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On infinitesimal projective transformations satisfying the certain conditions

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§ 1. Introduction.

We consider the following problem

PROBLEM. *Let M be a compact Riemannian manifold with positive constant scalar curvature. If M adminits a nonisometric infinitesimal projective transformation, then is M a space of positive constant curvature?*

For this problem, the following results are known.

THEOREM A. *Let M be a complete Riemannian manifold with parallel Ricci tensor. If M admits nonaffine infinitesimal projective transformations, then M is a space of positive constant curvature.* [1].

THEOREM B. *Let M be a compact Riemannian manifold with consant scalar curvature K. If the scalar curvature is nonpositive, then an infinitesimal projective transformation is a motion.* [2].

THEOREM C. *Let M be a compact Riemannian manifold satisfying a condition* $\nabla_k K_{ji} - \nabla_j K_{ki} = 0$, $(K \neq 0)$, *where* ∇_k, K_{ji} *denote a covariant derivative and Ricci tensor, respectively. The projective Killing vector* v^h *can be decomposed uniquely as follows,*

$$v^h = w^h + q^h ,$$

where w^h *and* q^h *are Killing vector and gradient projective Killing vector, respectively.* [2].

THEOREM D. *Let M be a compact Riemannian manifold satisfying a condition* $\nabla_k K_{ji} - \nabla_j K_{ki} = 0$, $(K \neq 0)$. *If M admits nonisometric infinitesimal projective transformations, then M is a space of positive constant curvature.* [2].

The purpose of this paper is to prove the following theorems

THEOREM 1. *Let M be a complete, connected and simply connectected Riemannian manifold with positive constant scalar curvature. If a projective Killing vector* v^h *is decomposable as follows,*

$$v^h = w^h - \frac{n(n-1)}{2K} f^h ,$$

where w^h *and* $\frac{n(n-1)}{2K} f^h$ *are a Killing vector and a non-zero gradient projective Killing vector, respectively, then* M *is isometric to a sphere of radius* $\sqrt{\frac{n(n-1)}{K}}$.

THEOREM 2. *Let* M *be a compact Riemannian manifold with constant scalar curvature and let* v^h *be a projective Killing vector. Put* $f=\nabla_i v^i/n+1$. *Then the following conditions are equivalent.*

(1 $w^h = v^h + \frac{n(n-1)}{2K} f^h$ *is a Killing vector,*

(2) $Z_{kji}^{h} f^k = 0$, *where* $Z_{kji}^{h} = K_{kji}^{h} + \frac{K}{n(n-1)} (\delta_j^h g_{ki} - \delta_k^h g_{ji})$, *and* K_{kji}^{h} *denotes the Riemannian curvature tensor,*

(3) $G_{ji} f^j = 0$, *where* $G_{ji} = K_{ji} - \frac{K}{n} g_{ji}$.

A vector field v^h is called an infinitesimal projective transformation or a projective Killing vector if it satisfies

$$\mathfrak{L}\left\{ {h \atop ji} \right\} = \nabla_j \nabla_i v^h + K_{kji}^{h} v^k = \delta_j^h \varphi_i + \delta_i^h \varphi_j , \tag{1.1}$$

where $\mathfrak{L}$, $\left\{ {h \atop ji} \right\}$, φ_i denote Lie derivation with respect to v^h, Christoffel's symbol and associated vector, respectively. From this equations, we get following results

$$\begin{aligned} \mathfrak{L} K_{kji}^{h} &= -\delta_k^h \nabla_j \varphi_i + \delta_j^h \nabla_k \varphi_i , \\ \mathfrak{L} K_{ji} &= -(n-1) \nabla_j \varphi_i , \\ \nabla^i \nabla_i v_j + K_{ji} v^i &= 2\varphi_j , \\ \nabla_j (\nabla_i v^i) &= (n+1) \varphi_j . \end{aligned}$$

We have $f_j = \varphi_j$, where f_j means $\nabla_j f$, therefore φ_j is a gradient vector and in the following discussions, we use f_j instead of φ_j.

§ 2. Proof of Theorem 1.

LEMMA 1. *Let* M *be a complete, connected and simply connected Riemannian manifold of dimension* n. *In order that* M *admits a nontrivial solution* ϕ *for the system of differential equations*

$$\nabla_k \nabla_j \phi_i + K(2\phi_k g_{ji} + \phi_j g_{ik} + \phi_i g_{kj}) = 0 , \quad K>0 , \quad \phi_i = \nabla_i \phi ,$$

it is necessary and sufficient that M be isometric with a sphers S^n of radius $\frac{1}{\sqrt{K}}$ in Euclidean $(n+1)$-space.

For this Lemma, see [3].

LEMMA 2. *If $v_h = w_h - \frac{n(n-1)}{2K} f_h$, then we have*

$$\nabla_h \nabla_j f_i + \frac{K}{n(n-1)} (2 f_h g_{ji} + f_j g_{hi} + f_i g_{hj}) = 0 .$$

PROOF. Substituting v_h into (1. 1), since w_h is the Killing vector, we obtain

$$\nabla_j \nabla_i f_h + K_{kjih} f^k = -\frac{2K}{n(n-1)} (g_{hj} f_i + g_{hi} f_j) . \tag{2. 1}$$

Since $\nabla_i f_h = \nabla_h f_i$, we hove

$$\begin{aligned} 0 &= \nabla_j \nabla_i f_h - \nabla_j \nabla_h f_i \\ &= -K_{kjih} f^k - \frac{2K}{n(n-1)} (g_{hj} f_i + g_{hi} f_j) + K_{kjhi} f^k \\ &\qquad + \frac{2K}{n(n-1)} (g_{ji} f_h + g_{hi} f_j) \\ &= -2K_{kjih} f^k - \frac{2K}{n(n-1)} (g_{hj} f_i - g_{ji} f_h) . \end{aligned}$$

Substituting this result into (2. 1), we get

$$\nabla_j \nabla_i f_h + \frac{K}{n(n-1)} (2 f_j g_{hi} + f_i g_{hj} + f_h g_{ij}) = 0 .$$

From Lemma 1, and Lemma 2, we have Theorem 1.

§ 3. Proof of Theorem 2.

In this section we assume M is compact and the scalar curvature is constant.

LEMMA 3. *If $w^h = v^h + \frac{n(n-1)}{2K} f^h$ is a Killing vector, then we have $Z^h_{kji} f^k = 0$.*

This is obious from the proof of Theorem 1.

LEMMA 4. *If $Z^h_{kji} f^k = 0$, then we obtain $G_{ji} f^j = 0$.*

This proof is trivial.

LEMMA 5. *There is the following equation,*

$$(n-1)\Delta^2 f+2K\Delta f+2K_{ji}\nabla^j f^i=0,$$

where Δ means $g^{ji}\nabla_j\nabla_i$.

For this Lemma, see [2].

Lemma 6. *If $G_{ji}f^j=0$, then we have $\Delta f=-\dfrac{2(n+1)}{n(n-1)}Kf$.*

Proof is the same as that in page 266, [2].

LEMMA 7. *A necessary and sufficient condition for a vector field w^h in M to be a Killing vector is $\nabla_i w^i=0$ and $\nabla^j\nabla_j w^h+K_i^h w^i=0$.*

For thisLemma, see [4].

LEMMA 8. *If $G_{ji}f^j=0$, then we get $v^h=w^h-\dfrac{n(n-1)}{2K}f^h$.*

PROOF. If we put $w^h=v^h+\dfrac{n(n-1)}{2K}f^h$, then we have

$$\begin{aligned}\nabla^i w_i &= \nabla^i v_i+\frac{n(n-1)}{2K}\Delta f\\ &=(n+1)f-(n+1)f\\ &=0,\end{aligned}$$

$$\begin{aligned}\nabla^j\nabla_j w^i+K_i^j w_j &= \nabla^j\nabla_j v^i+K_i^j v_j+\frac{n(n-1)}{2K}\{\nabla^j\nabla_j f_i+K_i^j f_j\}\\ &=2f_i+\frac{n(n-1)}{2K}\left\{-\frac{2(n+1)}{n(n-1)}Kf_i+\frac{2K}{n}f_i\right\}\\ &=0.\end{aligned}$$

Therefore w_i is a Killing vector from the Lemma 7. Consequently we arrive at the complete proof of Theorem 2 by means of Lemma 3, Lemma 4 and Lemma 8.

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