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# An example of a regular Cantor set whose difference set is a Cantor set with positive measure

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*Dedicated to Professor Haruo Suzuki on his 60th birthday*

## § 0. Introduction

In this paper, we give an example of a regular Cantor set whose self-difference set is a Cantor set and, at the same time, has a positive measure. This is a counter example of one of the questions posed by J. Palis related to homoclinic bifurcation of surface diffeomorphisms.

In [PT], Palis-Takens investigated homoclinic bifurcation in the following context. Let $M$ be a closed 2-dimensional manifold. We say a $C^r$-diffeomorphism $\phi: M\to M$ is *persistently hyperbolic* if there is a $C^r$-neighborhood $\mathscr{U}$ of $\phi$ such that for every $\psi\in\mathscr{U}$, the non-wandering set $\Omega(\psi)$ is a hyperbolic set (refer [PM] for the definitions and the notations of the terminologies of dynamical systems). Let $\{\phi_\mu\}_{\mu\in\mathbf{R}}$ be a 1-parameter family of $C^2$-diffeomorphisms on $M$. We say $\{\phi_\mu\}_{\mu\in\mathbf{R}}$ has a *homoclinic $\Omega$-explosion* at $\mu=0$ if:

(i) For $\mu<0$, $\phi_\mu$ is persistently hyperbolic;

(ii) For $\mu=0$, the non-wandering set $\Omega(\phi_0)$ consists of a (closed) hyperbolic set $\tilde{\Omega}(\phi_0)=\lim_{\mu\uparrow 0}\Omega(\phi_\mu)$ together with a homoclinic orbit of tangency $\mathscr{O}$ associated with a fixed saddle point $p$, so that $\Omega(\phi_0)=\tilde{\Omega}(\phi_0)\cup\mathscr{O}$ ; the product of the eigenvalues of $d\phi_0$ at $p$ is different from one in norm;

(iii) The separatrices have quadratic tangency along $\mathscr{O}$ unfolding generically; $\mathscr{O}$ is the only orbit of tangency between stable and unstable separatrices of periodic orbits of $\phi_0$.

Let $\Lambda$ be a basic set of a diffeomorphism $\psi$ on $M$. $d^s(\Lambda)$ ($d^u(\Lambda)$) denotes the Hausdorff dimension in the transversal direction of the stable (unstable) foliation of the stable (unstable) manifold of $\Lambda$ (refer [PM] for the precise definition), and is called the stable (unstable) *limit capacity*. Let $B$ denote the set of values $\mu>0$ for which $\phi_\mu$ is not persistently hyperbolic.

The result of Palis-Takens is;

THEOREM [1]. *Let* $\{\phi_\mu;\mu\in\boldsymbol{R}\}$ *be a family of diffeomorphisms of* $M$ *with a homoclinic* $\Omega$*-explosion at* $\mu=0$. *Suppose that* $d^s(\Lambda)+d^u(\Lambda)<1$, *where* $\Lambda$ *is the basic set of* $\phi_0$ *associated with the homoclinic tangency. Then*

$$\lim_{\delta\to 0}\frac{m(B\cap[0,\delta])}{\delta}=0$$

*where* $m$ $(\cdot)$ *denotes Lebesgue measure.*

This result states that, in the case of $d^s(\Lambda)+d^u(\Lambda)<1$, the measure of the parameters for which bifurcation occurs is relatively small. For the next step, the case of $d^s(\Lambda)+d^u(\Lambda)>1$ comes into question. In the proof of the theorem above, one of the essential points is the question of how two Cantor sets intersect each other when the one Cantor set is slided. In [P], Palis posed the following questions.

(Q. 1): For affine Cantor sets $X$ and $Y$ in the line, is it true that $X-Y$ either has measure zero or contains intervals?

(Q. 2): Same for regular Cantor sets, where for two subsets $X,Y$ of $\boldsymbol{R}$.,

$$X-Y=\{x-y|x\in X,y\in Y\}.$$

This can be also written as;

$$X-Y=\{\mu\in\boldsymbol{R}|X\cap(\mu+Y)\neq\phi\},$$

namely, $X-Y$ is the set of parameters at which $X$ and $Y$ have an intersection point when $Y$ is slided. Refer [L] for more detailed and intelligible exposition for these questions.

Cantor set $\mathscr{C}$ in $\boldsymbol{R}$ is called affine (regular or $C^r$ for $1<r\leq\infty$) if $\mathscr{C}$ is difined with finite number of expanding affine ($C^2$ or $C^r$) maps, namely;

DEFINITION. *Let* $\mathscr{C}$ *be a Cantor set on a closed interval* $I$. *For* $1\leq r\leq\infty$, $\mathscr{C}$ *is called* $C^r-$*Cantor set if there are closed disjoint intervals* $I_1,\ldots,I_k$ *on* $I$ *and onto* $C^r-$*maps* $g_i: I_i\to I$ *for all* $1\leq i\leq k$ *such that;*

(i) $|g_i'(x)|>1 \quad \forall x\in I_i$,

(ii) $\mathscr{C}=\bigcap_{n=0}^{\infty}\{\bigcup_{\sigma\in\Sigma_n^k} g_{\sigma(1)}^{-1}g_{\sigma(2)}^{-1}\cdots g_{\sigma(n)}^{-1}(I)\}$,
*where* $\Sigma_n^k=\{\sigma:\{1,\cdots,n\}\to\{1,\cdots,k\}\}$.

Our result in this paper claims that there is a counter example of (Q. 2), namely;

THEOREM. *There exists a $C^\infty$-Cantor set $\mathscr{C}$ such that*
(i) $m(\mathscr{C}-\mathscr{C})>0$,
(ii) $\mathscr{C}-\mathscr{C}$ *is a Cantor set.*

As is mentioned above, if the sum of limit capacities is less than 1, then the measure of parameter values at which the diffeomorphism has a homoclinic tangency is asymptotically zero. What happens if $d^s(\Lambda)+d^u(\Lambda)>1$? One of the questions about this situation is the following.

(Q. 3) : Is there any one-parameter family of plane diffeomorphisms $f_\mu$ such that the measure of parameters of homoclinic tangency is positive while the set of parameters of persistent hyperbolicity is dense?

Our example $\mathscr{C}$ of the theorem might be applicable to construct an example for this (Q. 3). However, when we try to embed the Cantor set $\mathscr{C}$ as transversal Cantor sets of the stable and unstable foliations of a family of diffeomorphisms, new bumps of stable and unstable manifolds grow up after the first tangency and it is very difficult to know whether those new and very thinly stretched bumps have tangency or not. Thus, the application of our theorem to plane diffeomorphisms does not seem to be easy. In the case of $d^s(\Lambda)+d^u(\Lambda)>1$, for the practical application to homoclinic bifurcation, the problems of "genericity" or "openness" may have more importance.

One will see in the proof of this theorem that $\mathscr{C}$ is constructed very artificially and cannot be defined as an *analytic* Cantor set. Therefore, this theorem may not give any clue to (Q. 1), i. e. the affine case. In fact, the affine case seems to have an essential difficulty of these problems.

In [MO], P. Mendes and F. Oliveira have got some partial answers to the affine case. P. Larsson [L] proved that if the sum of the Hausdorff dimensions is bigger than 1, then almost surely, the difference set of two "random Cantor sets" contains an interval.

In the succeeding sections, we shall give the proof of our theorem.

## § 1. Definition of the Cantor sets $\mathscr{C}(s)$, $\mathscr{D}(s)$

One of the most typical methods of constructing Cantor sets is recursive process of "removing the middle part of interval". By assigning the ratio of the length of the intervals which are left in each step, we can construct a Cantor set depending on a given sequence of positive numbers as follows.

DEFINITION 1.1. *Let $I=[x_1, x_2]$ be a closed interval and $\lambda$ a real number with $0<\lambda<\frac{1}{2}$. We define,*

$$I_0(\lambda\,; I)=[x_1,\ x_1+\lambda(x_2-x_1)]$$
$$I_1(\lambda\,; I)=[x_2-\lambda(x_2-x_1),\ x_2].$$

DEFINITION 1.2 (*Cantor set* $\mathscr{C}(s)$). *Let* $I^0=[0,1]$ *and* $s=(\lambda_1, \lambda_2, \lambda_3, \cdots)$ *be a one sided sequence of real numbers with* $0<\lambda_i<\frac{1}{2}$ *for all* $i\geq 1$. *We define the Cantor set* $\mathscr{C}(s)$ *as follows.*

*Let* $I_0^1=I_0(\lambda_1\,; I^0)$, $I_1^1=I_1(\lambda_1\,; I^0)$ *and* $I^1=I_0^1\cup I_1^1$. $\Delta_n$ *denotes the set of all sequences of* 0 *and* 1 *of length* $n$. *When* $I_\beta^{n-1}$ *'s are defined for all* $\beta\in\Delta_{n-1}$, *we define ;*

$$I_{\beta 0}^n=I_0(\lambda_n\,; I_\beta^{n-1})$$
$$I_{\beta 1}^n=I_1(\lambda_n\,; I_\beta^{n-1}).$$

*Inductively, we can define* $I_\alpha^n$ *for all* $\alpha\in\Delta_n$ *and for all* $n\geq 0$. *Define*

$$I^n=\bigcup_{\alpha\in\Delta_n} I_\alpha^n$$

*and*

$$\mathscr{C}(s)=\bigcap_{n\geq 0} I^n.$$

*This is clearly a Cantor set by the definition.*

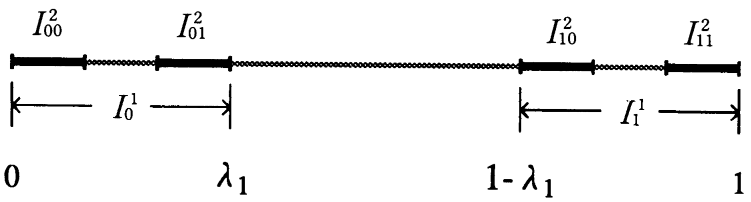


Figure 1

Our Cantor set $\mathscr{C}$ in the main theorem will be given as one of such Cantor sets $\mathscr{C}(s)$ for an appropriate $s=(\lambda_1, \lambda_2, \cdots)$. Our next claim is that if $0<\lambda_i<\frac{1}{3}$ for all $i$, then the difference set $\mathscr{C}(s)-\mathscr{C}(s)$ is also a Cantor set with a neat structure.

DEFINITION 1.3. *Let* $J=[x_1, x_2]$ *and* $0<\lambda<\frac{1}{3}$, *We define,*

$$J_0(\lambda\,; J)=[x_1, x_1+\lambda(x_2-x_1)]$$
$$J_1(\lambda\,; J)=\left[\frac{x_1+x_2}{2}-\frac{\lambda}{2}(x_2-x_1),\ \frac{x_1+x_2}{2}+\frac{\lambda}{2}(x_2-x_1)\right]$$
$$J_2(\lambda\,; J)=[x_2-\lambda(x_2-x_1),\ x_2].$$

DEFINITION 1.4. *Let* $J^0=[-1,1]$ *and* $s=(\lambda_1, \lambda_2, \lambda_3, \cdots)$ *be a one sided* *sequence of real numbers with* $0<\lambda_i<\frac{1}{3}$ *for all* $i\geq 1$. *Let*

$$J_0^1=J_0(\lambda_1\,;J^0)$$
$$J_1^1=J_1(\lambda_1\,;J^0)$$
$$J_2^1=J_2(\lambda_1\,;J^0)$$

*and* $\Pi_n$ *denote the set of all sequences of* $0,1,2$ *of length* $n$. *When* $J_\delta^{n-1}$*'s are defined for all* $\delta\in\Pi_{n-1}$, *we define ;*

$$J_{\delta 0}^n=J_0(\lambda_n\,;J_\delta^{n-1})$$
$$J_{\delta 1}^n=J_1(\lambda_n\,;J_\delta^{n-1})$$
$$J_{\delta 2}^n=J_2(\lambda_n\,;J_\delta^{n-1}).$$

*Inductively, we can define* $J_\gamma^n$ *for all* $\gamma\in\Pi_n$ *and for all* $n\geq 0$. *Define*

$$J^n=\bigcup_{\gamma\in\Pi_n} J_\gamma^n$$

*and*

$$\mathscr{D}(s)=\bigcap_{n\geq 0} J^n$$

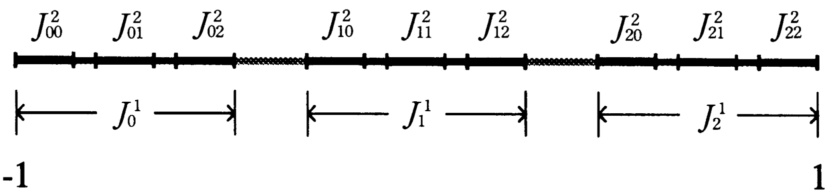


Figure 2

THEOREM A. *A. Let* $s=(\lambda_1,\lambda_2,\lambda_3,\cdots)$ *be a sequence of real numbers with* $0<\lambda_i<\frac{1}{3}$ *for all* $i\geq 1$. *then,*

$$\mathscr{C}(s)-\mathscr{C}(s)=\mathscr{D}(s).$$

PROOF :

$$\mathscr{C}(s)-\mathscr{C}(s)=(\bigcap_{n\geq 0} I^n)-(\bigcap_{n\geq 0} I^n).$$

By a straightforward argument, it can be seen that,

$$(\bigcap_{n\geq 0} I^n)-(\bigcap_{n\geq 0} I^n)=\bigcap_{n\geq 0}(I^n-I^n).$$

Therefore, it is enough to show that

$$I^n-I^n=J^n \qquad \forall\, n\geq 0.$$

By the definition of $I^n$ and $J^n$, that is an easy consequence of the following lemma 1. 5.

LEMMA 1.5. *For all* $n\geq 0$,

(i) *for all* $\alpha, \beta\in\Delta_n$, *there exists a* $\gamma\in\Pi_n$ *such that*

$$I_\alpha^n - I_\beta^n = J_\gamma^n$$

(ii) *for all* $\gamma\in\Pi_n$, *there exist* $\alpha, \beta\in\Delta_n$ *such that*

$$J_\gamma^n = I_\alpha^n - I_\beta^n.$$

PROOF:

Note that for arbitrary two closed intervals $I=[x_1, x_2]$ and $J=[y_1, y_2]$, $I-J=[x_1-y_2, x_2-y_1]$.

We prove (i) and (ii) simultaneously by induction.

When $n=0$, the statement holds, because $I^0-I^0=J^0$. Assume that the statement is valid for $n$.

Let $\alpha, \beta\in\Delta_{n+1}$ and $\alpha=\tilde{\alpha}\alpha_{n+1}$, $\beta=\tilde{\beta}\beta_{n+1}$ for $\tilde{\alpha}, \tilde{\beta}\in\Delta_n$ and $\alpha_{n+1}, \beta_{n+1}=0$ *or* 1. Then, by the hypothesis of induction, there exists a $\tilde{\gamma}\in\Pi_n$ such that $I_{\tilde{\alpha}}^n - I_{\tilde{\beta}}^n = J_{\tilde{\gamma}}^n$.

On the other hand, let $\gamma\in\Pi_{n+1}$ and $\gamma=\tilde{\gamma}\gamma_{n+1}$ for some $\tilde{\gamma}\in\Pi_n$ and $\gamma_{n+1}=0$ *or* 1. Then, by the hypothesis of induction, there exist $\tilde{\alpha}, \tilde{\beta}\in\Delta_n$ such that $I_{\tilde{\alpha}}^n - I_{\tilde{\beta}}^n = J_{\tilde{\gamma}}^n$.

Thus in both cases (i) and (ii), the statement of lemma 1.5 is obtained from the following lemma 1.6.

LEMMA 1.6. *Suppose that*

$$I_{\tilde{\alpha}}^n - I_{\tilde{\beta}}^n = J_{\tilde{\gamma}}^n$$

*for* $\tilde{\alpha}, \tilde{\beta}\in\Delta_n$ *and* $\tilde{\gamma}\in\Pi_n$. *Then,*

$$J_{\tilde{\gamma}0}^{n+1} = I_{\tilde{\alpha}0}^{n+1} = I_{\tilde{\beta}1}^{n+1}$$
$$J_{\tilde{\gamma}1}^{n+1} = I_{\tilde{\alpha}0}^{n+1} - I_{\tilde{\beta}0}^{n+1} = I_{\tilde{\alpha}1}^{n+1} - I_{\tilde{\beta}1}^{n+1}$$
$$J_{\tilde{\gamma}2}^{n+1} = I_{\tilde{\alpha}1}^{n+1} - I_{\tilde{\beta}0}^{n+1}.$$

PROOF: Let $I_{\tilde{\alpha}}^n=[x_1, x_2]$ and $I_{\tilde{\beta}}^n=[y_1, y_2]$. Then, $J_{\tilde{\gamma}}^n=[x_1-y_2, x_2-y_1]$. Since the length of $I_{\tilde{\alpha}}^n$ and $I_{\tilde{\beta}}^n$ are the same, we denote $\ell_n=x_2-x_1=y_2-y_1$. By the definition,

$$I_{\tilde{\alpha}0}^{n+1}=[x_1, x_1+\lambda_{n+1}\ell_n]$$
$$I_{\tilde{\alpha}1}^{n+1}=[x_2-\lambda_{n+1}\ell_n, x_2]$$
$$I_{\tilde{\beta}0}^{n+1}=[y_1, y_1+\lambda_{n+1}\ell_n]$$
$$I_{\tilde{\beta}1}^{n+1}=[y_2-\lambda_{n+1}\ell_n, y_2].$$

Therefore we have,

$$I^{n+1}_{\tilde{\alpha}0}-I^{n+1}_{\tilde{\beta}0}=[x_1-y_1-\lambda_{n+1}\ell_n,\ x_1-y_1+\lambda_{n+1}\ell_n]$$
$$I^{n+1}_{\tilde{\alpha}0}-I^{n+1}_{\tilde{\beta}1}=[x_1-y_2,\ x_1-y_2+2\lambda_{n+1}\ell_n]$$
$$I^{n+1}_{\tilde{\alpha}1}-I^{n+1}_{\tilde{\beta}0}=[x_2-y_1-2\lambda_{n+1}\ell_n,\ x_2-y_1]$$
$$I^{n+1}_{\tilde{\alpha}1}-I^{n+1}_{\tilde{\beta}1}=[x_2-y_2-\lambda_{n+1}\ell_n,\ x_2-y_2+\lambda_{n+1}\ell_n].$$

On the other hand, by the definition,

$$J^{n+1}_{\tilde{\gamma}0}=[x_1-y_2,\ x_1-y_2+2\lambda_{n+1}\ell_n]$$
$$J^{n+1}_{\tilde{\gamma}1}=[\frac{1}{2}(x_1-y_2+x_2-y_1)-\lambda_{n+1}\ell_n,\ \frac{1}{2}(x_1-y_2+x_2-y_1)+\lambda_{n+1}\ell_n]$$
$$J^{n+1}_{\tilde{\gamma}2}=[x_2-y_1-2\lambda_{n+1}\ell_n,\ x_2-y_1].$$

Since $\frac{1}{2}(x_1-y_2+x_2-y_1)=x_1-y_1$, the statement is obtained. □

The combination of Theorem A and the following Theorem B yields our main Theorem.

THEOREM B. *There exists a sequence of real numbers* $s=(\lambda_1,\lambda_2,\lambda_3,\cdots)$ *with* $0<\lambda_i<\frac{1}{3}$ *for all* $i\geq 1$ *such that ;*

(i) $m(\mathscr{C}(s)-\mathscr{C}(s))>0$,
(ii) $\mathscr{C}(s)$ *is a* $C^\infty$*−Cantor set.*

In the rest of this paper, we shall prove this Theorem B.

## § 2. Positivity of the measure

In general, the measure of $\mathscr{D}(s)$ is given as follows.

LEMMA 2. 1. *Let* $s=(\lambda_1,\lambda_2,\lambda_3,\cdots)$ *be a sequence of real numbers such that* $0<\lambda_n<\frac{1}{3}$ *for all* $n\geq 1$. *Then,*

$$m(\mathscr{D}(s))=2(1-\sum_{n=0}^{\infty}(3^n(1-3\lambda_{n+1})\prod_{j=1}^{n}\lambda_j)).$$

PROOF: Let $w_n$ denotes the length of each interval of $J^n$. For example, $w_0=2$, $w_1=2\lambda_1$, $w_2=\lambda_2 w_1=2\lambda_1\lambda_2$. In general, $w_n=\lambda_n w_{n-1}$, and,

$$w_n=2\prod_{j=1}^{n}\lambda_j.$$

In each interval of $J^{n-1}$, there are three intervals of $J^n$ and therefore, there are two gaps in it. The sum of the lengths of these gaps in $J^{n-1}$ is

$$w_{n-1}-3w_n.$$

Since there are $3^{n-1}$ intervals in $J^{n-1}$, the sum of the lengths of the open gaps of the n-th level is

$$3^{n-1}(w_{n-1}-3w_n).$$

Therefore, the sum of the lengths of the all open gaps is,

$$\begin{aligned}&\sum_{n=0}^{\infty}3^n(w_n-3w_{n+1})\\ &=\sum_{n=0}^{\infty}3^n w_n(1-3\lambda_{n+1})\\ &=2\sum_{n=0}^{\infty}3^n(1-3\lambda_{n+1})\prod_{j=1}^{n}\lambda_j. \quad \square\end{aligned}$$

It is convenient to introduce another sequence of positive numbers to define the sequence $s=(\lambda_1, \lambda_2, \cdots)$ for $\mathscr{C}(s)$ in Theorem B.

DEFINITION 2.2. *Let $\rho=\{r_n\}_{n\geq 0}$ be a sequence of positive real numbers such that*

(1) $$\sum_{n=0}^{\infty}r_n<1.$$

*We define a sequence of positive real numbers $s(\rho)=\{\lambda_n\}_{n\geq 1}$ depending on $\rho$ as follows.*

(2) $$\begin{cases}\lambda_1=\dfrac{1}{3}(1-r_0)\\ \lambda_{n+1}=\dfrac{1}{3}\left(\dfrac{1-\sum_{i=0}^{n}r_i}{1-\sum_{i=0}^{n-1}r_i}\right)\end{cases}\qquad \forall n\geq 1.$$

It is clear that

(3) $$0<\lambda_n<\frac{1}{3}\qquad \forall n\geq 1.$$

LEMMA 2.3. *Let $\rho=\{r_n\}_{n\geq 0}$ and $s(\rho)=\{\lambda_n\}_{n\geq 1}$ be sequences as in Definition 2.2.*

(i) $$\sum_{i=0}^{n}r_i=1-3^{n+1}\prod_{j=1}^{n+1}\lambda_j\qquad \forall n\geq 0.$$

(ii) $$r_n=3^n(1-3\lambda_{n+1})\prod_{j=0}^{n}\lambda_j\qquad \forall n\geq 0.$$

*Where, we assume $\lambda_o=1$ for the simplicity of notation.*

PROOF: The proofs are straightforward by induction. $\square$

By this lemma 2. 3 and the proof of lemma 2. 1, one can see the real nature of the sequence $\rho=\{r_n\}_{n\geq 0}$. In fact, $r_n$ represents the ratio of the measure of the set of all the open gaps of the n+1-th leved of $\mathscr{D}(s(\rho))$ in $J^0=[-1, 1]$. Therefore, if $\sum_{n=0}^{\infty} r_n<1$, then by lemma 2. 3 (ii) and lemma 2. 1, $m(\mathscr{D}(s(\rho)))>0$. Thus, what we have to do is to define a special sequence $\rho=\{r_n\}_{n\geq 0}$ so that $\sum_{n=0}^{\infty} r_n<1$ and $\mathscr{C}(s(\rho))$ may be defined as a $C^\infty$-Cantor set.

In order to define $\rho=\{r_n\}_{n\geq 0}$, we need to fix a $C^\infty$-function $h(t)$ on [0, 1] with the following properties.

(i) $h(t)\geq 0 \qquad 0\leq \forall t\leq 1$,

(ii) $\int_0^1 h(t)dt=1$,

(iii) for all $n\geq 0$, $\begin{cases} \lim\limits_{t\downarrow 0} h^{(n)}(t)=0, \\ \lim\limits_{t\uparrow 1} h^{(n)}(t)=0, \end{cases}$

where $h^{(n)}$ denotes the $n$-th derivative of $h$.

(For example,

$$h(t)=\begin{cases} \dfrac{e^{-\frac{1}{t(1-t)}}}{\int_0^1 e^{-\frac{1}{s(1-s)}}ds} & 0<t<1 \\ 0 & t=0, 1. \end{cases}$$

is one of such functions.)

For each integers $n\geq 0$, let

$$q_n=\max\{q_0, q_1, \cdots, q_{n-1}, 1, \sup_{t\in[0,1]}|h^{(n)}(t)|\}.$$

Clearly,

$$1\leq q_0\leq q_1\leq q_2\leq\cdots.$$

For $n\geq 0$, we define,

$$r_n=\frac{4^{-(n^2+2)}}{q_n}.$$

This $\rho=\{r_n\}_{n\geq 0}$ is exactly the sequence we need. Clearly.

$$\frac{1}{16}\geq r_0>r_1>r_2>\cdots.$$

Since $r_n\leq 4^{-(n^2+2)}\leq 4^{-(n+2)}$, we have

$$\sum_{n=0}^{\infty} r_n<\sum_{n=2}^{\infty}\frac{1}{4^n}=\frac{1}{12}. \tag{4}$$

Therefore, $\{r_n\}_{n\geq 0}$ satisfy (1). Let $s(\rho)=\{\lambda_n\}_{n\geq 1}$ be the sequence defined in definition 2.2.

By (2) and (4), we can easily see that $\lambda_n>\frac{1}{4}$ for all $n\geq 1$. So, together with (3), we have.

(5) $$\frac{1}{4}<\lambda_n<\frac{1}{3} \qquad \forall n\geq 1.$$

From now on, we fix $\rho$ and $s(\rho)$, and denote $s(\rho)$ just $s=(\lambda_1, \lambda_2, \cdots)$. In the following sections, we shall prove that $\mathscr{C}(s)$ is a $C^\infty$-Cantor set.

## § 3. The regularity of $\mathscr{C}(s)$

We prove the smoothness of $\mathscr{C}(s)$ as follows. We have to define $C^\infty$-functions $g_0$ and $g_1$ on $I_0^1$ and $I_1^1$ respectively such that the intersection of all the images of the compositions of $g_0^{-1}$ and $g_1^{-1}$ is equal to $\mathscr{C}(s)$. Since $\mathscr{C}(s)$ has the same structure on $I_0^1$ and $I_1^1$, $g_1$ has to be just a translation of $g_0$. Therefore, we have only to define a $C^\infty$-function $g$ on $I_0^1$, and put

$$\begin{cases} g_0(t)=g(t) & on\ [0, \lambda_1] \\ g_1(t)=g(t-1+\lambda\lambda_1) & on\ [1-\lambda_1, 1]. \end{cases}$$

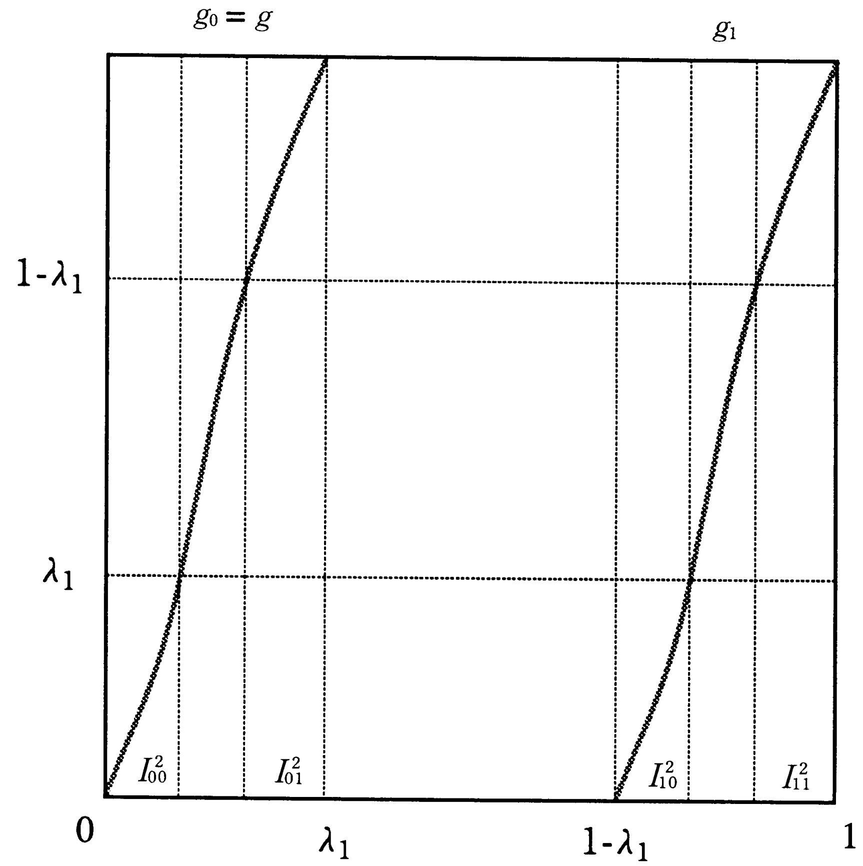


Figure 3

Let $U^0$ denote the open interval between $I_0^1$ and $I_1^1$, namely ;

$$U^0=I^0\backslash(I_0^1\cup I_1^1).$$

In general, let $U_\alpha^n(\alpha\in\Delta_n)$ denote the open interval between $I_{\alpha 0}^{n+1}$ and $I_{\alpha 1}^{n+1}$ in $I_\alpha^n$, namely ;

$$U_\alpha^n=I_\alpha^n\backslash({}_{\alpha 0}^{n+1}\cup I_{\alpha 1}^{n+1}).$$

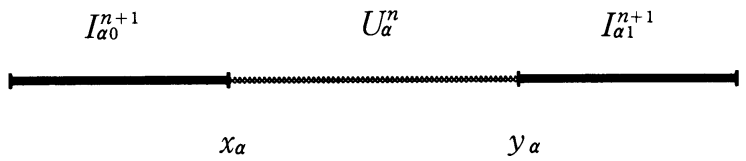


Figure 4

Let $\ell_n=\ell(I_\alpha^n)$, $u_n=\ell(U_\alpha^n)$ and $\overline{U_\alpha^n}=[x_\alpha, y_\alpha]$, where $\ell(\cdot)$ denotes the length of the interval. Then,

(6) $$u_n=\ell_n-2\ell_{n+1}.$$

Note that

(7) $$\ell_n=\lambda_n\ell_{n-1},$$

(8) $$\ell_n=\prod_{j=1}^{n}\lambda_j.$$

What is the shape of $g$ like? Since $g_0$ and $g_1$ have to define $\ell(s)$, $g^{-1}(I_\alpha^n)$ has to be exactly equal to $I_{0\alpha}^{n+1}$, and $g^{-1}(U_\alpha^n)=U_{0\alpha}^{n+1}$.

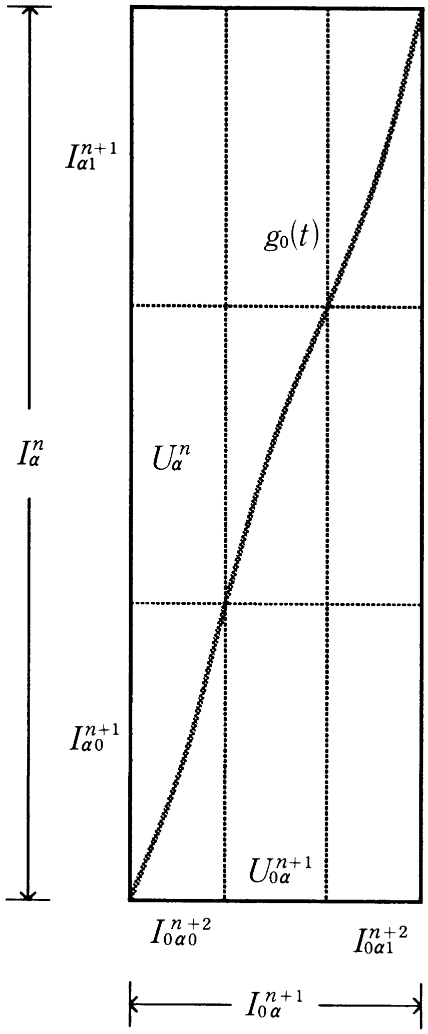


Figure 5

Note that

(9) $$u_n/u_{n+1}>3,$$

because by (6), (7) and Definition 2.2,

$$\begin{aligned} u_n-3u_{n+1} &= \ell_n-2\ell_{n+1}-3(\ell_{n+1}-2\ell_{n+2}) \\ &= \ell_n\{1+\lambda_{n+1}(6\lambda_{n+2}-5)\} \\ &= \ell_n\left\{1+\frac{1}{3}\left(\frac{1-\sum_{i=0}^{n} r_i}{1-\sum_{i=0}^{n-1} r_i}\right)\left\{2\left(\frac{1-\sum_{i=0}^{n+1} r_i}{1-\sum_{i=0}^{n} r_i}\right)-5\right\}\right\} \end{aligned}$$

(10) $$= \frac{\ell_n}{3(1-\sum_{i=0}^{n-1} r_i)}(3r_n-2r_{n+1}).$$

(10) is positive because $\{r_i\}$ is monotonically decreasing.

By (9), the average value of $g'(t)$ on $U_\alpha^n$ has to be more than 3. Therefore it is easier to define a $C^\infty$-function $f(t)$ which has a positive bump on each gap $U_\alpha^n$ and define $g$ as the integral

$$g(t)=\int_0^t (f(s)+3)ds.$$

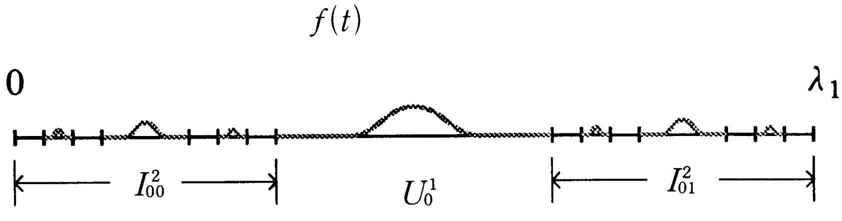


Figure 6

In order to define $f$, we introduce another sequence of positive real numbers ;

$$m_n=\frac{3(3r_{n-1}-2r_n)}{1-\sum_{i=0}^{n-1} r_i} \qquad \forall n\geq 1.$$

Since $\{r_n\}_{n\geq 0}$ is monotonically decreasing and by (4), $m_n>0$ for all $n\geq 1$. Moreover,

$$\begin{aligned}(11) \qquad m_n &< \frac{9r_{n-1}}{1-\sum_{i=0}^{n-1} h_i} \\ &< 10\cdot r_{n-1}.\end{aligned}$$

By (3) and (6), we have

$$u_n>\frac{\ell_n}{3}.$$

Let $[x'_\alpha, y'_\alpha]$ be the interval of lenght $\frac{\ell_n}{3}$ in the middle of $U^n_\alpha$ such that

$$[x'_\alpha, y'_\alpha]=\left[x_\alpha+\frac{1}{2}\left(u_n-\frac{\ell_n}{3}\right),\ y_\alpha-\frac{1}{2}\left(u_n-\frac{\ell_n}{3}\right)\right].$$

DEFINITION OF $f(t)$. *Recall that we have already defined a $C^\infty$-function $h(t)$ on $[0,1]$. We define $f(t)$ on $[0,\lambda_1]$ using this $h(t)$ as follows.*

(i) *On* $U^n_\alpha\cap[0,\lambda_1]$,

$$\begin{cases} f(t)=m_n h\left(\dfrac{t-x'_\alpha}{\frac{\ell_n}{3}}\right) & t\in[x'_\alpha, y'_\alpha] \\ f(t)=0 & \textit{otherwise.}\end{cases}$$

(ii) *On* $\mathscr{C}(s)\cap[0,\lambda_1]$, $f(t)=0$.

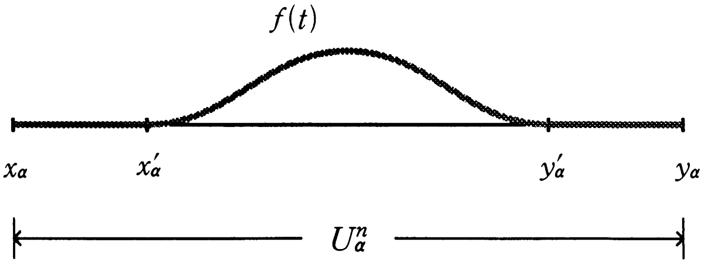


Figure 7

PROPOSITION 3.1. *$f(t)$ is a $C^\infty$-function on* $[0, \lambda_1]$.

PROOF:

For any $p \geq 0$, we define a function $f_p(t)$ as follows. Let $\Delta^0_n = \{\alpha = \alpha_1 \cdots \alpha_n \in \Delta_n | \alpha_1 = 0\}$ and $U = \bigcup_{n \geq 1, \alpha \in \Delta^0_n} U^p_\alpha$, Note that $U = [0, \lambda_1] \setminus \mathscr{C}(s)$. Since $f(t)$ is $C^\infty$ on $U$, $f^{(p)}(t)$ exists for any $p \geq 0$ on $U$. We define,

$$\begin{cases} f_p(t) = f^{(p)}(t) & \text{for } t \in U \\ f_p(t) = 0 & \text{otherwise } (i.e.\ t \in \mathscr{C}(s)). \end{cases}$$

Since $f_0 = f$, in order to show the smoothness of $f(t)$, we shall show that for any $p \geq 0$, $f_p$ is differentiable at any $t \in [0, \lambda_1]$ and $f'_p(t) = f_{p+1}(t)$. That implies $f$ is $C^\infty$.

Now we fix $p \geq 0$. Since $f_p$ is differentiable at any $t \in U$, we have only to show that at any $t \in \mathscr{C}(s) \cap [0, \lambda_1]$, $f_p$ is differentiable and $f'_p(t) = 0$.

Let $t_0 \in \mathscr{C}(s) \cap [0, \lambda_1]$. Since $f_p(t_0) = 0$, it is enough to show that

$$\lim_{t \to t_0} \frac{f_p(t)}{t - t_0} = 0 \tag{12}$$

Assume that $t_0 < t$, namely $t$ approaches to $t_0$ from the above. Therefore, we shall consider only the right side of $t_0$. The similar argument proves another case.

We shall prove that for any $\epsilon > 0$, there exists a $\delta > 0$ such that if $t - t_0 < \delta$, then

$$\frac{f_p(t)}{t - t_0} < \epsilon.$$

Let $\epsilon > 0$ be given. Let $n_0 \geq p + 2$ be an integer such that, for any $n \geq n_0$,

$$10 \cdot 3^{p+1} \cdot 4^{(n(p+1) - (n-1)^2 - 2)} < \epsilon.$$

If $t_0$ is the left end point of some $\overline{U^n_\alpha}$ and $t$ approaches to $t_0$ in $U^n_\alpha$, then (12) is clear because $f(t) \equiv 0$ in a neighbourhood of the end point of $\overline{U^n_\alpha}$.

If it is not the case, since $U_\alpha^{n'}$'s are connected components of the complement of the Cantor set $\mathscr{C}(s)$ and $t_0 \in \mathscr{C}(s)$, infinitely many $U_\alpha^{n'}$'s converge to $t_0$ from the right. Therefore, there exists a $\delta > 0$ satisfying the following property.

(*). *If* $t - t_0 < \delta$ *and* $[t_0, t] \cap U_\alpha^n \neq \emptyset$ *for some* $U_\alpha^n$, *then* $n \geq n_0$.

Suppose that $t - t_0 < \delta$ for this $\delta > 0$. If $t \in \mathscr{C}(s)$, then by the definition of $f_p$, $f_p(t) = 0$. Therefore, we assume that $t \in U_\alpha^n$ for some $n \geq n_0$ and $\alpha \in \Delta_n^0$.

Let $t_1$ be the left end point of $\overline{U_\alpha^n}$. Clearly, $t_0 < t_1 < t$ and

$$\frac{f_p(t)}{t - t_0} < \frac{f_p(t)}{t - t_1}.$$

Since $f_p$ is differentiable on $\overline{U_\alpha^n}$, by the mean value theorem,

$$\begin{aligned} \frac{f_p(t)}{t - t_1} &\leq \sup_{t \in U_\alpha^n} |f_p'(t)| \\ &= \sup_{t \in [x'_\alpha, y'_\alpha]} |f^{(p+1)}(t)| \\ &= m_n \left(\frac{3}{\ell_n}\right)^{p+1} \sup_{t \in [0,1]} h^{(p+1)}(t) \end{aligned} \tag{13}$$

By (5), (8) and (11), we have $\ell_n = \prod_{j=1}^n \lambda_j > \left(\frac{1}{4}\right)^n$ and $m_n < 10 \cdot r_{n-1}$. Therefore,

$$\begin{aligned} (13) &< 10 \cdot r_{n-1} \cdot 3^{p+1} \cdot (4^n)^{p+1}. \{\sup_{t \in [0,1]} h^{(p+1)}(t)\} \\ &= 10 \cdot \frac{4^{-((n-1)^2+2)}}{q_{n-1}} \cdot 3^{p+1} \cdot (4^n)^{p+1} \cdot \{\sup_{t \in [0,1]} h^{(p+1)}(t)\} \end{aligned} \tag{14}$$

Since $n - 1 \geq p + 1$, by the definition of $q_{n-1}$,

$$\begin{aligned} (14) &\leq 10 \cdot 3^{p+1} \cdot 4^{n(p+1)} \cdot 4^{-((n-1)^2+2)} \\ &= 10 \cdot 3^{p+1} \cdot 4^{(n(p+1)-(n-1)^2-2)} < \epsilon. \end{aligned}$$

□

## § 4. $g_0$ and $g_1$ define $\mathscr{C}(s)$

In the previous section, we defined the $C^\infty$-function $f$ and using it, defined $g_0$ and $g_1$. In this section, we shall prove that $\mathscr{C}(s)$ is defined by them, namely;

PROPOSITION 4.1.

$$\mathscr{C}(s) = \bigcap_{n \geq 0} \{\bigcup_{\sigma \in \Sigma_n^2} g_{\sigma(1)}^{-1} g_{\sigma(2)}^{-1} \cdots g_{\sigma(n)}^{-1}(I^0)\}.$$

For the proof, we need some lemmas.

LEMMA 4.2. *For any* $n\geq 1$ *and* $\alpha\in\Delta_n^0$,

$$\int_{U_\alpha^n} f(t)dt=\frac{1}{3}m_n\ell_n.$$

PROOF: It is straightforward by the definition of $f(t)$. □

LEMMA 4.3. *For all* $n\geq 1$,

$$\ell_{n-1}=g(\ell_n).$$

PROOF:

$$\begin{aligned} g(\ell_n)&=\int_0^{\ell_n}(f(t)+3)dt \\ &=3\ell_n+\int_0^{\ell_n}f(t)dt. \end{aligned} \tag{15}$$

In $[0, \ell_n]$, $f(t)$ has positive value only on countable number of open intervals $U_\alpha^k$ such that $U_\alpha^k\subset[0, \ell_n]$. Note that $U_\alpha^k\subset[0, \ell_n]$ for $\alpha=\alpha_1\cdots\alpha_k$ if and only if $k\geq n$ and $\alpha_1\cdots\alpha_n=0\cdots 0$ because $[0, \ell_n]=I_{0\cdots 0}^n$. Therefore, $k\geq n$, the number of $U_\alpha^k$'s in $[0, \ell_n]$ is $2^{k-n}$.

By lemma 4.2,

$$\begin{aligned} \int_0^{\ell_n}f(t)dt&=\sum_{U_\alpha^k\subset[0,\ell_n]}\left(\int_{U_\alpha^k}f(t)dt\right) \\ &=\sum_{i=n}^{\infty}2^{i-n}\cdot\frac{1}{3}m_i\ell_i \\ &=\sum_{i=n}^{\infty}2^{i-n}\ell_i\frac{3r_{i-1}-2r_i}{1-\sum_{j=1}^{i-1}r_j} \end{aligned} \tag{16}$$

By lemma 2.3(i), $\ell_i=\prod_{j=1}^{i}\lambda_j=\frac{1}{3^i}(1-\sum_{j=1}^{i-1}r_j)$. Therefore, by lemma 2.3 (ii),

$$\begin{aligned} (16)&=\sum_{i=n}^{\infty}2^{i-n}\frac{1}{3^i}(3r_{i-1}-2r_i) \\ &=2^{-(n-1)}\sum_{i=n}^{\infty}\left\{\left(\frac{2}{3}\right)^{i-1}r_{i-1}-\left(\frac{2}{3}\right)^{i}r_i\right\} \\ &=2^{-(n-1)}\lim_{k\to\infty}\left\{\left(\frac{2}{3}\right)^{n-2}r_{n-1}-\left(\frac{2}{3}\right)^{k}r_k\right\} \\ &=\frac{r_{n-1}}{3^{n-1}} \\ &=\frac{1}{3^{n-1}}3^{n-1}(1-3\lambda_n)\ell_{n-1} \end{aligned}$$

$$=\ell_{n-1}-3\ell_n.$$

Hence by (15), we have the statement of the lemma. □

Let $I_\alpha^n=[a_\alpha^n, b_\alpha^n]$.

LEMMA 4.4. *For any* $\alpha, \beta\in\Delta_n^0$,

$$\int_{I_\alpha^n} f(t)dt=\int_{I_\beta^n} f(t)dt.$$

PROOF: By the definition of $\mathscr{C}(s)$ and $f(t)$, the statement is clear because $f(t+a_\alpha^n)=f(t+a_\beta^n)$ for any $0\leq t\leq b_\alpha^n-a_\alpha^n=b_\beta^n-a_\beta^n$. □

PROOF OF PROPOSITION 4.1: What we have to show is that for any $n\geq 0$ and $\alpha\in\Delta_n$,

$$g_0(I_{0\alpha}^{n+1})=I_\alpha^n, \qquad g_1(I_{1\alpha}^{n+1})=I_\alpha^n.$$

We shall prove them by induction on $n$.

When $n=0$, it suffices to show that;

(17) $$g_0(I_0^1)=I^0, \qquad g_1(I_1^1)=I^o.$$

Since $g_0, g_1$ are monotonically increasing and $I_0^1=[0, \ell_1]$, $I_1^1=[1-\ell_1, 1]$, by the definition of $g_0, g_1$ and lemma 4.3, we have

$$\begin{cases} g_0(0), & g_0(\ell_1)=1 \\ g_1(1-\ell_1)=0, & g_1(1)=1. \end{cases}$$

This means (17).

Assume that statement be true for $n-1$. It is enough to show that;

(i) $g_0(a_{0\alpha}^{n+1})=a_\alpha^n$
(ii) $g_0(b_{0\alpha}^{n+1})=b_\alpha^n$
(iii) $g_1(a_{1\alpha}^{n+1})=a_\alpha^n$
(iv) $g_1(b_{1\alpha}^{n+1})=b_\alpha^n$.

Let $\alpha=\alpha'\alpha_n$. Then, by the hypothesis of induction,

$$g_0(I_{0\alpha'}^n)=I_{\alpha'}^{n-1}, \qquad g_1(I_{1\alpha'}^n)=I_{\alpha'}^{n-1}.$$

Namely;

(18) $$\begin{aligned} g_0(a_{0\alpha'}^n)&=a_{\alpha'}^{n-1}, & g_0(b_{0\alpha'}^n)&=b_{\alpha'}^{n-1}, \\ g_1(a_{1\alpha'}^n)&=a_{\alpha'}^{n-1}, & g_1(b_{1\alpha'}^n)&=b_{\alpha'}^{n-1}. \end{aligned}$$

First, we assume $\alpha_n=0$. Since $a_{0\alpha}^{n+1}=a_{0\alpha'}^n$, and $a_\alpha^n=a_{\alpha'}^{n-1}$, (i) is clear by (18). By lemma 4.3 and lemma 4.4, we have,

$$
\begin{aligned}
g_0(b_{0\alpha}^{n+1}) &= \int_0^{b_{0\alpha}^{n+1}} (f(t)+3)\,dt \\
&= \int_0^{a_{0\alpha}^{n+1}} (f(t)+3)\,dt + \int_{I_{0\alpha}^{n+1}} (f(t)+3)\,dt \\
&= a_\alpha^n + \ell_n \\
&= a_\alpha^n + (b_\alpha^n - a_\alpha^n) \\
&= b_\alpha^n.
\end{aligned}
$$

That proves (ii). The similar argument with $g_1$ gives (iii) and (iv).

Suppose $a_n=1$. Since $b_{0\alpha}^{n+1}=b_{0\alpha'}^n$ and $b_\alpha^n=b_{\alpha'}^{n-1}$, (ii) is clear. As to (i), we have,

$$
\begin{aligned}
g_0(a_{0\alpha}^{n+1}) &= \int_0^{a_{0\alpha}^{n+1}} (f(t)+3)\,dt \\
&= \int_0^{b_{0\alpha}^{n+1}} (f(t)+3)\,dt - \int_{a_{0\alpha}^{n+1}}^{b_{0\alpha}^{n+1}} (f(t)+3)\,dt \\
&= b_\alpha^n - \ell_n \\
&= b_\alpha^n - (b_\alpha^n - a_\alpha^n) \\
&= a_\alpha^n.
\end{aligned}
$$

That shows (i). The similar argument with $g_1$ gives (iii) and (iv). $\square$

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## References


[L] P. LARSSON, *The difference set of two Cantor sets,* Thesis-Uppsala University (1991).

[MO] P. MENDES, F. OLIVEIRA, *On the topological structure of the arithmetic sum of two Cantor sets,* (*preprint*).

[P] J. PALIS, *Fractional dimension and homoclinic bifurcations,* Colloquium-Hokkaido University (Ocotber, 1988).

[PM] J. PALIS, W. de MELO, *Geometric Theory of Dynamical Systems,* Springer-Verlag (1982).

[PT] J. PALIS, F. TAKENS, *Hyperbolicity and the creation of homoclinic orbits,* Annals of Math. **125** (1987), 337-374.


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