow Y$ is *bounded* (resp. *precompact*) if it sends some neighbourhood of 0 in X into a bounded (resp. precompact) subset of Y. Let $\mathscr{L}^{lb}(X, Y)$ (resp. $\mathscr{L}^{p}(X, Y)$) denote the space consisting of all bounded (resp. precompact) linear maps from X into Y. Then

$$\mathscr{L}^p(X, Y) \subseteq \mathscr{L}^{lb}(X, Y) \subseteq \mathscr{L}(X, Y) \text{ and } \mathscr{L}^{lb}(X, Y) = \mathscr{L}^p\big(X, Y(\sigma)\big)$$

where $Y(\sigma) = (Y, \sigma(Y, Y'))$.

Let $(E, C, \mathscr{T})$ be an ordered convex space. A linear map T from E into Y is called a *L-nuclear* map if there exists a L-prenuclear seminorm p on E such that $\{Tx : p(x) \leqslant 1\}$ is a bounded subset of Y.

The set consisting of all L-nuclear linear mappings is denoted by $\mathscr{L}^{ln}(E, Y)$. If E is locally decomposable, then $\mathscr{L}^{ln}(E, Y) \subseteq \mathscr{L}^{lb}(E, Y)$, and hence L-nuclear linear maps are continuous.

LEMMA 5.1. *For a locally solid space $(E, C, \mathscr{T})$, the following assertions hold*:

(a) *$\mathscr{L}^{ln}(E, Y)$ is a vector subspace of $\mathscr{L}^l(E, Y)$ and*

$$\mathscr{L}^{ln}(E, Y) = \mathscr{L}^{lb}\big(E(\sigma_S), Y\big) = \mathscr{L}^p\big(E(\sigma_S), Y(\sigma)\big)$$

where $E(\sigma_S) = (E, C, \sigma_S(E, E'))$ and $Y(\sigma) = (Y, \sigma(Y, Y'))$. If, in addition, Y is normable then $\mathscr{L}^{ln}(E, Y) = \mathscr{L}^l(E, Y)$.

(b) *$\mathscr{L}^{pn}(E, Y) \subseteq \mathscr{L}^{ln}(E, Y)$.*

Proof. By theorem 4.1, the assertion (a) holds; while (b) follows from Proposition 4.10.

It is remarkable that the space $\mathscr{L}^{ln}(E, Y)$ does not depend upon the topologies on E and Y, but olny on the dual pairs $\langle E, E' \rangle$ and $\langle Y, Y' \rangle$.

Proposition 5.2. *For a locally solid space $(E, C, \mathscr{T})$, the following statements are equivalent.*

(a) $\mathscr{T} = o(E, E')$.

(b) *$\mathscr{L}^{ln}(E, Y) = \mathscr{L}^{lb}(E, Y)$ for any locally convex space Y.*

(c) *$\mathscr{L}^{lb}(E, Y) \subset \mathscr{L}^l(E, Y)$ for any locally convex space Y.*

(d) *$\mathscr{L}^p(E, Y) \subset \mathscr{L}^{ln}(E, Y)$ for any locally convex space Y.*

Proof. The implication (a)$\Rightarrow$(b) follows from Lemma 5.1, while (c)$\Rightarrow$(a) follows from Corollary 4.2. Therefore (a), (b) and (c) are equivalent. Clearly (b) implies (d), therefore we complete the proof by showing that (d) implies $\mathscr{L}(E, Z) = \mathscr{L}^l(E, Z)$ for any normed space Z. In fact, if $T \in \mathscr{L}(E, Z)$ then $T \in \mathscr{L}^p(E, Z(\sigma))$ where $Z(\sigma) = (Z, \sigma(Z, Z'))$. On the other hand, by Mackey's theorem, $\mathscr{L}^{ln}(E, Z) = \mathscr{L}^{ln}(E, Z(\sigma))$. We conclude from

$$\mathscr{L}^p\big(E, Z(\sigma)\big) \subset \mathscr{L}^{ln}\big(E, Z(\sigma)\big) = \mathscr{L}^{ln}(E, Z) \subset \mathscr{L}^l(E, Z)$$

that $\mathscr{L}(E, Z) = \mathscr{L}^l(E, Z)$.

LEMMA 5.3. *Let E and F be locally solid spaces, Then the following assertions hold:*

(a) *If $T\in\mathscr{L}^{ln}(E, X)$ and $S\in\mathscr{L}(X, Y)$ then $S\circ T\in\mathscr{L}^{ln}(E, Y)$.*

(b) *If $S\in\mathscr{L}(E, F)$ is positive and if $T\in\mathscr{L}^{ln}(E, Y)$ then $T\circ S\in\mathscr{L}^{ln}(E, Y)$.*

The proof is straightforward and will be omitted.

Before giving a representation of L-nuclear mappings, we need the following result which is of interest in itself.

LEMMA 5.4. *Let $(E, C, \mathscr{T})$ be an ordered convex space for which C is generating, $h\in C'$ and suppose that*

$$r(x)=\inf\left\{h(u): u\in C \text{ with } -u\leqslant x\leqslant u\right\} \qquad (x\in X).$$

Then E_r is a base normed space.

Proof. For any $Q_r(u)\in Q_r(C)$, we can assume without loss of generality that $u\in C$. For any $Q_r(u)$, $Q_r(w)$ in $Q_r(C)$, we have

$$\left\|Q_r(u)+Q_r(w)\right\|_r=r(u+w)=h(u+w)=\left\|Q_r(u)\right\|_r+\left\|Q_r(w)\right\|_r$$

which shows that the norm $\|.\|_r$ is additive on $Q_r(C)$. On the other hand, if $\|Q_r(x)\|_r<1$, then there is $u\in C$ with $-u\leqslant x\leqslant u$ such that $h(u)<1$. Since $-Q_r(u)\leqslant Q_r(x)\leqslant Q_r(u)$ and since

$$\left\|Q_r(u)\right\|_r=r(u)=h(u)<1\,,$$

it follows that the open unit ball Σ in E_r is absolutely dominated (i.e., $\Sigma\subseteq S(\Sigma)$). The additivity of $\|.\|_r$ insures that Σ is solid. Therefore, by [16, (9.5)], E_r is a base normed space.

As a consequencc of Lemma 5.3, we conclude that if $(E, C, \mathscr{T})$ is an L-nuclear locally solid space, then there is a family $\{(E_\alpha, C_\alpha, \|.\|_\alpha): \alpha\in\Gamma\}$ of base normed spaces and a family $\{T_\alpha: \alpha\in\Gamma\}$ of positive continuous linear mappings such that $\mathscr{T}$ is the projective topology with respect to $\{(E_\alpha, C_\alpha, \|.\|_\alpha, T_\alpha): \alpha\in\Gamma\}$.

THEOREM 5.5. *Let $(E, C, \mathscr{T})$ be a locally solid space and suppose that $T\in\mathscr{L}(E, Y)$. Then the following statements are equivalent:*

(a) *$T\in\mathscr{L}^{ln}(E, Y)$.*

(b) *There exists $f\in C'$ such that for any o-neighbourhood U in Y there is $\alpha\geqslant 0$ for which $T'(U^o)\subseteq\alpha[-f, f]$.*

(c) *There exists $h\in C'$ such that for any continuous seminorm q on Y there is $\alpha_q\geqslant 0$ for which*

$$q(Tx)\leqslant\alpha_q \inf\left\{h(y): y\in C \textit{ with } -y\leqslant x\leqslant y\right\} \qquad (x\in E).$$

(d) *$T'\in\mathscr{L}^o(Y'(\beta), E')$, where $\mathscr{L}^o(Y'(\beta), E')$ is the space consisting of all linear maps that send some $\beta(Y', Y)$-neighbourhoods of 0 in Y' into an order-bounded subset of E'.*

(e) *T is the composite of a sequence of continuous linerr mappings*

$$E\xrightarrow{Q}F\xrightarrow{\hat{T}}X\xrightarrow{J}Y,$$

where F is a base normed space, X is a normed space, Q is positive and $\hat{T}$ is cone-absolutely summing (or L-nuclear).

Proof. The equivalence of (a), (b) and (c) follow from Theorem 4.1, and the implication (e)$\Rightarrow$(a) follows from Lemmas 5.1 and 5.3. We complete the proof by showing that (a)$\Leftrightarrow$(d)$\Rightarrow$(e).

(a)$\Rightarrow$(d): Let p be an L-prenuclear seminorm on E such that $B=\{Tx\in Y: p(x)\leqslant 1\}$ is a bounded subset of Y, let $V=\{x\in E: p(x)\leqslant 1\}$ and suppose that f, in C' is such that

$$p(x)\leqslant \sup\left\{g(x): g\in[-f,f]\right\} \qquad (x\in E).$$

Then $V^o\subseteq[-f,f]$. Since $B^o=(T(V))^o=(T')^{-1}(V^o)$, we conclude from $T'(B^o)\subseteq V^o\subseteq[-f,f]$ that $T'\in\mathscr{L}^o(Y'(\beta), E')$.

(d)$\Rightarrow$(a): Let B be a convex, circled bounded subset of Y, and let f, in C', be such that $T'(B^o)\subseteq[-f,f]$. Since $[-f,f]^o\subseteq T^{-1}(B^{oo})=T^{-1}(B)$, it follows that $T([-f,f]^o)\subset B$. Clearly the gauge of $[-f,f]^o$ is an L-prenuclear seminorm on E, hence $T\in\mathscr{L}^{ln}(E, Y)$.

(a)$\Rightarrow$(e): Let p be an L-prenuclear seminorm on E such that $B=\{Tx\in Y: p(x)\leqslant 1\}$ is a bounded subset of Y. Then $p^{-1}(0)\subseteq T^{-1}(0)$, and thus there exists a continuous linear map S from E_p onto $Y(B)$ (Since $\|S\|\leqslant 1$) such that $T=J_B\circ S\circ Q_p$. On the other hand, since p is L-prenuclear, by Theorem 4.1, there exists $h\in C'$ such that

$$p(x)\leqslant \inf\left\{h(u): u\in C,\ x\in[-u, u]\right\} \qquad (x\in E).$$

Let $r(x)=\inf\{h(u): u\in C,\ x\in[-u, u]\}$ $(x\in E)$. Then $p\leqslant r$ and $Q_{p,r}: E_r\to E_p$ is cone-absolutely summing. Define $\hat{T}=S\circ Q_{p,r}$. $\hat{T}$ is a cone-absolutely summing map from E_r onto $Y(B)$ and

$$T=J_B\circ\hat{T}\circ Q_r.$$

Clearly Q_r is positive. Lemma 5.4 insures that E_r is a base normed space. Therefore (a) implies (e).

According to the proof of the implication (a)$\Rightarrow$(e) and Lemmas (5.1) and (5.3), we see that $T\in\mathscr{L}(E, Y)$ is L-nuclear if and only if there exists an L-prenuclear seminorm q on E and $S\in\mathscr{L}(E_q, F)$ such that $T=S\circ Q_q$.

Corollary 5.6. *For a locally solid space* $(E, C, \mathscr{T})$, *the identity map* $I: E\to E$ *is L-nuclear if and only if* E' *contains an order unit and* $\mathscr{T}=o(E, E')$.

A linear map T from E into F is said to be *order-bounded* if it maps some order-bounded set in E into an order-bounded set in F. Denote by $L^b(E, F)$ the space consisting of all order-bounded linear mappings from E into F. If E and F are Riesz spaces and if F is order-complete then $L^b(E, F)$ is an order-complete Riesz space under the natural ordering (see Peressini [7, p. 22]). We denote by $\mathscr{L}^o(E, Y)$ the space consisting of all linear mappings $T: X\to F$ each of which sends some o-neighbourhood in E into an order-bounded subset of Y. If E is also a locally convex Riesz space and if F is an order-complete locally o-convex Riesz space, then Peressini [8] has shown that $\mathscr{L}^o(E, F)\subseteq\mathscr{L}^{lb}(E, F)$ and that $\mathscr{L}^o(E, F)$ is a lattice ideal in $L^b(E, F)$.

Proposition 5.7. *Let F be a locally convex Riesz space, X a locally convex space and suppase that T is a linear map from X into F. Then the following assertions hold.*

(a) *If* $T\in\mathscr{L}^o(X, F)$ *then* $T'\in\mathscr{L}^{ln}(F'(\sigma_S), X'(\beta))$ *and* $T''\in\mathscr{L}^o(X''_e, [F])$, *where* T'' *is the second adjoint map of* T, $[F]$ *is the l-ideal* (i.e., *solid subspace of* F'') *in* F'' *generated by* F *and* X''_e *is the bidual equipped with the natural topology.*

(b) *If X is infrabarrelled, F is an l-ideal in* F'' *and if* $T'\in\mathscr{L}^{ln}(F'(\sigma_S), X'(\beta))$, *then* $T\in\mathscr{L}^o(X, F)$.

Proof. (a) There is a convex, circled, o-neighbourhood U in X and $0\leqslant y\in F$ such that $T(U)\subseteq[-y, y]$. It is easily seen that $T'([-y, y^o])\subseteq U^o$. Hence $T'\in\mathscr{L}^{ln}(F'(\sigma_S), X'(\beta))$ because the gauge of $[-y, y]^o$ is L-prenuclear. If $U^{o\pi}$ (resp. $[-y, y]^{o\pi}$) is the polar of U^o (resp. $[-y, y]^o$) taken in X'' (resp. F''), then

$$U^{o\pi}\subseteq\left(T'\left([-y, y]^o\right)\right)^\pi=(T'')^{-1}\left([-y, y]^{o\pi}\right)$$

and so $T''(U^{o\pi})\subseteq[-y, y]^{o\pi}$. Notice that the set $\{\phi\in Y'': -y\leqslant\phi\leqslant y\}$ is a $\sigma(Y'', Y')$-closed, order-bounded subset of $[F]$. Therefore, $T''\in\mathscr{L}^o(X''_e, [F])$ because $U^{o\pi}$ is a neighbourhood of 0 in X''_e.

(b) We now assume that X is infrabarreled and that F is an l-ideal in F''. If $T' \in \mathscr{L}^{ln}(F'(\sigma_S), X'(\beta))$ then there is $0 \leqslant y \in F$ such that $M = T'(W)$ is a $\beta(X', X)$-bounded, convex, circled subset of X', where $W = \{g \in F' : \langle y, |g| \rangle \leqslant 1\}$. It is easily seen that $T(M^o) \subseteq [-y, y]$, where M^o is the polar of M taken in X. Since X is infrabarrelled, M^o is a neighbourhood of 0 in E, therefore $T \in \mathscr{L}^o(X, F)$, and the proof is complete.

A linear map $T: X \to Y$ is said to be *prenuclear* (resp. *quasi-nuclear*, *s-type* [10]) if there exists a prenuclear (resp. quasi-nuclear, strongly nuclear) seminorm p on X such that $\{Tx \in Y : p(x) \leq 1\}$ is a bounded subset of Y.

The set consisting of all prenuclear (resp. quasi-nuclear, s-type) linear mappings is denoted by $\mathscr{L}^{pn}(X, Y)$ (resp. $\mathscr{L}^{qn}(X, Y)$, $s(X, Y)$). It is clear that $\mathscr{L}^{pn}(X, Y) \subseteq \mathscr{L}^{lb}(X, Y)$, therefore prenuclear linear mappings are continuous.

LEMMA 5.8. *The following assertions hold*:

(a) $\mathscr{L}^{pn}(X, Y)$ *is a vector subspace of* $\mathscr{L}^{lb}(X, Y)$ *and*

$$s(X, Y) \subseteq \mathscr{L}^{qn}(X, Y) \subseteq \mathscr{L}^{pn}(X, Y) \subseteq \mathscr{L}^{s}(X, Y);$$

$$\mathscr{L}^{pn}(X, Y) = \mathscr{L}^{pn}\Big(X, Y(\sigma)\Big). \tag{5.1}$$

If, in addition, F is normable then $\mathscr{L}^{pn}(X, Y) = \mathscr{L}^{s}(X, Y)$.

(b) $\mathscr{L}^{lb}(X(\sigma), Y) \subseteq \mathscr{L}^{qn}(X, Y)$.

Proof. The assertion (b) is obvious. By Corollary 4.6, $\mathscr{L}^{pn}(X, Y)$ is a vector subspace of $\mathscr{L}^{lb}(X, Y)$. In view of Corollary 4.9 and Theorems 4.5 and 3.4, the formula (5.1) is valid. Finally, if F is normable then it is true that $\mathscr{L}^{pn}(X, Y) = \mathscr{L}^{s}(X, Y)$. Therefore the proof is complete.

It is easy to see that the definition of quasi-nuclear maps coincides with the usual definition (see Pietsch [9]) in the normed space case. Therefore, in view of Pietsch [9, (3.2.10) and (2.4.4)], $\mathscr{L}^{qn}(X, Y)$ is, in general, not equal to $\mathscr{L}^{pn}(X, Y)$. At the end of this section we shall give an example to show that $\mathscr{L}^{pn}(X, Y)$ is, in general, not equal to $\mathscr{L}^{s}(X, Y)$.

Proposition 5.9. *The following statements are equivalent.*

(a) X *is nuclear.*
(b) $\mathscr{L}^{pn}(X, Y) = \mathscr{L}^{lb}(X, Y)$ *for any locally convex space* Y.
(c) $\mathscr{L}^{lb}(X, Y) \subset \mathscr{L}^{s}(X, Y)$ *for any locally convex space* Y.
(d) $\mathscr{L}^{p}(X, Y) \subseteq \mathscr{L}^{pn}(X, Y)$ *for any locally convex space* Y.

Proof. The equivalence of (a), (b) and (c) follow from Lemma 5.8; obviously (b) implies (d). According to Corollary 4.6, by a similar argument given in the proof of (d)$\Rightarrow$(a) in Proposition 5.2, we have that (d) implies (a).

Parts of the following result are due to Randtke [10, p. 97].

LEMMA 5.10. *Let Z be a locally convex space, and suppose that $T\in\mathscr{L}(X, Y)$, $S\in\mathscr{L}(Y, Z)$. If one of them is prenuclear (resp. quasi-nuclear, s-type) then $S\circ T$ is prenuclear (resp. quasi-nuclear, s-type).*

Proof. This follows from Corollary 4.3.

Brudovskiĭ [1, 2] has given some characterizations of s-type mappings. We now present a representation of prenuclear linear mappings as follows:

THEOREM 5.11. *Let T be a continuous linear map from X into Y. The following statements are equivalent.*

(a) $T\in\mathscr{L}^{pn}(X, Y)$.

(b) *There exists a convex, circled, prenuclear subset M of X' such that for any continuous seminorm q on Y there is $\alpha_q\geqslant 0$ for which the inequality*

$$q(Tx)\leqslant \alpha_q r(x)$$

holds for any $x\in X$, where r is the gauge of the polar M° of M.

(c) *There exists a $\sigma(X', X)$-closed equicontinuous subset B on X' and a positive Radon measure μ on B such that for any continuous seminorm q on Y there is $\alpha_q\geqslant 0$ for which*

$$q(Tx)\leqslant \alpha_q\int_B\left|\langle x, x'\rangle\right| d\mu(x') \qquad (x\in X).$$

(d) *There exists a convex, circled, o-neighbourhood V in X such that for any continuous seminorm q on Y there is $\alpha_q\geqslant 0$ for which the inequality*

$$\sum_{i=1}^{n} q(Tx_i)\leqslant \alpha_q \sup\left\{\sum_{i=1}^{n}\left|\langle x_i, x'\rangle\right| : x'\in V^\circ\right\}$$

holds for any finite subset $\{x_1, \cdots, x_n\}$ of X.

(e) *T is the composite of a sequence of continuous linear mappings*

$$X\xrightarrow{Q} Z\xrightarrow{\hat{T}} G\xrightarrow{J} Y$$

where Z and G are normed spaces and $\hat{T}$ is absolutely summing (or prenuclear).

Proof. T is prenuclear if and only if there exists a prenuclear seminorm p on X such that for any continuous seminorm p on Y there is $\alpha_q\geqslant 0$ for which

$$q(Tx)\leqslant \alpha_q p(x) \qquad (x\in X).$$

Then, in view of Theorem 4.5, the statements (a)—(d) are mutually equivalent. On the other hand, by Lemmas 5.8 and 5.10, (e) implies (a). It remains to verify that (a) implies (e).

Let $T\in\mathscr{L}^{pn}(X, Y)$ and let p be a prenuclear seminorm on X such that $B=\{Tx : p(x)\leqslant 1\}$ is a bounded subset of Y. The boundedness of B insures that $p^{-1}(0)\subseteq T^{-1}(0)$, so there exists a continuous linear map S from X_p onto $Y(B)$ (since $\|S\|\leqslant 1$) such that $T=J_B\circ S\circ Q_p$. On the other hand, since p is prenuclear, by Theorem 4.5, there exists a continuous seminorm r on X with $p\leqslant r$ such that $Q_{p,r}: X_r\to X_p$ is absolutely summing. Define

$$\hat{T}=S\circ Q_{p,r}.$$

Then $\hat{T}$ is an absolutely summing map from X_r onto $Y(B)$ and

$$T=J_B\circ\hat{T}\circ Q_r.$$

Therefore (a) implies (e).

Recall that a locally convex space X is nuclear if and only if each equicontinuous subset of X' is prenuclear or, equivaledtly, each continuous seminorm on X is prenuclear (see Schaefer [11, p. 178]).

Corollary 5.12. *The identity map $I: X\to X$ is prenuclear if and only if X is a normable and finite dimensional space.*

Proof. If I is a prenuclear linear map, then by Theorem 5.9, there exists a convex, circled, prenuclear subset M of X' such that for any continuous seminorm q on X there is $\alpha_q\geqslant 0$ such that

$$q(x)\leqslant\alpha_q\, p(x) \qquad (x\in X),$$

where p is the gauge of M^o. Therefore the topology on X is determined by the single seminorm p; consequently X is normable. On the other hand, M is the unit ball in the Banach dual X' which is prenuclear, thus X is nuclear, and surely must be finite dimensional.

Conversely, if X is normable and finite dimensional, then X is nuclear. In view of Corollary 3.6, the identy map is absolutely summing, and hence must be prenuclear by making use of Lemma 5.8.

Corollary 5.13. (Dvoretzky and Rogers). *A normed space X is finite dimensional if and only if every summable family in X is absolutely summable.*

Proof. In view of Lemma 5.8, absolutely summing mappings are prenuclear. Therefore this result follows from Corollaries 5.12 and 3.6.

Corollary 5.14. *If $T\in\mathscr{L}^{pn}(X, Y)$ and $S\in\mathscr{L}^{pn}(Y, Z)$ then $S\circ T$ is a* *nuclear mapping from X into Z.*

Proof. By Theorem 5.11 the mappings T and S can be decomposed in the following way:

$$X \xrightarrow{Q_1} X_1 \xrightarrow{\hat{T}} X_2 \xrightarrow{J_1} Y$$

$$Y \xrightarrow{Q_2} Y_1 \xrightarrow{\hat{S}} Y_2 \xrightarrow{J_2} Z,$$

where X_1, X_2, Y_1 and Y_2 are normed spaces, $\hat{T}\in\mathscr{L}^s(X_1, X_2)$ and $\hat{S}\in\mathscr{L}^s(Y_1, Y_2)$.

Define

$$L = Q_2\circ J_1\circ\hat{T}.$$

Then $L\in\mathscr{L}^s(X_1, Y_1)$, and hence by Pietsh [9, (3.3.5)], $\hat{S}\circ L$ is a nuclear map from X_1 inio Y_2 because X_1, Y_1 and Y_2 are normed spaces. As

$$S\circ T = J_2\circ\hat{S}\circ L\circ Q_1,$$

we conclude that $S\circ T$ is a nuclear mapping from X into Z.

Corollary 5.15. *If $T\in\mathscr{L}^{qn}(X, Y)$ and $S\in\mathscr{L}^{qn}(Y, Z)$ then $S\circ T$ is a nuclear mapping from X into Z.*

Proof. This follows from Lemma 5.8 and Corollary 5.14.

Examples 5.16. (a) It is well-known that the space c_0 consisting of all null-sequences of real numbers is a Banach lattice equipped with the usual norm and usual ordering, and that l^1 is its topological dual without order unit. According to Corollary 3.3, $c_0(\sigma_s)=(c_0, \sigma_s(c_0, l^1))$ is an L-nuclear space and hence $I\in\mathscr{L}^l(c_0(\sigma_s), c_0(\sigma_s))$, where I is the identity map. But Corollary 5.6 shows that $I\notin\mathscr{L}^{ln}(c_0(\sigma_s), c_0(\sigma_s))$, and Proposition 5.2 indicates that $I\notin\mathscr{L}^{lb}(c_0(\sigma_s), c_0(\sigma_s))$, $I\notin\mathscr{L}^p(c_0(\sigma_s), c_0(\sigma_s))$. Therefore we conclude that $\mathscr{L}^{ln}(E, Y)\neq\mathscr{L}^l(E, Y)$, $\mathscr{L}^l(E, Y)\neq\mathscr{L}^{lb}(E, Y)$ and that $\mathscr{L}^{lb}(E, Y)\neq\mathscr{L}(E, Y)$.

(b) For any infinite dimensional locally convex space X, it is well-known that $X(\sigma)=(X, \sigma(X, X'))$ is a nuclear space. By Corollary 3.6, $I\in\mathscr{L}^s(X(\sigma), X(\sigma))$, but corollary 5.12 shows that $I\notin\mathscr{L}^{pn}(X(\sigma), X(\sigma))$. Therefore we conclude that $\mathscr{L}^{pn}(X, Y)\neq\mathscr{L}^s(X, Y)$.

(c) Since the dual space of l^1 is an AM-space with order unit, it follows from Corollary 5.6 that $I\in\mathscr{L}^{ln}(l^1, l^1)$. As l^1 is not nuclear, it follows that $I\notin\mathscr{L}^s(l^1, l^1)$ and surely $I\notin\mathscr{L}^{pn}(l^1, l^1)$. Therefore we conclude that $\mathscr{L}^{pn}(E, Y)\neq\mathscr{L}^{ln}(E, Y)$.

6. Lattice properties of L-nuclear mappings

In this section $(E, C, \mathscr{T})$ and $(F, K, \mathscr{P})$ are assumed to be locally convex

Riesz spaces. Recall that $(F, K, \mathscr{P})$ is *boundedly order-complete* if every $\mathscr{P}$-bounded subset of F which is directed upwards has a supremum, and that $(F, K, \mathscr{P})$ is *locally order-complete* if $\mathscr{P}$ has a neighbourhood base at 0 consisting of convex, solid and order-complete subsets of F. In [15], we have shown that if $(F, K, \mathscr{P})$ is both locally and boundedly order-complete then $\mathscr{L}^l(E, F)$ is an l-ideal in $L^b(E, F)$ (i.e., $\mathscr{L}^l(E, F)$ is a solid subspace of $\mathscr{L}^b(E, F)$.

Proposition 6.1. *If $(F, K, \mathscr{P})$ is both boundedly and locally order-complete then $\mathscr{L}^{ln}(E, F)$ is an l-ideal in $L^b(E, F)$.*

Proof. Suppose that $T\in\mathscr{L}^{ln}(E, F)$. Since $\mathscr{L}^{ln}(E, F)\subseteq\mathscr{L}^l(E, F)$, it then follows from [7] that $|T|$ exists in $L^b(E, F)$, where $|T|$ is defined by

$$|T|(u)=\sup\left\{\sum_{i=1}^{n}|Tu_i| : u=\sum_{i=1}^{n}u_i,\ u_i\in C\right\} \text{ for all } u\in C\,.$$

We now show that $|T|\in\mathscr{L}^{ln}(E, F)$. Let f, in C', be such that for any continuous Riesz seminorm r on F there exists $\alpha_r\geqslant 0$ for which the inequality

$$r(Tx)\leqslant\alpha_r\left\langle|x|, f\right\rangle \tag{6.1}$$

holds for all $x\in E$. According to the hypothesis, by [5, Lemma 1] there exists a continuous Riesz seminorm q on F such that

$$r\Big(|T|u\Big)\leqslant\sup\left\{q\Big(\sum_{i=1}^{n}|Tu_i|\Big) : u=\sum_{i=1}^{n}u_i,\ u_i\in C\right\} \tag{6.2}$$

holds for any $u\in C$. For this q there exists $\alpha_q\geqslant 0$ such that the inequality (6.1) holds for all $x\in E$, therefore for any finite subset $\{u_1, \cdots, u_n\}$ of C with $u=\sum_{i=1}^{n}u_i$, we obtain

$$q\Big(\sum_{i=1}^{n}|Tu_i|\Big)\leqslant\sum_{i=1}^{n}q(Tu_i)\leqslant\alpha_q\Big\langle\sum_{i=1}^{n}u_i, f\Big\rangle=\alpha_q\langle u, f\rangle$$

and thus

$$r\Big(|T|u\Big)\leqslant\alpha_q\langle u, f\rangle \text{ for all } u\in C\,.$$

Now for any $x\in E$, there is

$$r\Big(|T|x\Big)\leqslant r\Big(|T|x^+\Big)+r\Big(|T|x^-\Big)\leqslant\alpha_q\left\langle|x|, f\right\rangle$$

which implies that $|T|\in\mathscr{L}^{ln}(E, F)$.

Finally, it is not hard to see that $\mathscr{L}^{ln}(E, F)$ is a solid subspace of $L^b(E, F)$. Therefore the proof is complete.

Corollary 6.2. *If $(F, K, \mathscr{P})$ is a complete L-nuclear space then $\mathscr{L}^{ln}(E, F)$ is an l-ideal in $L^b(E, F)$.*

Proof. Since $\mathscr{P}=o(F, F')$, it follows from [16, (13.9)] that F is $\mathscr{P}$-complete if and only if $(F, K, \mathscr{P})$ is both boundedly and locally order-complete. The result now follows from Proposition 6.1.

Corollary 6.3. *If $(E, C, \mathscr{T})$ is an L-nuclear space and if $(F, K, \mathscr{P})$ is both boundedly and locally order-complete (in particular, F is a complete L-nuclear space), then $\mathscr{L}^{lb}(E, F)$ is an l-ideal in $L^b(E, F)$.*

Proof. This follows from Proposition 5.2 and Prosition 6.1.

It is known from Theorem 5.5 that the mapping $T \rightsquigarrow T'$ is an order isomorphism from $\mathscr{L}^{ln}(E, F)$ onto $\mathscr{L}^{o}(F'(\beta), E')$, where T' is the adjoint mrp of T. Since $(F', K' \beta(F', F))$ is a locally convex Riesz space and since $(E', C', \sigma(E', E))$ is an order-complete locally o-convex Riesz space (i.e., an order-complete Riesz space for which C' is a normal cone in $(E', \sigma(E', E))$), by Peressini [8, Proposition 3], $\mathscr{L}^{o}(F'(\beta), E')$ is an l-ideal in $L^b(F', E')$ and is contained in $\mathscr{L}(F'(\beta), E'(\sigma))$. Therefore for any $T\in\mathscr{L}^{ln}(E, F)$, the absolute-value $|T'|$ of T', defined by

$$\begin{aligned} |T'|\,(g) &= \sup\left\{T'(h): |h|\leqslant g,\, h\in F'\right\} \\ &= \sup\left\{\sum_{i=1}^{n} T'(g_i): g_i\in K',\ \sum_{i=1}^{n} g_i = g\right\} \qquad (g\in K') \end{aligned}$$

exists in $\mathscr{L}^{o}(F'(\beta), E')$; consequently, there exists a unique element in $\mathscr{L}^{ln}(E, F)$, denoted by $a(T)$, such that

$$\big(a(T)\big)' = |T'|\,. \tag{6.3}$$

On the other hand, for any $u\in C$ and any $g\in K'$, since

$$\left\langle \big(a(T)\pm T\big)\,u,\, g\right\rangle = \left\langle u,\, \big(|T'|\pm T'\big)\,g\right\rangle \geqslant 0\,,$$

$a(T)\pm T$ are positive L-nuclear maps from E into F; further, if $S\in\mathscr{L}^{ln}(E, F)$ is such that $\pm T\leqslant S$, then it is easily seen that $a(T)\leqslant S$. Therefore $a(T)$ is the absolute-value of T. Moreover, we have the following relation

$$T\big([-u, u]\big)\subseteq\big[-a(T)\,u,\, a(T)\,u\big] \qquad (u\in C)$$

which shows that $T\in L^b(E, F)$. We conclude that $\mathscr{L}^{ln}(E, F)$ is an order-complete Riesz space under the lattice operations $a(.)$ defined by the formula (6.3) and is only a subspace of $L^b(E, F)$ because the map $T\rightsquigarrow T'$ becomes an l-isomorphism from $\mathscr{L}^{ln}(E, F)$ onto $\mathscr{L}^{o}(F'(\beta), E')$. (It should be noted that $L^b(E, F)$ may not be a Riesz space.)

Now if $L^b(E, F)$ is a Riesz space and if T, in $\mathscr{L}^{ln}(E, F)$, is such that the map $|T|$ defined by

$$|T|(u) = \sup\left\{\sum_{i=1}^{n} |Tu_i| : u_i \in C, \ \sum_{i=1}^{n} u_i = u\right\} \qquad (u \in C)$$

exists in $L^b(E, F)$, then $|T| = a(T) = |T'|'$ and $\mathscr{L}^{ln}(E, F)$ is an l-ideal in $L^b(E, F)$. In fact, $|T| \leqslant a(T)$ and hence

$$|T|' \leqslant \big(a(T)\big)' = |T'| . \tag{6. 4}$$

The existences of $|T|$ also insures that $\pm T \leqslant |T|$, hence $\pm T' \leqslant |T|'$ or, equivalently,

$$|T'| \leqslant |T|' .$$

Combining this with the formula (6.4), we obtain $|T'| = |T|'$; consequently,

$$|T| = a(T) = |T'|' .$$

Finally, if $0 \leqslant S \leqslant T$ where $T \in \mathscr{L}^{ln}(E, F)$ and $S \in L^b(E, F)$, there exists $f \in C'$ such that for any continuous Riesz seminorm r on F there is $\alpha_r \geqslant 0$ for which

$$r(Tx) \leqslant \alpha_r \langle |x|, f \rangle$$

holds for all $x \in E$. It then follows that

$$r(Sx) \leqslant r(Sx^+) + r(Sx^-) \leqslant \alpha_r \langle |x|, f \rangle \text{ for all } x \in E .$$

Therefore $\mathscr{L}^{ln}(E, F)$ is an l-ideal in $L^b(E, F)$. We may summarize in the following result.

Proposition 6. 4. *$\mathscr{L}^{ln}(E, F)$ is an order-complete Riesz space and is a subspace of $L^b(E, F)$. If, in addition, $L^b(E, F)$ is a Riesz space and $|T|$ exists in $\mathscr{L}^{ln}(E, F)$ for any $T \in \mathscr{L}^{ln}(E, F)$, then $\mathscr{L}^{ln}(E, F)$ is an l-ideal in $L^b(E, F)$ and the equalities*

$$|T|' = |T'| , \qquad |T| = |T'|'$$

hold for all $T \in \mathscr{L}^{ln}(E, F)$.

Let $\mathscr{L}$ be a locally convex topology on a Riesz space (F, K). May and Chivukula [5] called that $(F, K, \mathscr{L})$ has *property* (C) if it satisfies the following two conditions:

(i) every $\mathscr{L}$-compact subset of F has a supremum in F;

(ii) for any continuous seminorm r on F, there exists a continuous seminorm q on F such that for any $\mathscr{L}$-compact subset B of F the following inequality holds

$$r(\sup B) \leqslant \sup \{q(b) : b \in B\}.$$

Let G be an ordered vector space. Recall that a subspace G_1 of G is called an *o-ideal* in G if it follows from $0 \leqslant x \leqslant u$ with $u \in G_1$ and $x \in G$ that $x \in G_1$. Therefore a Riesz subspace of E is an l-ideal if and only if it is an o-ideal. According to Theorem 4.1, it is easily see that $\mathscr{L}^{ln}(E, F)$ and $\mathscr{L}^l(E, F)$ are always o-ideals in $\mathscr{L}(E, F)$. Also Proposition 6.2 shows that $\mathscr{L}^{ln}(E, F)$ is always an o-ideal in $L^b(E, F)$.

Proposition 6.5. *Let E be an l-ideal in the bidual E'' of E, (F, K) a Riesz space and suppose that the locally convex space $(F, \mathscr{L})$ has the property* (C). *Then the following assertions hold.*

(a) *$\mathscr{L}^l(E, F)$ is a Riesz space under the lattice operation*

$$|T|(u) = \sup \{Tx : |x| \leqslant u\} \qquad (u \in C), \tag{6.5}$$

and is contained in $L^b(E, F)$ (note that $L^b(E, F)$ is not necessarily a Riesz space).

(b) *$\mathscr{L}^{ln}(E, F)$ is an l-ideal in $\mathscr{L}^l(E, F)$.*

(c) *If F is order-complete then $\mathscr{L}^l(E, F)$ is an l-ideal in $L^b(E, F)$.*

Proof. (a) Since E is an l-ideal in E'', it follows from [16, (13.5)] that each order-interval in E is $\sigma(E, E')$-compact. If $T \in \mathscr{L}^l(E, F)$, T is continuous with respect to $\sigma(E, E')$ and $\mathscr{L}$, hence $T([-u, u])$ is a $\mathscr{L}$-compact subset of F for any $u \in C$. By the hypothesis, the supremum

$$|T|(u) = \sup \{Tx : |x| \leqslant u\}$$

exists for any $u \in C$. As $|T|$ is positive, $|T| \in L^b(E, F)$ and so $\mathscr{L}^l(E, F) \subseteq L^b(E, F)$. Further we show that $|T|$ is cone-absolutely summing. In fact, for any continuous seminorm r on F, there exists a continuous seminorm q on F such that

$$r\big(|T|u\big) \leqslant \sup \{q(Tx) : |x| \leqslant u\} \qquad (u \in C). \tag{6.6}$$

On the other hand, since $T \in \mathscr{L}^l(E, F)$, by Corollary 4.8, there is $f \in C'$ such that

$$q(Tx) \leqslant \big\langle |x|, f \big\rangle \text{ for all } x \in X.$$

Combining this with the inequality (6.6), we obtain

$$r\big(|T|u\big) \leqslant \langle u, f \rangle \text{ for all } u \in C.$$

Now for any $x \in E$, we have

$$r\left(|T|x\right)\leqslant r\left(|T|x^+\right)+r\left(|T|x^-\right)\leqslant\left\langle |x|, f\right\rangle.$$

Consequently, $|T|\in\mathscr{L}^l(E,F)$ in view of Corollary 4.8; therefore $\mathscr{L}^l(E,F)$ is a Riesz space and is contained in $L^b(E,F)$.

(b) Since $\mathscr{L}^{ln}(E,F)\subseteq\mathscr{L}^l(E,F)$, for any $T\in\mathscr{L}^{ln}(E,F)$, $|T|$ exists in $\mathscr{L}^l(E,F)$, and there is $f\in C'$ such that for any continuous seminorm q on F there exists $\alpha_q\geqslant 0$ for which the inequality

$$q(Tx)\leqslant\alpha_q\left\langle |x|, f\right\rangle$$

holds for all $x\in E$. Now it is easily seen that $|T|\in\mathscr{L}^{ln}(E,F)$. Therefor $\mathscr{L}^{ln}(E,F)$ is a Riesz subspace of $\mathscr{L}^l(E,F)$. It was noted that $\mathscr{L}^{ln}(E,F)$ is always an o-ideal in $\mathscr{L}^l(E,F)$. Hence $\mathscr{L}^{ln}(E,F)$ is an l-ideal in $\mathscr{L}^l(E,F)$.

(c) The order-completeness of F insures that $L^b(E,F)$ is an order-complete Riesz space under the usual lattice operations defined by the equality (6.5). According to the conclusion (a), $\mathscr{L}^l(E,F)$ is a Riesz subspace of $L^b(E,F)$ and surely an l-ideal in $L^b(E,F)$.

7. A characterization of $\mathscr{L}^{lb}(l^1\langle A,E\rangle, l^1[A\ Y])$

Let $(E,C,\mathscr{T})$ be a locally solid space. It is easily seen that $T\in\mathscr{L}^{lb}(E,Y)$ if and only if there exists a continuous seminorm p on E such that for any continuous seminorm q on Y there is $\alpha_q\geqslant 0$ for which

$$q(Tu)\leqslant\alpha_q p(u)\qquad(u\in C).$$

Suppose now that $T\in\mathscr{L}(E,Y)$. Then by the above remark, $T_A\in\mathscr{L}^{lb}(l^1\langle A,E\rangle, l^1[A,Y])$ if and only if there exists a continuous seminorm p on E such that for any continuous seminorm q on Y there is $\alpha_q\geqslant 0$ for which the inequality

$$\sum_{i=1}^n q(Tu_i)\leqslant\alpha_q p\left(\sum_{i=1}^n u_i\right)\tag{7.1}$$

holds for any finite subset $\{u_1,\cdots,u_n\}$ of C. For convenience of expression, we write

$$\mathscr{L}^{sl}(E,Y)=\left\{T\in\mathscr{L}(E,Y): T_N\in\mathscr{L}^{lb}\left(l^1\langle N,E\rangle, l^1[N,Y]\right)\right\}.$$

It is easily seen that

$$\mathscr{L}^{ln}(E,Y)\subseteq\mathscr{L}^{sl}(E,Y)\subseteq\mathscr{L}^{lb}(E,Y)\cap\mathscr{L}^l(E,Y).$$

and that if Y is normable then

$$\mathscr{L}^{ln}(E,Y)=\mathscr{L}^{sl}(E,Y)=\mathscr{L}^l(E,Y).$$

Elements in $\mathscr{L}^{sl}(E, Y)$ are called strongly *cone-absolutely summing mappings.*

THEOREM 7.1. *Let $(E, C, \mathscr{T})$ be a locally solid space and suppose that $T\in\mathscr{L}(E, Y)$. Then $T\in\mathscr{L}^{sl}(E, Y)$ if and only if there exists a strongly monotone, continuous seminorm p on E and $\hat{T}\in\mathscr{L}(E_p, Y)$ with $T=\hat{T}\circ Q_p$ such that $\hat{T}$ maps positive summable families in E_p into absolutely summable families in Y.*

Proof. Necessity. Let p be a continuous strongly monotone seminorm on E such that the inequality (7.1) holds, and suppose that $B=\{Tx\in Y: p(x)\leqslant 1\}$. Then $p^{-1}(0)\subseteq T^{-1}(0)$, and hence there exists a $\hat{T}\in\mathscr{L}(E_p, Y)$ such that $T=\hat{T}\circ Q_p$. Let $(Q_p(u_i), A)\in C_\epsilon(A, E_p)$. Without loss of generality one can assume that $u_i\in C$. For any $\delta>0$ there exists $\alpha_0\in\mathscr{T}(A)$ such that

$$p\Big(\sum_{i\in\alpha} u_i\Big)=\Big\|\sum_{i\in\alpha} Q_p(u_i)\Big\|_p\leqslant\delta \tag{7.2}$$

whenever $\alpha\in\mathscr{T}(A)$ with $\alpha\cap\alpha_0=\phi$. Therefore inequalities (7.1) and (7.2) insure that $(\hat{T}(Q_p(x_i)), A)$ is absolutely summable in Y.

Sufficiency. Let $V=\{x\in E: p(x)\leqslant 1\}$ and let Σ be the closed unit ball in the normed space E_p. Denote by V^o the polar of V taken in E', and by Σ^π the polar of Σ taken in the Banach dual space E_p'. It follows from the isometry of the adjoint Q_p' from E_p' onto $E'(V^o)$ that $V^o=Q_p'(\Sigma^\pi)$, and hence that

$$\varepsilon_p(x_i, A)=\varepsilon_{\|.\|_p}\big(Q_p(x_i), A\big)\ \text{for all}\ (x_i, A)\in l'(A, E).$$

Letting

$$\mathscr{U}=\Big\{\big(Q_p(u_i), A\big)\in C_\epsilon(A, E_p): \varepsilon_{\|.\|_p}\big(Q_p(u_i), A\big)\leqslant 1\Big\}$$

and

$$\hat{T}_A\big(Q_p(x_i), A\big)=\Big(\hat{T}\big(Q_p(x_i)\big), A\Big)\ \text{for all}\ \big(Q_p(x_i), A\big)\in l^1(A, E_p).$$

A similar argument given in the proof of Pietsch [9, (2.1.2)] shows that $\hat{T}_A(\mathscr{U})$ is a bounded subset of $l^1[A, Y]$. Therefore, for any continuous seminorm q on Y, there is $\mu_q\geqslant 0$ such that the following inequality holds.

$$\pi_q\Big(\hat{T}\big(Q_p(u_i)\big), A\Big)\leqslant\mu_q\varepsilon_{\|.\|_q}\big(Q_p(u_i), A\big)=\mu_q\varepsilon_p(u_i, A) \tag{7.3}$$

whenever $(u_i, A)\in C_\epsilon(A, E)$. We conclude from the formula (7.3) and from Lemma 2.4 that $T\in\mathscr{L}^{sl}(E, Y)$.

Corollary 7.2. *Let $(E, C, \|.\|)$ be a normed, locally solid space and Y a normed space. Then a linear map $T: E\to Y$ is L-nuclear (or cone-*

absolutely summing) if and only if T maps positive summable families in E into absolutely summable families in Y.

As an application of Corollaries 7.2 and 5.6, we obtain the following interesting result, which should be compared with the theorem of Dvoretzky-Rogers.

Corollary 7.3. *In a normed, locally solid space E for which E' has no any order- unit, there are positive summable families which are not absolutely summable.*

Proof. Let E be a normed, locally solid space for which E' has no any order-unit. If every positive summable families is absolutely summabl, then Corollary 7.2 insures that the identity map is L-nuclear. Therefore by Corollary 5.6, E' has an order-unit, contrary to the hypothesis. This contradition shows that there are positive summable families in E which are not absolutely summable.

Corollary 7.4. *Let E and F be locally solid spaces. If $T\in\mathscr{L}^{sl}(E, F)$ and if $S\in\mathscr{L}(F, Y)$ can be factorized through a normed space, then $S\circ T\in\mathscr{L}^{ln}(E, Y)$. In particular the product of two strongly cone-absolutely summing mappings is L-nuclear.*

Proof. Let G be a normed space such that S is the compose of a sequence of continuous linear mappings

$$F\xrightarrow{Q}G\xleftarrow{\hat{S}}Y.$$

By Theorem 7.1, the mapping T can be decomposed in the following way

$$E\xrightarrow{Q_p}E_p\xrightarrow{\hat{T}}F$$

where p is a continuous, strongly monotone seminorm on E, and $\hat{T}\in\mathscr{L}(E_p, F)$ maps positive summable sequences in E_p into absolutely summable sequences in Y. Define $H=Q\circ\hat{T}$, sincs Q is continuous, H maps positive summable sequences in E_p into absolutely summable sequences in F. By Corollary 7.2, H is an L-nuclear map from E_p into G. On the other hand, since Q_p is positive and since

$$S\circ T=\hat{S}\circ H\circ Q_p,$$

we conclude from Lemma 5.3 that $S\circ T$ is L-nuclear.

Let X and Y be locally convex spaces and suppose that $T\in\mathscr{L}(X, Y)$. Then it is easily seen that $T_A\in\mathscr{L}^{lb}(l^1(A, X), l^1[A, Y])$ if and only if there exists a convex, circled, o-neighbourhood V in X such that for any continuous seminorm q on Y there is $\alpha_q\geqslant 0$ for which the inequality

$$\sum_{i=1}^{n} q(Tx_i) \leqslant \alpha_q \sup\left\{\sum_{i=1}^{n}\left|\langle x_i, x'\rangle\right| : x' \in V^0\right\} \tag{7.4}$$

holds for any finite subset $\{x_1, \cdots, x_n\}$ of X; conseqently $T \in \mathscr{L}^{lb}(X, Y)$.

For simplicity of notation, we write

$$\mathscr{L}^{ss}(X, Y) = \left\{T \in \mathscr{L}(X, Y) : T_N \in \mathscr{L}^{lb}\left(l^1(N, X), l^1[N, Y]\right)\right\}.$$

Clearly we have

$$\mathscr{L}^{pn}(X, Y) \subseteq \mathscr{L}^{ss}(X, Y) \subseteq \mathscr{L}^{lb}(X, Y) \cap \mathscr{L}^{s}(X, Y).$$

THEOREM 7.5. *Let $T \in \mathscr{L}(X, Y)$. Then $T_A \in \mathscr{L}^{lb}(l^1(A, X), l^1[A, Y])$ if and only if there exists a convex circled, o-neighbourhood V in X and $\hat{T} \in \mathscr{L}(X_V, Y)$ such that*

$$T = \hat{T} \circ Q_V \text{ and } \hat{T}_A \in L\left(l^1(A, X_V), l^1[A, Y]\right).$$

Proof. Necessity. Let V be a conuex, circled, o-neighbourhood in X such that the inequality (7.4) holds. Let p be the gauge of V and $B = T(V)$. According to the inequality (7.4), B is a bounded subset of Y and so $p^{-1}(0) \subseteq T^{-1}(0)$; thus there exists $\hat{T} \in \mathscr{L}(X_V, Y)$ such that $T = \hat{T} \circ Q_V$. Since Q'_V is an isometry from X'_V onto $X'(V^o)$, by making use of the formula (7.4), it is not hard to see that $\hat{T}_A \in L(l^1(A, X_V), l^1[A, Y])$.

Sufficiency. Since X_V is a normed space, it follows from Theorem 3.5 that $\hat{T}_A \in \mathscr{L}(l^1(A, X_V), l^1[A, Y])$, and hence that $\hat{T} \in \mathscr{L}^s(X_V, Y)$. Let Σ denote the unit ball in X_V. Then $Q'_V(\Sigma^o) = V^o$. For any continuous seminorm q on Y there is $\alpha_q \geqslant 0$ such that the inequality

$$\sum_{i=1}^{n} q\left(\hat{T}\left(Q_V(x_i)\right)\right) \leqslant \sup\left\{\sum_{i=1}^{n}\left|\left\langle Q_V(x_i), f\right\rangle\right| : f \in \Sigma^o\right\}$$

holds for any finite subset $\{Q_V(x_1), \cdots, Q_V(x_n)\}$ of X_V. Now it is easy to see that the formula (7.4) holds, therefore $T_A \in \mathscr{L}^{lb}(l^1(A, X), l^1[A, Y])$.

Corollary 7.6. *If $T \in \mathscr{L}^{ss}(X, Y)$ and $S \in \mathscr{L}^{ss}(Y, Z)$ then $S \circ T \in \mathscr{L}^{pn}(X, Z)$.*

Proof. By Theorem 6,1, the mappings T and S can be decomposed in the following way:

$$X \xrightarrow{Q_V} X_V \xrightarrow{\hat{T}} Y$$

$$Y \xrightarrow{Q_W} Y_W \xrightarrow{\hat{S}} Z,$$

where V and W are convex, circled o-neighbourhoods in X and Y resp. and $\hat{T} \in \mathscr{L}(X_V, Y)$ (resp. $\hat{S} \in \mathscr{L}(Y_W, Z)$) sends summable families in X_V (resp.

Y_W) into absolutely summable families in Y (resp. Z). Define

$$L = Q_W \circ \hat{T}\,.$$

Since Q_W is continuons, $Q_W(l^1[A, Y]) \subseteq l^1[A, Y_W]$, thus

$$L\left(l^1(A, X_V)\right) \subseteq l^1[A, Y_W]\,.$$

Since X_V and Y_W are normed spaces, it follows from Theorem 3.5 and Lemma 5.8 that $L \in \mathscr{L}^{pn}(X_V, Y_W)$. In view of Lemma 5.10 and

$$S \circ T = \hat{S} \circ L \circ Q_V\,,$$

we conclude that $S \circ T \in \mathscr{L}^{pn}(X, Y)$.

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