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On codimensions of maximal ideals in cohomology rings

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1. Introduction

Throughout this paper, let G be a finite group, K a field of characteristic $p>0$, and M a finitely generated left KG-module. For the cohomology ring

$$\mathrm{E}_G(M)=\mathrm{Ext}_{KG}^*(M, M)\simeq \mathrm{H}^*(G, \mathrm{End}_K(M))$$

of M, Carlson [5, 7] conjectured that if K is algebraically closed, then every maximal ideal in $\mathrm{E}_G(M)$ contains the kernel of the restriction map to some cyclic shifted subgroup. This conjecture was proved in [12] by using almost commutativity of restriction maps and cup products. Note that the conjecture implies a theorem of Avrunin and Scott [3] which associates the module varieties with the rank varieties. One of the purposes of this paper is to extend Carlson's conjecture as follows:

THEOREM A. *Assume that K is algebraically closed. Then, for a maximal ideal $\mathfrak{M}$ in $\mathrm{E}_G(M)$, there exist a cyclic shifted subgroup U and a maximal ideal $\mathfrak{N}$ in $\mathrm{E}_U(M)$ such that $\mathfrak{M}$ contains $\mathrm{res}_{G,U}^{-1}(\mathfrak{N})$.*

Here $\mathrm{res}_{G,U}$ is the restriction map. Although there is another proof based on Carlson's conjecture as in [14], we give a proof by directly extending the original one as anounced in [12].

On the other hand, Carlson [5] showed that each simple $\mathrm{E}_G(M)$-module has finite dimension over K for an arbitrary field K. Another purpose of this paper is to prove the following theorem.

THEOREM B. *If S is a simple $\mathrm{E}_G(M)$-module, then $\dim_K S\leq\dim_K M$.*

We shall prove Theorem B as follows. Suppose that M has a decomposition $M=M_1\oplus M_2\oplus\cdots\oplus M_n$ of KG-submodules. We regard $\mathrm{E}_G(M)$ as a matrix ring whose(i, j)-entry is $\mathrm{Ext}_{KG}^*(M_j, M_i)$. We shall show in Section 3 that we can interpret the Jacobson radical of $\mathrm{E}_G(M)$ by means of each entry by considering the functor $\mathrm{Ext}_{KG}^*(-, M)$ and its certain subfunctors. This argument follows Green's methods [10, Appendix] for Hom functors. In Section 4, we calculate the radical in the cyclic group case and give a proof of Theorem B by using Theorem A.

NOTATION. For a ring A, rad A denotes the Jacobson radical of A. Max(A) denotes the set of maximal ideals of A. For an ideal I of A, set $\sqrt{I}=\{a\in A | a^c\in I$ for some $c>0\}$. This is only a subset unless A is commutative.

2. Maximal ideals and cup, res

In this section we give a proof of Theorem A. At first we recall some basic concepts. They are detailed in [6, 4].

Let K be the trivial KG-module, and H a subgroup of G. Then there are two graded ring homomorphisms

$$\begin{aligned}\mathrm{cup}_{G,M}&:\mathrm{E}_G(K)\longrightarrow \mathrm{E}_G(M),\\ \mathrm{res}_{G,H,M}&:\mathrm{E}_G(M)\longrightarrow \mathrm{E}_H(M)\end{aligned}$$

which are induced from the inclusions $K\hookrightarrow \mathrm{End}_K(M)$ and $KH\hookrightarrow KG$, respectively. The former coincides with the cup product with the identity element of $\mathrm{E}_G(M)$. Evens [9] showed that $\mathrm{E}_G(K)$ is finitely generated as a K-algebra, and $\mathrm{E}_H(M)$ becomes a finitely generated $\mathrm{E}_G(K)$-module by $\mathrm{E}_G(K)\xrightarrow{\mathrm{res}}\mathrm{E}_H(K)\xrightarrow{\mathrm{cup}}\mathrm{E}_H(M)$.

On the other hand, if $p>2$, then the subalgebra $\mathrm{E}_G^{\mathrm{ev}}(K)=\bigoplus_{n\geq 0}\mathrm{Ext}_{KG}^{2n}(K, K)$ is contained in the center of $\mathrm{E}_G(K)$, and $\rho^2=0$ for each odd degree homogeneous element ρ. So $\mathrm{Max}(\mathrm{E}_G(K))\simeq\mathrm{Max}(\mathrm{E}_G^{\mathrm{ev}}(K))$. If $p=2$, then $\mathrm{E}_G(K)$ itself is commutative. In both cases, we can treat $\mathrm{E}_G(K)$ as if it were a commutative noetherian graded ring, and can consider a 'lying over' problem between $\mathrm{E}_G(K)$ and $\mathrm{E}_H(K)$.

We recall the definition of shifted subgroups. Let $E=\langle x_1,x_2,\cdots,x_n\rangle$ be an elementary abelian p-subgroup of rank n of G. A set of linearly independent elements $\alpha_1,\alpha_2,\cdots,\alpha_m$ in K^n defines a subgroup $\langle u_1,u_2,\cdots,u_m\rangle$ of the unit group of KG, where

$$u_i=1+\sum_{j=1}^{n}\alpha_{ij}(x_j-1),\quad \alpha_i=(\alpha_{i1}\cdots\alpha_{in}).$$

It is an elementary abelian p-group of rank m. Such a subgroup is called a *shifted* subgroup of G. In the case $m=1$, it is particularly called a *cyclic shifted* subgroup. We can define the restriction map from G to a shifted subgroup H as above. Evens' result also holds, that is, $\mathrm{E}_H(M)$ is finitely generated over $\mathrm{E}_G(K)$.

Let $U=\langle u\rangle$ be a cyclic group of order p. We choose the standard elements

$$\eta_u \in \mathrm{Ext}^1_{KU}(K,K), \quad \zeta_u \in \mathrm{Ext}^2_{KU}(K,K)$$

corresponding to the generator u as in [6]. Then

$$\mathrm{E}_U(K)=\begin{cases} K[\eta_u] & \text{if } p=2, \\ K[\zeta_u]\otimes_K \Lambda(\eta_u) & \text{otherwise,} \end{cases}$$

where $K[\zeta_u]$ is the polynomial ring and $\Lambda(\eta_u)$ is the exterior algebra. We define a maximal ideal $\mathfrak{N}_u$ of $\mathrm{E}_U(K)$ by

$$\mathfrak{N}_u=\begin{cases} (\eta_u-1) & \text{if } p=2, \\ (\zeta_u-1)+\mathrm{rad}\ \mathrm{E}_U(K) & \text{otherwise.} \end{cases}$$

Let $\mathfrak{C}=\{u\in KG \mid \langle u\rangle$ is a cyclic shifted subgroup$\}$, on which G acts by conjugation. We can define a map $\mathfrak{C}\cup\{1\}\to \mathrm{Max}(\mathrm{E}_G(K))$ which maps u to $\mathrm{res}^{-1}(\mathfrak{N}_u)$, and 1 to the homogeneous maximal ideal $\mathrm{E}^+_G(K)$. The following is Carlson's version of Quillen's stratification theorem.

THEOREM 2.1 ([13, 6]). *If K is algebraically closed, then the induced map $\mathfrak{C}/G\cup\{1\}\to\mathrm{Max}(\mathrm{E}_G(K))$ is bijective.*

For the remainder of this section, let H be a subgroup or a shifted subgroup of G. For convenience, we write ρ_M and ρ_H instead of $\mathrm{cup}_{G,M}(\rho)$ and $\mathrm{res}_{G,H}(\rho)$ respectively. Also for subsets of $\mathrm{E}_G(K)$, we use the same notation. When H is shifted, since the K-algebra inclusion $KH \hookrightarrow KG$ is not a Hopf algebra homomorphism, the diagram

$$\begin{array}{ccc} \mathrm{E}_G(K) & \xrightarrow{\mathrm{res}} & \mathrm{E}_H(K) \\ {\scriptstyle\mathrm{cup}}\downarrow & & \downarrow{\scriptstyle\mathrm{cup}} \\ \mathrm{E}_G(M) & \xrightarrow[\mathrm{res}]{} & \mathrm{E}_H(M) \end{array}$$

is not commutative. However it is 'almost' commutative as follows:

THEOREM 2.2 ([12]). $((\rho^p)_M)_H=((\rho^p)_H)_M$ *for all* $\rho\in\mathrm{E}_G(K)$.

To show Theorem A, we introduce some notation. Let

$$\mathrm{J}_G(M)=\mathrm{Ker}(\mathrm{cup}:\ \mathrm{E}_G(K)\longrightarrow\mathrm{E}_G(M)),$$
$$\mathrm{V}_G(M)=\{P\in\mathrm{Max}(\mathrm{E}_G(K)) \mid \mathrm{J}_G(M)\subset P\}, \quad \text{and}$$
$$\mathrm{Ker}_{G,H}(M)=\mathrm{Ker}(\mathrm{res}:\ \mathrm{E}_G(M)\longrightarrow\mathrm{E}_H(M)).$$

Then $\mathrm{J}_G(M)$ is the annihilator of $\mathrm{E}_G(M)$, and its support $\mathrm{V}_G(M)$ is isomorphic to $\mathrm{Max}((\mathrm{E}_G(K))_M)$. We write V_G for $\mathrm{V}_G(K)=\mathrm{Max}(\mathrm{E}_G(K))$. We note that

(1) $(\mathrm{E}_G^{\mathrm{ev}}(K))_M$ is contained in the center of $\mathrm{E}_G(M)$;

(2) $\mathrm{Ker}_{G,H}(M)$ is a homogeneous ideal contained in $\mathrm{E}_G^+(M)=\bigoplus_{n\geq 1}\mathrm{Ext}_{KG}^n(M,M)$;

(3) $\mathrm{E}_G^+(M)$ may not be a maximal ideal.

The following lemma is well known (see [2, Corollary 2.5] for the proof).

LEMMA 2.3. *Let R be a commutative ring, I a proper ideal of R, and L a finitely generated faithful R-module. Then $IL \subsetneq L$.*

Hence the following maps are well-defined;

$$\mathrm{cup}^*: \mathrm{Max}(\mathrm{E}_G(M)) \longrightarrow \mathrm{V}_G, \qquad \mathrm{cup}^*(\mathfrak{M})=\mathrm{cup}^{-1}(\mathfrak{M}),$$
$$\mathrm{res}^*: \mathrm{V}_H \longrightarrow \mathrm{V}_G \qquad \mathrm{res}^*(Q)=\mathrm{res}^{-1}(Q).$$

Moreover, again by Lemma 2.3, the image of cup* is $\mathrm{V}_G(M)$ and the image of res* is $\{P\in\mathrm{V}_G|\mathrm{Ker}_{G,H}(K)\subset P\}$. Well-definedness of cup* implies that every maximal ideal of $\mathrm{E}_G(M)$ has finite codimension over K.

LEMMA 2.4. *For $P\in\mathrm{res}^*(\mathrm{V}_H)$, let $(\mathrm{res}^*)^{-1}(P)=\{Q_1, Q_2, \cdots, Q_n\}$. Then*

$$\bigcap_{i=1}^n \mathrm{res}^{-1}(Q_i\mathrm{E}_H(M)) \subset\sqrt{P\mathrm{E}_G(M)+\mathrm{Ker}_{G,H}(M)} \quad \textit{in } \mathrm{E}_G(M).$$

PROOF. Since $\mathrm{E}_H(K)$ is almost commutative and finitely generated as a K-algebra, we have $\cap_i Q_i=\sqrt{P\mathrm{E}_H(K)}$. By the Artin-Rees lemma, there exists a positive integer c such that

$$\bigcap_i (Q_i^c\mathrm{E}_H(M))\subset(\bigcap_i Q_i)\mathrm{E}_H(M)=(\sqrt{P\mathrm{E}_H(K)})\mathrm{E}_H(M)\subset\sqrt{P\mathrm{E}_H(M)},$$

where we can interpret P as both $(P_H)_M$ and $(P_M)_H$, by Theorem 2.2. Also by the same lemma, there exists a positive integer d such that $P^d\mathrm{E}_H(M)\cap(\mathrm{E}_G(M))_H\subset(P\mathrm{E}_G(M))_H$. This completes the proof. □

LEMMA 2.5. *For $P\in\mathrm{res}^*(\mathrm{V}_H)$, we have $\mathrm{Ker}_{G,H}(M)\subset\sqrt{P\mathrm{E}_G(M)}$.*

PROOF. We may assume that G is a p-group, since the restriction maps to Sylow p-subgroups are monic. We use induction on the order of G. We may assume that there is a maximal subgroup S of G such that H is a (shifted) subgroup of S, otherwise the restriction to H is isomorphism. Let $(\mathrm{res}_{G,S}^*)^{-1}(P)=\{Q_1, Q_2, \cdots, Q_n\}$. Then, by the assumption of induction and by Lemma 2.4,

$$\mathrm{Ker}_{G,H}(M)=\mathrm{res}_{G,S}^{-1}(\mathrm{Ker}_{S,H}(M))\subset\bigcap_i\mathrm{res}_{G,S}^{-1}(\sqrt{Q_i\mathrm{E}_S(M)})$$

$$\subset \sqrt{P\mathrm{E}_G(M)+\mathrm{Ker}_{G,S}(M)}.$$

Hence we may assume $H=S$.

By a result of Alperin and Evens [1], there exists $\beta \in \mathrm{Ker}_{G,H}(K)$ such that homogeneous elements of $\mathrm{Ker}_{G,H}(M)$ are contained in $\sqrt{\beta \mathrm{E}_G(M)}$, and hence in $\sqrt{P\mathrm{E}_G(M)}$. Choose finitely many homogeneous generators of $\mathrm{Ker}_{G,H}(M)$ as an $\mathrm{E}_G(M)$-module. Their images in $\mathrm{E}_G(M)/P\mathrm{E}_G(M)$ generate a finitely generated nil multiplicative subsemigroup. So it is nilpotent by Levitzki's theorem [11, pp. 199]. Hence $\mathrm{Ker}_{G,H}(M)$ is nilpotent in $\mathrm{E}_G(M)/P\mathrm{E}_G(M)$. This completes the proof. □

By the above lemmas, we have the following:

PROPOSITION 2.6. *For* $P \in \mathrm{res}^*(\mathrm{V}_H)$, *let* $(\mathrm{res}^*)^{-1}(P)=\{Q_1, Q_2, \cdots, Q_n\}$. *Then*

$$\bigcap_{i=1}^{n} \mathrm{res}^{-1}(Q_i\mathrm{E}_H(M)) \subset \sqrt{P\mathrm{E}_G(M)} \quad \textit{in } \mathrm{E}_G(M).$$

We rewrite Theorem A as follows. Note that if K is algebraically closed, then by Theorem 2.1 there certainly exists such a cyclic shifted subgroup H as in the following theorem.

THEOREM A'. *Let* $\mathfrak{M}$ *be a maximal ideal of* $\mathrm{E}_G(M)$, *and let* $P=\mathrm{cup}^*(\mathfrak{M}) \in \mathrm{V}_G(M)$. *Suppose that* H *is a subgroup or a shifted subgroup of* G *such as* $P \in \mathrm{res}^*_{G,H}(\mathrm{V}_H)$. *Then there is a maximal ideal* $\mathfrak{N}$ *of* $\mathrm{E}_H(M)$ *such that* $\mathrm{res}^{-1}_{G,H}(\mathfrak{N}) \subset \mathfrak{M}$.

PROOF. Let $(\mathrm{res}^*_{G,H})^{-1}(P)=\{Q_1, Q_2, \cdots, Q_s\}$, and $(\mathrm{cup}^*_{H,M})^{-1}(Q_i)=\{\mathfrak{N}_{i1}, \mathfrak{N}_{i2}, \cdots, \mathfrak{N}_{it_i}\}$. Then we have $(\cap_j \mathfrak{N}_{ij})/Q_i\mathrm{E}_H(M)=\mathrm{rad}(\mathrm{E}_H(M)/Q_i\mathrm{E}_H(M))$ in the finite dimensional K-algebra, for each i. Hence Proposition 2.6 implies

$$\bigcap_{ij} \mathrm{res}^{-1}(\mathfrak{N}_{ij}) \subset \bigcap_i \sqrt{\mathrm{res}^{-1}(Q_i\mathrm{E}_H(M))} \subset \sqrt{P\mathrm{E}_G(M)} \subset \mathfrak{M}.$$

Therefore $\mathfrak{M}$ contains some $\mathrm{res}^{-1}(\mathfrak{N}_{ij})$. This completes the proof. □

3. Ideal subfunctors of Ext

The argument in this section follows Green's method [10, Appendix] for Hom functors.

Let mod KG be the category of finitely generated left KG-modules, and Mod K the category of vector spaces over K. Let Mmod KG denote the category of K-linear contravariant functors from mod KG to Mod K. Thus objects are those contravariant functors F : mod $KG \rightarrow$ Mod K whose induced maps $\mathrm{Hom}_{KG}(X, Y)\to\mathrm{Hom}_K(FY, FX)$ are K-linear. Morphisms are natural transformations. For example, $\mathrm{Hom}_{KG}(-, M)$ and $\mathrm{Ext}^*_{KG}(-, M)$ are objects of Mmod KG. Mmod KG is a K-linear (i.e. morphism sets are K-vector spaces, and their composition maps are K-bilinear) abelian category. If F' is a subfunctor of F, then we write $F'\subset F$ (*subfunctors* are pointwisely defined).

Throughout this section, let M, N, X, Y be objects in mod KG. Ω denotes the Heller operator, namely, $\Omega(X)$ is the kernel of the projective cover of X. We define $\Omega^0(X)$ as its core, and inductively $\Omega^{n+1}=\Omega\Omega^n$. For non-negative integers i, n, consider the canonical homomorphism

$$\gamma^i_n(X): \mathrm{Ext}^i_{KG}(\Omega^n(X), M)\longrightarrow\mathrm{Ext}^{i+n}_{KG}(X, M).$$

When $i>0$, $\gamma^i_n(X)$ is an isomorphism which maps the class of

$$0\to M\to B_{i-1}\to B_{i-2}\to\cdots\to B_0\to\Omega^n(X)\to 0$$

to the class of the Yoneda splice

$$0\to M\to B_{i-1}\to\cdots\to B_0\to P_{n-1}\to\cdots\to P_0\to X\to 0,$$

where $\cdots\to P_1\to P_0\to X\to 0$ is a minimal projective resolution of X. When $i=0$ and $n>0$, $\gamma^0_n(X)$ is the canonical epimorphism because $\mathrm{Ext}^0_{KG}=\mathrm{Hom}_{KG}$. On the other hand, $\gamma^0_0(X)$ is the canonical monomorphism induced from the splitting inclusion $\Omega^0(X)\to X$.

We have a graded $\mathrm{E}_G(M)$-homomorphism

$$\gamma_n(X)=\prod_i\gamma^i_n(X): \mathrm{Ext}^*_{KG}(\Omega^n(X), M)\to\mathrm{Ext}^*_{KG}(X, M)$$

of degree n, and a natural transformation

$$\gamma_n: \mathrm{Ext}^*_{KG}(\Omega^n(-), M)\to\mathrm{Ext}^*_{KG}(-, M)$$

in Mmod KG.

DEFINITION 3.1. A subfunctor F of $\mathrm{Ext}^*_{KG}(-, M)$ is called a *right ideal subfunctor* when $\gamma_n(F\Omega^n)\subset F$ for all $n\geq 0$, that is,

$$\begin{array}{ccc}\mathrm{Ext}^*_{KG}(\Omega^n(X), M) & \xrightarrow{\gamma_n(X)} & \mathrm{Ext}^*_{KG}(X, M)\\ \uparrow{\scriptstyle\mathrm{incl}} & & \uparrow{\scriptstyle\mathrm{incl}}\\ F\Omega^n(X) & \xrightarrow[\gamma_n(X)]{} & F(X)\end{array}$$

is a well-defined commutative diagram for all n and X. Then we write $F\leq\mathrm{Ext}^*_{KG}(-, M)$.

DEFINITION 3. 2. Suppose $F \leq \mathrm{Ext}^*_{KG}(-, M)$ and $F' \leq \mathrm{Ext}^*_{KG}(-, N)$. A natural transformation α: $F \to F'$ is called a *right ideal morphism* when $\alpha\gamma_n = \gamma_n \alpha \Omega^n$ for all $n \geq 0$, that is,

$$\begin{tikzcd} F\Omega^n(X) \arrow[r, "\gamma_n(X)"] \arrow[d, "\alpha(\Omega^n(X))"'] & F(X) \arrow[d, "\alpha(X)"] \\ F'\Omega^n(X) \arrow[r, "\gamma_n(X)"'] & F'(X) \end{tikzcd}$$

is commutative for all n and X. We write the class of right ideal morphisms from F to F' by $[F, F']$.

Dually we can define *left ideal* subfunctors and morphisms by

$$\lambda_n: \mathrm{Ext}^*_{KG}(M, -) \to \mathrm{Ext}^*_{KG}(M, \Omega^n(-)).$$

The below arguments also hold for left ideal subfunctors.

PROPOSITION 3. 3. *If* $F \leq \mathrm{Ext}^*_{KG}(-, M)$, *then* $F(X)\mathrm{Ext}^*_{KG}(Y, X) \subset F(Y)$, *namely,*

$$\begin{tikzcd} \mathrm{Ext}^*_{KG}(X, M) \times \mathrm{Ext}^*_{KG}(Y, X) \arrow[r] & \mathrm{Ext}^*_{KG}(Y, M) \\ F(X) \times \mathrm{Ext}^*_{KG}(Y, X) \arrow[r] \arrow[u, "\mathrm{incl}"] & F(Y) \arrow[u, "\mathrm{incl}"'] \end{tikzcd}$$

is a well-defined commutative diagram, where the horizontal maps are the composition maps.

PROOF. Given $\rho \in \mathrm{Ext}^*_{KG}(Y, X)$, choose $f \in \mathrm{Hom}_{KG}(\Omega^n(Y), X)$ such that ρ is the class of f. Then in the commutative diagram

$$\begin{tikzcd} \mathrm{Ext}^*_{KG}(X, M) \arrow[r, "f^\#"] & \mathrm{Ext}^*_{KG}(\Omega^n(Y), M) \arrow[r, "\gamma_n(Y)"] & \mathrm{Ext}^*_{KG}(Y, M) \\ F(X) \arrow[r, "F(f)"'] \arrow[u, "\mathrm{incl}"] & F(\Omega^n(Y)) \arrow[r, "\gamma_n(Y)"'] \arrow[u, "\mathrm{incl}"'] & F(Y), \arrow[u, "\mathrm{incl}"'] \end{tikzcd}$$

$\gamma_n(Y)f^\#$ coincides with multiplication by ρ from the right hand. $\square$

Here we state an interesting lemma without proof, although this is not necessary for our later argument.

LEMMA 3. 4. *The following hold.*

(1) *If* $F_1, F_2 \leq \mathrm{Ext}^*_{KG}(-, M)$ *then* F_1+F_2, $F_1 \cap F_2 \leq \mathrm{Ext}^*_{KG}(-, M)$.

(2) (Homomorphism theorem) *Suppose that* $F \leq \mathrm{Ext}^*_{KG}(-, M)$, $F' \leq \mathrm{Ext}^*_{KG}(-, N)$ *and* $\alpha \in [F, F']$. *Then both* $\mathrm{Ker}\,\alpha$ *and* $\mathrm{Im}\,\alpha$ *are right ideals, and there is a natural correspondence*

$$\{S \leq \mathrm{Ext}^*_{KG}(-, M) \mid \mathrm{Ker}\,\alpha \subset S \subset F\} \simeq \{S' \leq \mathrm{Ext}^*_{KG}(-, N) \mid S' \subset \mathrm{Im}\ \alpha\}.$$

(3) (Yoneda's lemma) *For* $F \leq \mathrm{Ext}^*_{KG}(-, M)$, *there is a natural* K*-linear isomorphism*

$$[\mathrm{Ext}^*_{KG}(-, N), F] \simeq F(N).$$

If $F \leq \mathrm{Ext}^*_{KG}(-, M)$, then $F(X)$ is a right $\mathrm{E}_G(X)$-module by Proposition 3.3. Let $\mathfrak{R}(\mathrm{Ext}^*_{KG}(-, M))$ be the class of right ideal subfunctors of $\mathrm{Ext}^*_{KG}(-, M)$, and $\mathfrak{R}(\mathrm{E}_G(M))$ the set of right ideals of $\mathrm{E}_G(M)$. We define two maps $\mathfrak{R}(\mathrm{Ext}^*_{KG}(-, M)) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \mathfrak{R}(\mathrm{E}_G(M))$ by

$$\alpha(F) = F(M)$$
$$\beta(I)(X) = \{\zeta \in \mathrm{Ext}^*_{KG}(X, M) \mid \zeta \mathrm{Ext}^*_{KG}(M, X) \subset I\}.$$

Well-definedness of β follows from the fact that

$$(\gamma_n(X)(\zeta))\rho = \zeta(\lambda_n(X)(\rho)) \quad \text{in } \mathrm{E}_G(M)$$

for $\zeta \in \mathrm{Ext}^*_{KG}(\Omega^n(X), M)$ and $\rho \in \mathrm{Ext}^*_{KG}(M, X)$. Not that α and β may not be bijections. The following are easily verified.

(1) $F \subset \beta\alpha(F)$.

(2) If $\alpha(F) = \mathrm{E}_G(M)$, then $F = \mathrm{Ext}^*_{KG}(-, M)$.

(3) If F is maximal, then $\beta\alpha(F) = F$.

(4) If I is maximal, then $\beta(I)$ is maximal.

(5) For a set $\{I_\lambda \mid \lambda \in \Lambda\}$ of right ideals, $\beta(\bigcap_\lambda I_\lambda) = \bigcap_\lambda \beta(I_\lambda)$.

So α and β induce a one to one correspondence between the maximal objects in $\mathfrak{R}(\mathrm{Ext}^*_{KG}(-, M))$ and the maximal objects in $\mathfrak{R}(\mathrm{E}_G(M))$. Let

$$\mathrm{rad}\ \mathrm{Ext}^*_{KG}(-, M) = \bigcap_{\max F} F,$$

where F runs through maximal right ideal subfunctors. By (5), $\bigcap_{\max F} F$ coincides with $\beta(\mathrm{rad}\,\mathrm{E}_G(M))$, and

$$(\mathrm{rad}\,\mathrm{Ext}^*_{KG}(-, M))(M) = \bigcap_{\max F} F(M) = \mathrm{rad}\,\mathrm{E}_G(M).$$

The same fact holds for left ideal subfunctors. In particular, since

$$(\mathrm{rad\,Ext}^*_{KG}(-, M))(N)=(\mathrm{rad\,Ext}^*_{KG}(N, -))(M),$$

denote this by $\mathrm{rad\,Ext}^*_{KG}(N, M)$. When $M=M_1\oplus M_2$, it is easy to show that

$$\mathrm{rad\,Ext}^*_{KG}(-, M)=\mathrm{rad\,Ext}^*_{KG}(-, M_1)\oplus \mathrm{rad\,Ext}^*_{KG}(-, M_2)$$

as in the case of radicals of modules [8, Ex 5. 11]. Hence we get

THEOREM 3. 5. *Let* $M=M_1\oplus M_2\oplus \cdots \oplus M_n$. *Then*

(1) $\mathrm{rad\,Ext}^*_{KG}(-, M)=\bigoplus_{i=1}^{n}\mathrm{rad\,Ext}^*_{KG}(-, M_i)$.

(2) *If we regards* $\mathrm{E}_G(M)$ *as the matrix ring* $\{(\zeta_{ij})\mid \zeta_{ij}\in \mathrm{Ext}^*_{KG}(M_j, M_i)\}$ *of size* $n\times n$, *then*

$$\mathrm{rad\,E}_G(M)=\{(\zeta_{ij})\mid \zeta_{ij}\ \mathrm{Ext}^*_{KG}(M_i, M_j)\subset \mathrm{rad\,E}_G(M_i)\}.$$

Carlson [5] showed that rad $\mathrm{E}_G(M)$ is nilpotent. This fact implies the equality

$$\mathrm{rad\,E}_G(M)=\{\zeta\in \mathrm{E}_G(M)\mid \zeta\rho \text{ is nilpotent for all } \rho\in \mathrm{E}_G(M)\},$$

and hence the second statement of Theorem 3. 5. However we do not use his result.

It is well known that the Jacobson radical of the matrix ring over *a ring* can be interpreted by means of each entry. The second statement of Theorem 3. 5 for $\mathrm{End}_{KG}(M)$ is one of the central ideas in Clifford theory. It is interesting to find properties corresponding to the fact that $\mathrm{End}_{KG}(M)$ is local for an indecomposable module M.

4. Cyclic case

We calculate radical functors in the simplest case, and give a proof of Theorem B.

Let $U=\langle u\rangle$ be a cyclic group of order p, and $V_i=KU/(\mathrm{rad}\,KU)^i$ for positive integers i. Then $V_1, V_2, \cdots, V_{p-1}$ are the non-projective indecomposable KU-modules with $\dim_K V_i=i$. $V_1=K$ is the unique simple KU-module. They are uniserial (i.e. they have unique composition series), and have a common projective cover KU. The Heller operator Ω acts by $\Omega(V_i)=V_{p-i}$. For l with $1\leq l\leq \min(i, j)$, let $f_{ijl}: V_j\to V_i$ be the canonical homomorphism of rank l. We write simply f_l instead of f_{ijl}. Then they are K-basis of $\mathrm{Hom}_{KU}(V_j, V_i)$, namely

$$\mathrm{Hom}_{KU}(V_j, V_i)=<f_l: V_j\to V_i\mid 1\leq l\leq \min(i, j)>_K.$$

It is easy to show that $f_l: V_j\to V_i$ is projective, that is, f_l factors through the projective cover KU, if and only if $l \leq i+j-p$. Let $\hom_{KU}(V_j, V_i)$ denote the factor space of $\mathrm{Hom}_{KU}(V_j, V_i)$ by the subspace of projective homomorphisms. Then

$$\hom_{KU}(V_j, V_i) = <\bar{f}_l \mid i+j-p < l \leq \min(i, j)>_K,$$

where we use bar convention. In particular, we have

$$\dim_K \hom_{KU}(V_j, V_i) = \begin{cases} \min(i, j) & \text{if } i+j \leq p, \\ p-\max(i, j) & \text{otherwise.} \end{cases}$$

We regard $\mathrm{Ext}^n_{KU}(V_j, V_i)$ as $\hom_{KU}(\Omega^n(V_j), V_i)$ for positive n, so its K-vector space structure is determined.

There is an isomorphism

$$\Omega : \hom_{KU}(V_j, V_i) \to \hom_{KU}(\Omega(V_j), \Omega(V_i) = \hom_{KU}(V_{p-j}, V_{p-i}),$$

which maps $\bar{f}_l$ to $\overline{\Omega(f_l)} = \bar{f}_{l+p-(i+j)}$. It is the lifting map in the sense of making the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega(V_j) & \longrightarrow & KU & \longrightarrow & V_j & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \Omega(f_l)} & & \Big\downarrow{\scriptstyle f_{l+p-i}} & & \Big\downarrow{\scriptstyle f_l} & & \\ 0 & \longrightarrow & \Omega(V_i) & \longrightarrow & KU & \longrightarrow & V_i & \longrightarrow & 0 \end{array}$$

of projective covers commutative. Thus in a similar way to Proposition 3.3, it is easy to show that the diagram

$$\begin{array}{ccc} \mathrm{Ext}^m_{KU}(V_j, V_i) \times \mathrm{Ext}^n_{KU}(V_k, V_j) & \longrightarrow & \mathrm{Ext}^{m+n}_{KU}(V_k, V_i) \\ \Big\| & & \Big\| \\ \hom_{KU}(\Omega^m(V_j), V_i) \times \hom_{KU}(\Omega^n(V_k), V_j) & & \\ \Big\downarrow{\scriptstyle 1 \times \Omega^m} & & \Big\| \\ \hom_{KU}(\Omega^m(V_j), V_i) \times \hom_{KU}(\Omega^{m+n}(V_k), \Omega^m(V_j)) & \longrightarrow & \hom_{KU}(\Omega^{m+n}(V_k), V_j) \end{array}$$

of the composition maps is commutative. Thus for

$$\bar{f}_s \in \hom_{KU}(\Omega^m(V_j), V_i) = \mathrm{Ext}^m_{KU}(V_j, V_i),$$
$$\bar{f}_t \in \hom_{KU}(\Omega^n(V_k), V_j) = \mathrm{Ext}^n_{KU}(V_k, V_j),$$

the composition $\bar{f}_s \bar{f}_t$ in $\mathrm{Ext}^{m+n}_{KU}(V_k, V_i)$ is the class of the composition

$$\Omega^{m+n}(V_k) \xrightarrow{\Omega^m(f_t)} \Omega^m(V_j) \xrightarrow{f_s} V_i,$$

where Ω^m is Ω if m is old, and ineffective otherwise.

Since V_i is indecomposable, an endomorphism of V_i is nilpotent if and only if it is not an isomorphism. So $\mathrm{rad}\,\mathrm{End}_{KU}(V_i)$ has codimension 1. The same fact holds for homogeneous elements of $\mathrm{E}_U(V_i)$. To prove this, suppose that $\bar{f} \in \mathrm{Ext}^n_{KU}(V_i, V_i)$ for positive n. Since $\bar{f}^2$ is the class of $f \cdot \Omega^n(f): V_i \to \Omega^n(V_i) \to V_i$, $\bar{f}$ is nilpotent if and only if f is not an isomorphism.

Hence,

$$\bigoplus_{n \geq 0} \{\bar{f} \in \mathrm{Ext}^n_{KU}(V_i, V_i) \mid f : \Omega^n(V_i) \to V_i \text{ is not an isomorphism}\}$$

is a homogeneous nil ideal, and the factor ring by this ideal is a polynomial ring generated by the image of the class σ_i of the identity map of V_i, whose degree is 1 if $p=2$, or 2 otherwise. Therefore the above ideal coincides with $\mathrm{rad}\,\mathrm{E}_G(V_i)$. Note that in the case of $i=1$, the generator σ_1 is equal to η_u or ζ_u of Section 2, and it is mapped to σ_i by cup: $\mathrm{E}_U(K) \to \mathrm{E}_U(V_i)$. So cup induces a K-algebra isomorphism

$$R = K[\sigma_1] \simeq \mathrm{E}_U(V_i)/\mathrm{rad}\ \mathrm{E}_U(V_i).$$

The following is easy to show by the same arguments as above.

PROPOSITION 4.1. *For any i, j with $1 \leq i, j \leq p-1$, we have*

$$\mathrm{rad}\,\mathrm{Ext}^*_{KU}(V_j, V_i) = \bigoplus_{n \geq 0} \{\bar{f} \in \mathrm{Ext}^n_{KU}(V_j, V_i) \mid f : \Omega^n(V_j) \to V_i \textit{ is not an isomorphism}\}.$$

Note that f is not an isomorphism unless $j=i$ and n is even, nor unless $j=p-i$ and n is odd.

PROPOSITION 4.2. *We have*

$$\mathrm{Ext}^*_{KU}(V_j, V_i)/\mathrm{rad}\,\mathrm{Ext}^*_{KU}(V_j, V_i) \simeq \begin{cases} R & \textit{if } j=i \textit{ or } j=p-i, \\ 0 & \textit{otherwise.} \end{cases}$$

as R-modules for $1 \leq i, j \leq p-1$.

Let V be a finitely generated KU-module, and have a decomposition $V = \bigoplus_{i=1}^{p} V_i^{m_i}$, where m_i are their multiplicities. Then, by Theorem 3.5 and Proposition 4.2, we have

$$\mathrm{E}_U(V)/\mathrm{rad}\,\mathrm{E}_U(V) \simeq \begin{cases} \mathrm{M}_{m_1}(R) \oplus \mathrm{M}_{m_2}(K) & \text{if } p=2, \\ \bigoplus_{i=1}^{(p-1)/2} \mathrm{M}_{m_i+m_{p-i}}(R) \oplus \mathrm{M}_{m_p}(K) & \text{otherwise,} \end{cases}$$

where $\mathrm{M}_m(R)$ is the matrix ring over R of size $m \times m$.

Combining these isomorphisms and Theorem A, we have the following main theorem of this section.

THEOREM 4.3. *Let G be a finite group, K an algebraically closed field, and M a KG-module. Suppose that $\mathfrak{M}$ is a maximal ideal of* $\mathrm{E}_G(M)$, *and* $\mathrm{E}_G(M)/\mathfrak{M}\simeq \mathrm{M}_d(K)$ *for a positive integer d. Then there is a cyclic shifted subgroup U such that*

$$d\leq\begin{cases}\max\{m_1, m_2\} & \text{if } p=2,\\ \max\{m_i+m_{p-i}, m_p \mid 1\leq i\leq (p-1)/2\} & \text{otherwise,}\end{cases}$$

for the decomposition $M=\bigoplus_{i=1}^{p} V_i^{m_i}$ as KU-modules.

If K is an arbitrary field and $\bar{K}$ is an algebraic closure of K, then every maximal ideal of $\mathrm{Ext}^*_{KG}(M, M)$ is contained in some maximal ideal of $\bar{K}\otimes_K \mathrm{Ext}^*_{KG}(M, M)\simeq \mathrm{Ext}^*_{\bar{K}G}(\bar{K}\otimes_K M, \bar{K}\otimes_K M)$. Therefore Theorem B follows straightforward from Theorem 4.3. Note that if M is a direct sum of some copies of the trivial KG-module K, then there exists a maximal ideal which satisfies the equality in the theorem.

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