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A Santalo's formula in L-P

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Abstract It is show that a formula by Santalō on hyperbolic space of curvature -1 holds for Lorentz-Poincarē upper half space with curvature 1.

Introduction We call *L-P* plane, or simply *L-P*, relating to Lorentz-Poincarē, to the upper half space with the metric $ds^2=\frac{dx^2-dy^2}{y^2}$. The curvature of the *L-P* plane is 1.

If z is a complex variable, the group $SL(2)$ acts on the upper half plane $\mathrm{Im}(z)>0$ as the transformation group.

$$z'=\frac{az+b}{pz+q} \qquad aq-bp=1$$

Where a, b, p, q are real numbers. This is the classical Poincarē model for non-euclidean hyperbolic geometry. In the first section we introduce the double numbers, see [1], [7] and [8]. The referee observed that the reference [6], pag. 166, is appropiate. We show that substitution in the above transformation of the complex variable by a double number variable we obtain the Lorentz-Poincarē geometry. We also find relationship between double numbers, curvature and geodesics. Our main results is the integral formula in the third section.

Along the second section we obtain different expressions for the density of points, pair of points, geodesics, pair of geodesics, and kinematic density as is customary in integral geometry. Some of them will be used in the following section.

1. Double numbers in L-P

Let *L-P* plane be the upper half plane of Lorentz-Poincarē that means, the upper half plane $y>0$ with the metric

$$(1) \qquad ds^2=\frac{dx^2-dy^2}{y^2}$$

Considering [1] and [7], we find an interesting relation between this metric and the so-called double numbers.

As a generalization of complex numbers, Benz, [1], and Yaglom, [7],

[8], call $z=x+jy$ with $x, y\in \boldsymbol{R}$, $j^2=1$, $j\neq\pm 1$: this set is a commutative ring. On the natural way, the conjugate of z, $\bar{z}$, is $\bar{z}=x-jy$, $y>0$.

LEMMA. *Let z be a double number. The element of arc of the L-P plane is*

$$ds^2=4\frac{dz\Lambda d\bar{z}}{(z-\bar{z})^2} \tag{2}$$

and (2) is invariant under Moebius transformations.

PROOF. Routine calculation gives (2).

Applying a Moebius transformation to double numbers we get $z'=\frac{az+b}{pz+q}$ with $a, q, p, b\in\boldsymbol{R}$, $aq-pb=1$, z, z' double numbers.

We obtain

$$\begin{aligned} z'-\bar{z}' &= \frac{(az+b)(p\bar{z}+q)-(\bar{a}\bar{z}+b)(pz+q)}{(pz+q)(p\bar{z}+q)} \\ &= \frac{pb(\bar{z}-z)+aq(z+\bar{z})}{(pz+q)(p\bar{z}+q)}=\frac{(aq-pb)-(z-\bar{z})}{(pz+q)(p\bar{z}+q)}. \end{aligned}$$

On the other side, from the expression of z' we get

$$dz'=\frac{(a(pz+q)-p(az+b))}{(pz+q)}dz=\frac{dz}{(pz+q)^2},$$

and conjugating we have

$$d\bar{z}'=\frac{d\bar{z}}{(p\bar{z}+q)^2},$$

and

$$dz'\Lambda d\bar{z}'=\frac{dz\Lambda d\bar{z}}{(pz+q)^2(p\bar{z}+q)^2}.$$

Finally,

$$ds^2=4\frac{dz\Lambda d\bar{z}}{(z-\bar{z})^2}=4\frac{dz'\Lambda d\bar{z}'}{(z'-\bar{z}')^2}.$$

Any point (x, y) in the L-P plane can be parametrized by a double number.

Let us consider $x=jv$, $y=e^{-ju}$ with $u, v\in\boldsymbol{R}$, where e^{-ju} is the exponential function defined by the serie

$$e^{-ju}=\sum_{k=0}^{\infty}\frac{(-ju)^k}{k!}$$

then we have $dx=j\,dv$, $dy=-j.\,e^{-ju}du$. Replacing in (1) we get

(3) $$ds^2=\frac{dv^2-e^{-2ju}du^2}{e^{-2ju}}=-du^2+e^{2ju}dv^2$$

which we call the polar form of the element of arc. It is well known that the Gauss curvature K is given by

$$K=\frac{-1}{2\sqrt{EG}}\left(\left(\frac{E_v}{\sqrt{EG}}\right)_v+\left(\frac{G_u}{\sqrt{EG}}\right)_u\right)$$

where E, G are the coefficients of the first fundamental form and the subscripts denote partial derivation.

We can rewrite (3) as

$$ds^2=(i)^2du^2+e^{2ju}dv^2, \quad i^2=-1.$$

Considering the coefficients of the first fundamental form $E=i^2$, $G=e^{2ju}$ an easy computation gives

$$K=\frac{-1}{2\sqrt{-e^{2ju}}}\left(\left(\frac{2j\,e^{2ju}}{\sqrt{-e^{2ju}}}\right)_u\right)=1.$$

Therefore, the L-P plane with metric (3) has constant Gauss curvature 1. If we do not want to mix double and complex numbers we can take the complex parametrization $x=iv$, $y=e^{-iu}$ then $ds^2=du^2+e^{2iu}dv^2$ and also, $K=1$.

At any point (x_0, y_0) of the L-P plane, the metric (1) is associated to the following inner product, if $P=(p_1, p_2)$ and $Q=(q_1, q_2)$ with $p_2>0$, $q_2>0$

(4) $$<P, Q>=\frac{p_1q_1-p_2q_2}{y_0^2}$$

and the norm of P, is $\|P\|=\dfrac{\sqrt{|p_1^2-p_2^2|}}{y_0}$.

Then essentially, the L-P inner product coincides with the Lorentzian one of [2], and [3].

In n-dimensional spacetime, it is usual to give a time orientation by saying a timelike vector $X(<X, X><0)$ is future pointing if its n-coordinate is positive; in our case, every vector of the L-P plane is future pointing and the lemma of [3] holds in the following way:

LEMMA. *Let P and Q be timelike vectors in the L-P plane then*

(i) $<P, Q>\leq 0$

(ii) $P+Q$ *is a timelike vector*

(iii) $-<P, Q>\geq\|P\|\,\|Q\|$, *and the equality holds if and only if* $Q=$

cP for some $c>0$.

(iv) $\|P+Q\|\geq\|P\|+\|Q\|$ *and the equality holds if and only if* $Q=cP$ *for* $c>0$.

We conclude that every results related to pure angles and triangles, see [3], holds in the L-P plane.

From [4] we have another point of view. Nomizu considers the group of matrices G_2 consisting of all 2×2 matrices of the form

$$\begin{bmatrix} p & q \\ 0 & 1 \end{bmatrix} \quad p>0, \quad q\in \boldsymbol{R}$$

and a diffeomorphism of G_2 onto the L-P plane given by $g\in G_2 \longrightarrow (q, p)$. The action of G_2 on the L-P plane is

$$(5) \qquad (x, y) \longrightarrow (px+q, py)=u(p, q).$$

We find that the metric (1) is invariant by the action (5) of G_2 and corresponds to a left invariant Lorentz metric on the G_2 of constant Gauss curvature 1.

Let $(x(t), y(t))$ be a geodesic with t affine parameter. The equations are (see [4])

$$(6) \qquad \frac{d^2x}{dt^2}=2\frac{(dx)}{dt}\frac{(dy)}{dt}/y$$

$$\frac{d^2x}{dt^2}=\left(\left(\frac{dx}{dt}\right)^2\left(\frac{dy}{dt}\right)^2\right)/y.$$

The geodesics are classified in:

1) null, given by null lines

2) spacelike, given by the curves $x(u)=b+a.\,shu$, $y(u)=a\cdot chu$ with the arc-lenght parameter t is given by $t=t(u)=\int_{u_0}^{u}\frac{du}{ch\,u}$

3) timelike, given by the vertical lines $x=b$, $y=ae^{ct}$ respect to the affine parameter t and also, by two half branches of hyperbolas ($y>0$) parametrized by $x(u)=b\pm a\cdot chu$ $y(u)=shu$ $y>0$ and the proper-time parameter t measured from $u=u_0>0$ being $t(u)=\int_{u_0}^{u}\frac{du}{shu}$.

According to the Yaglom's classification, [8] pag. 223, these spacelike geodesics are the great circles of an Euclidean hyperbolid of two sheets; in fact our geodesics are the great circles of one sheet of an Euclidean hyperboloid of two sheets. The distance can be defined as we define angular measure in a pencil of lines in the Minkowskian plane; this metric is hyperbolic. Therefore, its integral geometry is well known from [5], chap. 17.

Also, according to Yaglom, [8], the non-null timelike geodesics can be obtained intersecting the hyperbolic hemicylinder with planes which planes which contain the z-axe.

In this way, we can consider the L-P plane with the trigonometry given by the relations

$$a=b+c$$
$$\frac{A}{sh\ a}=\frac{B}{sh\ b}=\frac{C}{sh\ c}$$
$$A^2=B^2+C^2+2BC\ ch\ a$$

where A, B, C are angles of a pure triangle, [3], and a, b, c, its opposite sides.

This trigonometry corresponds to the co-Minkowskian geometry which, in fact, is dual to the Lorentzian trigonometry, [3], [8]. The motions of the co-Minkowskian plane are induced by the motions of the three-dimensional space

$$x'=ch\ x+sh\ y+a$$
$$y'=sh\ x+ch\ y+b$$
$$z'=u\cdot x+v\cdot y+w\cdot z+c.$$

with $y>0$, $y'>0$.

2. Densities in L-P

We already showed that the group $SL(2)$ keeps the form (2) invariant. In fact, the group $SL(2)$ acts on the L-P plane as a transformation group:

$$z'=\frac{az+b}{pz+q},\ aq-pb=1\ \ a, b, p, q\in \boldsymbol{R} \text{ and } z', z \text{ double numbers.}$$

Its Maurer-Cartan forms are equal to those of $SL(2)$ acting on the classical hyperbolic model of Poincaré. Then from [5], pag. 174, we have

$$w_1=q\ da-b\ dp,\ w_2=q\ db-b\ dq,\ w_3=-p\ da+a\ dp.$$

Keeping in mind that $d(aq-pb)=0$, we get

$$(7)\qquad \begin{aligned} da&=a\ w_1+b\ w_2 & dp&=p\ w_1+q\ w_3\\ db&=-b\ w_1+a\ w_2 & dq&=-q\ w_1+p\ w_2.\end{aligned}$$

The structure equations are

$$dw_1=-w_2\Lambda w_3,\ dw_2=-2w_1\Lambda w_3,\ dw_3=-2w_3\Lambda w_1$$

The isotropy group G_0 at $z=j$ $(j^2=1)$ is a characterized by $a=q$, $b=p$ which are equivalent to $w_1=0$, $w_2-w_3=0$. Since $d(w_1\Lambda(w_2-w_3))=0$, following [5], we have that the invariant density of G/G_0 is the density of points of the L-P, i. e.

$$(8)\qquad d(G/G_0)=dp=(w_1\Lambda(w_2-w_3))$$
$$=q^2 da\Lambda db+b^2 dp\Lambda dq-bq(da\Lambda dq+dp\Lambda db)+dp\Lambda da.$$

Also, since $ds^2=\dfrac{dx^2-dy^2}{y^2}$, we know that density of points $P\in L$-P (or its element of area) is given by

$$(9)\qquad dP=\frac{dx\Lambda dy}{y^2}.$$

For geodesics

a) From [4], we can parametrize the non-null timelike geodesics by

$$(10)\qquad x(u)=b+a\ ch\ u \qquad y(u)=a\ sh\ u.$$

The tangent vector is $\left(\dfrac{dx}{du}, \dfrac{dy}{du}\right)$ and its lenght $\dfrac{1}{sh\ u}$. Its proper time parameter t will be $t=t(u)=\displaystyle\int_{u_0}^{u}\frac{1}{sh\ r}dr=1n\left(tgh\frac{u}{2}-tgh\frac{u_0}{2}\right)$ for $u_0>0$.

We identify $\begin{bmatrix} y & x \\ 0 & 1\end{bmatrix}\in G_2$ with $(x, y)\in L$-P and applying it to geodesics we have

$$(11)\qquad \begin{bmatrix} a\ sh\ u & b\pm a\ ch\ u \\ 0 & 1\end{bmatrix}$$

The action defined by (5) can be rewritten as

$$\begin{bmatrix} p & q \\ 0 & 1\end{bmatrix}\begin{bmatrix} a\ sh\ u & b\pm a\ ch\ u \\ 0 & 1\end{bmatrix}=\begin{bmatrix} pa\ sh\ u & pb\pm pa\ ch\ u+q \\ 0 & 1\end{bmatrix}.$$

It follows that

$$(12)\qquad x'=pb+q\pm pa\ ch\ u, \quad y'=pa\cdot sh\ u.$$

The geodesics remain invariant under the transformation if and only if $p=1$, $q=0$. Thus the subgroup H of G_2 which leaves the geodesics invariant is $H=\{id\}$; then we have a bijection between the set geodesics of the L-P plane and the homogeneous space G_2/H. The Lie algebra of H is $\mathscr{H}=\{0\}$ and the Lie algebra of G_2 is

$$\mathscr{J}=\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}0&1\\0&0\end{bmatrix}\right\}.$$

We call $\mathscr{M}$ the subspace of $\mathscr{J}$ such that $\mathscr{J}=\mathscr{M}\oplus\mathscr{M}$ and $Ad(H)\mathscr{M}=\mathscr{M}$. Trivially $\mathscr{M}=\mathscr{J}$ and $dp\Lambda dq$ is the 2-form invariant by $Ad(H)$.

On the other side, it is known that the forms of Maurer-Cartan of a group G are given by $A^{-1}\cdot dA$ for $A\in G$. Thus, if

$$A=\begin{bmatrix}p&q\\0&1\end{bmatrix}\text{ we have }A^{-1}=\begin{bmatrix}1/p&-q/p\\0&1\end{bmatrix}\ dA=\begin{bmatrix}dp&dq\\0&0\end{bmatrix}$$

and

$$A^{-1}\cdot dA=\begin{bmatrix}\dfrac{dp}{p}&\dfrac{dq}{p}\\0&0\end{bmatrix}.$$

Therefore the density of geodesics is

$$(13)\qquad dG=\frac{dp\Lambda dq}{p^2}$$

b) We will find another expression for dG. The equation of the non null timelike geodesic G is

$$(x-b)^2-y^2=a^2$$

or equivalently

$$x=b+a\ ch\ u,\quad y=a\ sh\ u.$$

By the action (5), the geodesic G transforms into (12). We see that under the action $u(p,q)$, the coordinates a, b of G transform according to

$$a'=pa,\quad b'=pb+q$$

so that

$$da'=p\ da,\quad db'=p\ db$$

then

$$da'\Lambda \mathrm{db}'=P^2 da\Lambda db=\left(\frac{a'}{a}\right)^2 da\Lambda db$$

and we get

$$\frac{da'\Lambda db'}{(a')^2}=\frac{da\Lambda db}{a^2}.$$

This 2-form is invariant under the transformation (12). We obtained

$$(14)\qquad dG=\frac{da\Lambda db}{a^2}$$

c) Now we want to express the density in terms of the affine parameter u. We already have (11)

$$A=\begin{bmatrix} a\ sh\ u & b\pm a\ ch\ u \\ 0 & 1 \end{bmatrix} \text{ and } A^{-1}=\begin{bmatrix} \dfrac{1}{a\ sh\ u} & \dfrac{-b\ a\ ch\ u}{a\ sh\ u} \\ 0 & 1 \end{bmatrix}$$

hence

$$A^{-1}\cdot dA=\begin{bmatrix} \dfrac{ch\ u}{sh\ u}du+\dfrac{da}{a} & \dfrac{-ub\mp ch\ u\ da\mp a\ sh\ u\ du}{a\ sh\ u} \\ 0 & 0 \end{bmatrix}$$

From [5] it is known that

$$dG=-\frac{ch\ u}{a\ sh^2\ u}du\Lambda db\mp\frac{da\Lambda db}{a^2\ shu}\mp\frac{du\Lambda da}{a\ sh^2\ u}.$$

In part b) we showed (14), so we get

$$dG\left(1-\frac{1}{sh\ u}\right)=-\frac{ch\ u}{a\ sh^2\ u}du\Lambda db\mp\frac{du\Lambda da}{a\ sh^2\ u}$$

and, finally

$$(15)\qquad dG=\frac{ch\ u\ du\Lambda db\mp du\Lambda da}{a(sh\ u-1)}$$

d) It is known that the equation of G is $(x-b)^2-y^2=a^2$ and differenciating $(x-b)(dx-db)-y\ dy=a\ da$. Using this and (14) we obtain the density of geodesics in terms of the coordinates x, y, and of parameter b:

$$(16)\qquad dG=\frac{(x-b)dx\Lambda db\mp ydy\Lambda db}{((x-b)^2-y^2)^{3/2}}$$

e) We want to obtain an expression of dG in terms of double numbers.

The geodesics $|z|=1$ where $z=x+jy$, $j^2=1$ and $y>0$ is invariant under the subgroup G_1 of G given by $a=-q$, $b=p$ or $a=q$, $b=-p$.

In view of (7), this is equivalent to $w_1=0$, $w_2+w_3=0$ and $d(w_1\Lambda(w_2+w_3))=0$, then the invariant density of G/G_1 is

$$d(G/G_1)=w_1\Lambda(w_2+w_3).$$

An element of $SL(2)$ transforms the L-P geodesic $|z|=1$ on another one with center $(\xi, 0)$ and radius r. To obtain ξ and r we have that

$$\frac{az+b}{pz+q}=w \text{ then } \frac{(qw-b)(q\bar{w}-b)}{(-pw+a)(-p\bar{w}+a)}=1 \text{ and}$$
$$(q^2-p^2)w\ \bar{w}+(ap-bp)(w+\bar{w})+b^2-a^2=0$$

Since $aq-pb=1$, it follows that

$$\xi=\frac{ap-bp}{q^2-p^2} \text{ and } r=\frac{1}{q^2-p^2}$$

We know that up to a constant factor, the invariant density for set of geodesics is

$$d(G/G_1)=\frac{dr\Lambda d\xi}{2r^2}=dG \tag{17}$$

Density of pairs of points

Let P_1 and P_2 be two points of the L-P plane. There is a geodesic G which passes through these two points.

We consider

$$P_1\begin{cases}x_1=b\pm a\ ch\ u_1\\ y_1=a\ sh\ u_1\end{cases} \qquad P_2\begin{cases}x_2=b\pm a\ ch\ u_2\\ y_2=a\ sh\ u_2\end{cases}$$

They satisfy the equation

$$(x_1-b)^2-y_1^2=(x_2-b)^2-y_2^2=a^2$$

or equivalently

$$((x_1-b)+(x_2-b))(x_1-x_2)=y_1^2-y_2^2$$

that is

$$\frac{1}{2}\left(x_1+x_2-\frac{y_1^2-y_2^2}{x_1-x_2}\right)=b$$

where b is unique because $y_1>0$ and $y_2>0$. Substitution in the general equation gives

$$\frac{1}{2}\left(x_1-x_2+\frac{y_1^2-y_2^2}{x_1-x_2}\right)-y_1^2=a^2.$$

As $a>0$, a is also unique and the geodesic is well determined. Computing

$$dx_1=db\pm ch\ u_1\ da\pm a\ sh\ u_1\ du_1$$
$$dy_1=sh\ u_1\ da+a\ ch\ u_1\ du_1$$
$$dx_2=db\pm ch\ u_2 da\pm a\ sh\ u_2 du_2$$
$$dy_2=sh\ u_2 da+a\ ch\ u_2 du_2$$

From (9)

$$dP_i=\frac{dx_i\Lambda dy_i}{y_i^2} \quad \text{for } i=1,2 \quad \text{and } u_1>0.$$

$$dP_1=\frac{sh\ u_1 db\Lambda da-a\ da\Lambda du_1+a\ ch\ u_1 db\Lambda du_1}{a^2 sh^2 u_1}$$

and

$$dP_2=\frac{sh\ u_2 db\Lambda da-a\ da\Lambda du_2+a\ ch\ u_2 db\Lambda du_2}{a^2 sh^2 u_2}.$$

The density of pairs of points is

$$dP_1\Lambda dP_2=\frac{ch\ u_2-ch\ u_1}{a^2 sh^2 u_1 sh^2 u_2}du_1\Lambda du_2\Lambda da\Lambda db$$

or equivalently

$$dP_1\Lambda dP_2=\frac{ch\ u_2-ch\ u_1}{sh^2 u_1 sh^2 u_2}du_1\Lambda du_2\Lambda dG \tag{18}$$

where G is the geodesic through P_1 and P_2. As a kind of duality we look for.

Density of pair of geodesic

We assume G_1 and G_2 are each a branch of timelike geodesic. From (14) we know that density of geodesic $dG_i=\frac{da_i\Lambda db_i}{a_i^2}$ where b_i is its center and a_i its radius, $i:1,\ 2$.

Considering the geodesic defined by $(x-b_i)^2-y^2=a_i^2,\ i:1,\ 2$ we already know their densities, (16), i. e.,

$$dG_i=\frac{(x-b_i)dx\Lambda db_i-y\ dy\Lambda db_i}{((x-b_i)^2-y^2)^{3/2}}\qquad i=1,\ 2$$

and the density of pairs of geodesics is given by

$$dG_1\Lambda dG_2=\frac{(b_2-b_1)y\ db_1\Lambda db_2\Lambda dx\Lambda dy}{(((x-b_1)^2-y^2)((x-b_2)^2-y^2))^{3/2}}.$$

Equivalently,

$$dG_1\Lambda dG_2=\frac{(b_2-b_1)y^3 db_1\Lambda db_2\Lambda dP}{(((x-b_1)^2-y^2)((x-b_2)^2-y^2))^{3/2}}$$

where P is the point of intersection of G_1 and G_2.

Kinematic density

According to [5], chapters 15 and 18, and [2], the kinematic density is

$$\begin{aligned}dK=w_1\Lambda w_2\Lambda w_3&=(q\ da-b\ dp)\Lambda(q\ db-b\ dq)\Lambda(-p\ da+a\ dp)\\&=q(aq-pb)da\Lambda db\Lambda dp+b(aq-pb)da\Lambda dp\Lambda dq\end{aligned}$$

$$=(b\,dq-q\,db)\Lambda da\Lambda dp.$$

A motion in the L-P plane can be represented by

$$x'=ch\,\alpha\cdot x+sh\,\alpha\cdot y+a$$
$$y'=sh\,\alpha\cdot x+ch\,\alpha\cdot y+b$$

with $y>0$, $y'>0$. This is essentially the motion on the Lorentzian plane, [3]; consequently we can assert that Poincaré's formula holds as in [2] but only for two timelike curves. Thereupon, the statement of this result is.

PROPOSITION. Let Γ_0 and Γ_1 be two timelike curves. Suppose that the lenght of Γ_i is L_i and T_i is the tangent line to Γ_i at the point of intersection of these curves, for $i:0,\ 1$. Then

$$\int dK=4L_0L_1$$
$$\Gamma_0\cap\Gamma_1\neq\phi,\ 0<(T_0,\ T_1)<arc\,ch\,5$$

where $(T_1,\ T_0)$ denote the angle from T_1 to T_0.

3. An integral formula

THEOREM. *Let C be a simple, closed curve in the L-P plane, which is the border of convex set K of area F. Let σ be the lenght of the geodesic segment obtained by the intersection of branch of geodesic G with K then*

$$\int_{G\cap K\neq\phi}(-\sigma+sh\sigma)dG=\frac{1}{2}F^2$$

PROOF. Let P_1 and P_2 be two points of K and G the branch of geodesic determined by them. Let u_1 and u_2 be these abscisas on G of P_1 and P_2 varying on $[\alpha,\ \beta]$ with $\alpha>0$. It is well know that

$$\int dP_1\Lambda dP_2=F^2.$$
$$P_1,\ P_2\in K$$

From (18) we get

$$\int_{G\cap K=\phi}\int_\alpha^\beta\int_\alpha^\beta\frac{ch\,u_1-ch\,u_2}{sh^2u_1sh^2u_2}du_1\Lambda du_2\Lambda dG=F^2.$$

First, we will compute

$$\frac{1}{2}\int_\alpha^\beta\int_\alpha^\beta\frac{ch\,u_1-ch\,u_2}{sh^2u_1sh^2u_2}du_1\Lambda du_2$$
$$=\int_\alpha^\beta du_1\left(\int_\alpha^{u_1}\frac{ch\,u_1du_2}{sh^2u_1sh^2u_2}-\int_\alpha^{u_1}\frac{ch\,u_2du_2}{sh^2u_1sh^2u_2}\right)$$

$$=\int_\alpha^\beta\left(\frac{ch\ u_1}{sh^2 u}\left(-coth\ u_2\Big|_\alpha^{u_1}\right)-\frac{1}{sh^2 u_1}\left(-\frac{1}{sh\ u_2}\Big|_\alpha^{u_1}\right)\right)du_1$$
$$=\int_\alpha^\beta\left(\frac{ch\ u_1}{sh^2 u_1}(coth\ \alpha-coth\ u_1)-\frac{1}{sh^2 u_1}\left(\frac{1}{sh\ \alpha}-\frac{1}{sh\ u_1}\right)\right)du_1$$
$$=coth\ \alpha\left(-\frac{1}{sh\ u_1}\Big|_\alpha^\beta\right)-\int_\alpha^\beta\frac{ch\ u_1}{sh^3 u_1}du_1-\frac{1}{sh\ \alpha}\left(-coth\ u_1\Big|_\alpha^\beta\right)+\int_\alpha^\beta\frac{du_1}{sh^3 u_1}$$
$$=coth\ \alpha\left(\frac{1}{sh\ \alpha}-\frac{1}{sh\ \beta}\right)-\int_\alpha^\beta\frac{ch^2 u_1}{sh^3 u_1}du_1+\frac{coth\ \beta}{sh\ \alpha}+\frac{coth\ \alpha}{sh\ \alpha}+\frac{du_1}{sh^3 u_1}$$
$$=\frac{coth\ \alpha}{sh\ \beta}-\frac{coth\ \beta}{sh\ \alpha}+\int_\alpha^\beta\frac{1-ch^2 u_1}{sh^3 u_1}du_1$$
$$=\frac{coth\beta\cdot sh\beta-sh\alpha\ coth\alpha}{sh\beta\cdot sh\alpha}-\int_\alpha^\beta\frac{sh^2 u_1}{sh^3 u_1}du_1=\frac{ch\beta-ch\alpha}{sh\beta\ sh\alpha}-\int_\alpha^\beta\frac{du_1}{sh\ u}$$
$$=\frac{ch\beta-ch\alpha}{sh\beta\cdot sh\alpha}-ln\ th\frac{u_1}{2}\Big|_\alpha^\beta=\frac{ch\beta-ch\alpha}{sh\beta\cdot sh\alpha}+ln\left(\frac{th\alpha/2}{th\beta/2}\right).$$

Replacing this in (19) we have

$$\int_{G\cap K\neq\phi}\left(\frac{ch\beta-ch\alpha}{sh\alpha\ sh\beta}+ln\left(\frac{th\alpha/2}{th\beta/2}\right)\right)dG=\frac{1}{2}F^2.$$

As σ is the geodesic distance, we have

$$\int_\alpha^\beta ds=\sigma=ln\ th\frac{\beta}{2}-ln\ th\frac{\alpha}{2}$$

we want to express $\dfrac{ch\beta-ch\alpha}{sh\beta\cdot sh\alpha}$ in terms of σ.

Since

$$e^\sigma=\frac{th\beta/2}{th\alpha/2}=\frac{(ch\beta-1)(ch\alpha+1)}{sh\alpha\cdot sh\beta}$$

we obtain

$$\frac{ch\beta-ch\alpha}{sh\beta\cdot sh\alpha}=e^\sigma+\frac{1-ch\alpha\cdot ch\beta}{sh\alpha\cdot sh\beta}$$

and

$$\frac{ch\beta-ch\alpha}{sh\beta\cdot sh\alpha}=e^\sigma-\frac{1}{2}(e^\sigma+e^{-\sigma})=sh\ \sigma.$$

Coming back to (20) we find the thesis.

REMARKS. 1) Under the above hypothesis and fixing the radio of G we have

$$\int_{G\cap K\neq\phi}\sigma\ db=F$$

but we can say nothing about $\int \sigma \, dG$ because the integral $\int_{G \cap K \neq \phi} \frac{da}{a^2}$ depends on the positions of the convex set K.

2) This integral formula looks like a Santaló's one, [5] pag. 317, but the difference should be noted: The curvature of the L-P plane is 1.

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