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View from inside

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Abstract. In this paper, we define a perspective projection of a given immersed n-dimensional hypersurface as a C^∞ map via a C^∞ immersion from the given n-manifold to S^{n+1}, and characterize when and only when such a perspective projection is non-singular.

In order to obtain such characterizations, we consider an immersion from an n-dimensional manifold to S^{n+1}. We first obtain equivalent conditions for a given point P of S^{n+1} to be outside the union of tangent great hyperspheres of a given immersed n-dimensional manifold $\mathbf{r}(N)$ in S^{n+1} (Theorem 2.4). It turns out that if such a point P exists then the given manifold N must be diffeomorphic to S^n and in the case that $n \geq 2$ the given immersion $\mathbf{r} : N \to S^{n+1}$ must be an embedding. Then, we obtain characterizations of a perspective projection of a given immersed n-dimensional manifold to be non-singular.

Next, we obtain one more equivalent condition in terms of hedgehogs when the given N is S^n and the given immersion is an embedding (Theorem 3.3). We also explain why we consider these equivalent conditions for an embedding $\mathbf{r} : S^n \to S^{n+1}$ instead of an embedding $\widetilde{\mathbf{r}} : S^n \to \mathbb{R}^{n+1}$ in terms of hedgehogs.

Key words: perspective projection, perspective point, projective dual, dual hypersurface, hedgehog

1. Introduction

Both the following C_1, C_2 of Fig. 1 are simply closed plane curves with four inflection points but without singular points. At a glance, shapes of these two curves seem to have no differences. However, by drawing their tangent lines, we notice that the union of tangent lines for C_1 does not cover the plane, while the union of tangent lines for C_2 does cover the plane (see Fig. 2). These imply that we see no silhouette of the object C_1 by viewing it from the point P in the left side of Fig. 2, while we always see the silhouette of the object C_2 by viewing it from any point in the right side of Fig. 2. In other words, there exists a point P inside C_1 such that the perspective projection of C_1 from P seems to give no information on the shape of C_1 but the perspective projection of C_2 from any point inside it gives some information on the shape of C_2 as its silhouette. Hence, we can

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Figure 1. Both C_1, C_2 are simply closed curves with four inflection points and without singular points.

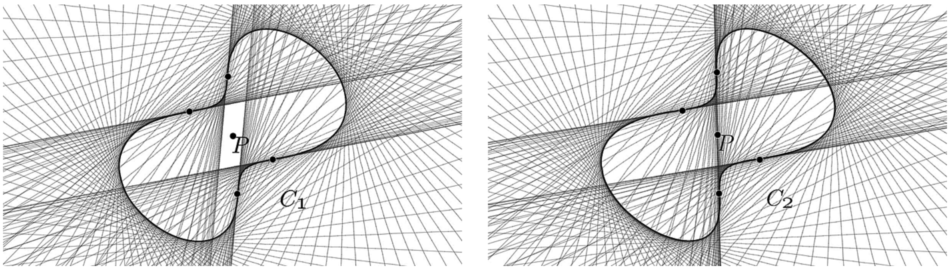

Figure 2. Left: The union of tangent lines for C_1 does not cover the plane. Right: The union of tangent lines for C_2 covers the plane.

say that "the view of C_1 from inside" is different from "the view of C_2 from inside".

In this paper we clarify the meaning of such a hidden difference by investigating conditions for a given point $P \in \mathbb{R}^{n+1}$ to be outside the union of tangent hyperplanes of an embedded n-dimensional manifold in $\mathbb{R}^{n+1}$. As a by-product, we see that the diffeomorphic type of an embedded n-manifold is determined by the existence of a point from where the perspective projection seems to give no information on the shape of it; in other words, a "view from inside" when it seems to be useless determines the diffeomorphic type of the n-manifold. In order to obtain such conditions we first introduce the notions of spherical perspective projections and projective duals and obtain several such conditions in Section 2. In Section 3, the notion of hedgehogs is introduced and several results concerning hedgehogs are stated and proved.

2. Spherical perspective projections and projective duals

Let S^{n+1} be the $(n+1)$-dimensional unit sphere in $\mathbb{R}^{n+2}$ $(n \geq 1)$. We let N and $\mathbf{r} : N \to S^{n+1}$ be a compact connected n-dimensional C^∞ manifold and a C^∞ immersion respectively. For a point $P \in S^{n+1}$, we put

$$E_P = \{X \in S^{n+1} \mid P \cdot X = 0\},$$

where the dot in the center means the standard scalar product of $P, X \in \mathbb{R}^{n+2}$. The set E_P may be regarded as the equator with respect to the north pole P. For a point $P \in S^{n+1}$ we can define the map $\pi_P : S^{n+1} - \{\pm P\} \to E_P$ which maps $X \in S^{n+1} - \{\pm P\}$ to the unique point Y such that $Y \in E_P \cap (\mathbb{R}P + \mathbb{R}X)$ and $X \cdot Y > 0$. Note that for a point P such that $P \cdot \mathbf{r}(x) \neq \pm 1$ for any $x \in N$, which means that $\{\pm P\} \subset S^{n+1} - \mathbf{r}(N)$, the restriction $\pi_P|_{\mathbf{r}(N)} : \mathbf{r}(N) \to E_P$ is well-defined. The map π_P and the restriction $\pi_P|_{\mathbf{r}(N)}$ are called the *spherical perspective projection relative to* P and the *spherical perspective projection of* $\mathbf{r}(N)$ *relative to* P respectively. The set of critical values of $\pi_P \circ \mathbf{r}$ is called the *silhouette of* $\mathbf{r}(N)$ *relative to* π_P. The silhouette is also called *outline* or *image contour* or *apparent contour* (for instance, see [3], [4], [5], [15]).

Lemma 2.1 *For a point $P \in S^{n+1}$ such that $P \cdot \mathbf{r}(x) \neq \pm 1$ for any $x \in N$, the silhouette of $\mathbf{r}(N)$ relative to π_P is the empty set if and only if $P \notin \cup_{x \in N} GH_{\mathbf{r}(x)}$ holds, where $GH_{\mathbf{r}(x)} = S^{n+1} \cap (\mathbb{R}\mathbf{r}(x) + d\mathbf{r}_x(T_xN))$.*

Proof of Lemma 2.1. Since $\mathbf{r}$ is a C^∞ immersion, a point $x \in N$ is a singular point of $\pi_P \circ \mathbf{r}$ if and only if $S^{n+1} \cap (\mathbb{R}P + \mathbb{R}\mathbf{r}(x)) \subset GH_{\mathbf{r}(x)}$. Thus, Lemma 2.1 follows. $\square$

Let P^{n+1} be the $(n+1)$-dimensional real projective space. For any 1 dimensional linear subspace $\ell \in P^{n+1}$, we identify ℓ with the set $\ell \cap S^{n+1}$.

Lemma 2.2 *The map $[\mathbf{n}] : N \to P^{n+1}$ given by*

$$[\mathbf{n}](x) = \bigcap_{Y \in GH_{\mathbf{r}(x)}} E_Y,$$

is well-defined under the above identification.

Proof of Lemma 2.2. For any $Y \in d\mathbf{r}_x(T_xN)$ there exists a C^∞ curve c :

$(-\epsilon, \epsilon) \to N$ such that $c(0) = x$ and $Y = \frac{d\mathbf{r}(c(t))}{dt}|_{t=0}$. By differentiating $\mathbf{r}(c(t))\cdot\mathbf{r}(c(t)) = 1$ with respect to $t \in (-\epsilon, \epsilon)$, we see that $\mathbf{r}(x)\cdot Y = 0$. Since $\mathbf{r}$ is a C^∞ immersion, the vector space $d\mathbf{r}_x(T_xN)$ is n-dimensional. Therefore, the intersection in the right-hand side is the set of two antipodal points in S^{n+1}, which can be identified with the 1-dimensional linear subspace passing through these points. $\square$

The map $[\mathbf{n}]$ is called the *projective dual* of $\mathbf{r}$ (for instance, see [2]). It should be noted that the projective dual of $\mathbf{r}$ may be singular in general though $\mathbf{r}$ is non-singular. Since the set

$$\bigcup_{x\in N}\Big(\bigcap_{Y\in GH_{\mathbf{r}(x)}} E_Y\Big)$$

is equal to

$$\big\{X \in S^{n+1} \mid \exists\, x \in N \text{ such that } X\cdot\mathbf{r}(x) = 0,\ X\cdot Y = 0 \quad \forall\, Y \in d\mathbf{r}_x(T_xN)\big\},$$

under the above identification the image $[\mathbf{n}](N)$ may be identified with the envelope of the family $\{E_{\mathbf{r}(x)} \mid x \in N\}$ (for the definition of the envelope, for instance see [3]).

For any $x \in N$ define $E_{[\mathbf{n}](x)} = E_X$ $(X \in [\mathbf{n}](x))$. By Lemma 2.2 $E_{[\mathbf{n}](x)}$ is well-defined and we have the following:

Lemma 2.3 *For any $x \in N$ the equality $E_{[\mathbf{n}](x)} = GH_{\mathbf{r}(x)}$ holds.*

If N is orientable, then the *spherical dual* $\mathbf{n} : N \to S^{n+1}$ of $\mathbf{r}$ is well-defined by choosing one point from among the set of two antipodal points $[\mathbf{n}](x) \subset S^{n+1}$ for any $x \in N$ compatible with the choice of orientation of N (see [1, p. 3], [12], [13, Section 18]).

A C^∞ map $f : N \to S^{n+1}$ (resp. $F : N \to P^{n+1}$) is said to be *hemispherical* if there exists a point $P \in S^{n+1}$ such that $E_P \cap f(N) = \emptyset$ (resp. $E_P \cap F(N) = \emptyset$) (here, $F(N)$ is regarded as the union of 1-dimensional linear subspaces of $\mathbb{R}^{n+2}$ parametrized by N), and such a point is called a *hemispherical point for f* (resp. F).

Theorem 2.4 *Let N be a compact connected n-dimensional C^∞ manifold and let $\mathbf{r} : N \to S^{n+1}$ be a C^∞ immersion. For a point $P \in S^{n+1}$ the following* **(a)** *and* **(b)** *are equivalent. Moreover, if $n \geq 2$, then the following*

(c), *too, is an equivalent condition.*

(a) *The silhouette of* $\mathbf{r}(N)$ *relative to* π_P *is the empty set.*
(b) *The point* P *is a hemispherical point for the projective dual* $[\mathbf{n}]$.
(c) *The map* $\pi_P \circ \mathbf{r}$ *is a* C^∞ *diffeomorphism.*

Note first that the condition **(c)** implies that the given manifold N is C^∞ diffeomorphic to S^n and that the given immersion $\mathbf{r}$ is an embedding. Note second that any one of **(a)**, **(b)**, **(c)** implies that $P \cdot \mathbf{r}(x) \neq \pm 1$ for any $x \in N$. Note third that as Fig. 3 shows the condition **(c)** is not a necessary condition for the point P satisfying the condition **(a)**, or equivalently the condition **(b)** when $n = 1$. Note fourth that the equivalence of the condition **(c)** implies that all higher dimensional spherical limaçons must be singular, where a higher dimensional spherical limaçon is a C^∞ map $\mathbf{r} : S^n \to S^{n+1}$ $(n \geq 2)$ which satisfy **(a)** or **(b)** but does not satisfy **(c)**.

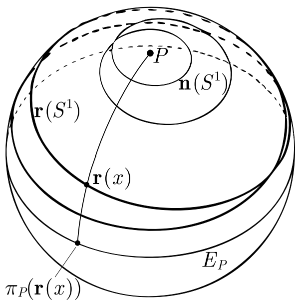

Figure 3. Spherical limaçon and its dual.

Proof of Theorem 2.4. Since $P \in E_Q$ holds if and only if $Q \in E_P$ holds for any two points $P, Q \in S^{n+1}$, $P \in E_{[\mathbf{n}](x)}$ holds if and only if $[\mathbf{n}](x) \subset E_P$ holds. Therefore, by virtue of Lemmas 2.1, 2.2 and 2.3, we have that **(a)** $\Leftrightarrow$ **(b)**.

Next, suppose that $\pi_P \circ \mathbf{r}$ is a C^∞ diffeomorphism. Then, by the definition of silhouettes, **(a)** is satisfied. Therefore, in order to finish the proof of Theorem 2.4 it is sufficient to show the following:

Lemma 2.5[1] *Let* N *be a compact connected* n*-dimensional* C^∞ *manifold*

[1]As pointed out by one of referees, in the case that $n = 2$ and N is oriented, Lemma 2.5 can be obtained easily by Quine's theorem [14].

and $\mathbf{r} : N \to S^{n+1}$ *be a* C^∞ *immersion. Suppose that* $n \geq 2$ *and* $P \notin \cup_{x \in N} E_{[\mathbf{n}](x)}$ *holds. Then,* $\pi_P \circ \mathbf{r}$ *is a* C^∞ *diffeomorphism.*

Proof of Lemma 2.5. Let $A \subset \pi_P(\mathbf{r}(N))$ be the set of regular values of the composition $\pi_P \circ \mathbf{r}$ and put $B = E_P - \pi_P(\mathbf{r}(N))$. We have that $A \cap B = \emptyset$. Since N is compact, we have that A and B are open. Note that $\pi_P \circ \mathbf{r}$ has no silhouettes by Lemma 1.1. Thus, we have that $A \neq \emptyset$ and $E_P = A \cup B$. Since E_P is connected, we have that $E_P = A$. Hence, $\pi_P \circ \mathbf{r}$ is surjective. Let X be a point of E_P. Since N is compact, as shown in [10, p. 8], the set $(\pi_P \circ \mathbf{r})^{-1}(X)$ is a finite set. Thus, we may put $(\pi_P \circ \mathbf{r})^{-1}(X) = \{x_1, \ldots, x_k\}$ $(k < \infty)$, where $x_i \neq x_j$ if $i \neq j$. Suppose that $k \geq 2$. Then, note that $x_1 \neq x_k$. Since N is connected there exists a continuous map $f : [0,1] \to N$ such that $f(0) = x_1$ and $f(1) = x_k$. Then, since $\pi_P \circ \mathbf{r} \circ f(0) = \pi_P \circ \mathbf{r} \circ f(1) = X$ and we have assumed that $n \geq 2$, there exists a homotopy $F : [0,1] \times [0,1] \to E_P$ between $\pi_P \circ \mathbf{r} \circ f$ and a constant map $X : [0,1] \to E_P$ $(X(s) = X)$ such that $F(s,0) = \pi_P \circ \mathbf{r} \circ f(s)$, $F(s,1) = X$ and $F(0,t) = F(1,t) = X$ for any $t \in [0,1]$. Since we have shown that $\pi_P \circ \mathbf{r}$ is non-singular and surjective, by the covering homotopy theorem (cf. [16]), there exist a continuous map $G : [0,1] \times [0,1] \to N$ such that $G(s,0) = f(s)$ and $\pi_P \circ \mathbf{r} \circ G(s,t) = F(s,t)$. Since $f(0) = x_1$, $f(1) = x_k$, $F(0,t) = F(1,t) = X$ and $F(s,1) = X$, we have that $G(0,t) = x_1$ and $G(1,t) = x_k$ and the connected subset $G([0,1],1)$ is contained in $(\pi_P \circ \mathbf{r})^{-1}(X)$. Hence and since the set $(\pi_P \circ \mathbf{r})^{-1}(X)$ is a finite set, the point $G(0,1) = x_1$ must be equal to $G(1,1) = x_k$, which contradicts the assumption that $x_1 \neq x_k$. Therefore we have that $k = 1$ and thus $\pi_P \circ \mathbf{r}$ is injective under the assumption that $n \geq 2$. Since we have proved that $\pi_P \circ \mathbf{r}$ is bijective, by the inverse function theorem the map $\pi_P \circ \mathbf{r}$ must be a C^∞ diffeomorphism. $\square$

Let $\widetilde{\mathbf{r}} : N \to \mathbb{R}^{n+1} \times \{1\}$ be a C^∞ immersion. By composing an appropriate parallel translation if necessary, we may assume that the image $\widetilde{\mathbf{r}}(N)$ does not contain the point $P = (0, \ldots, 0, 1) \in \mathbb{R}^{n+1} \times \{1\}$. Note that $\{P\} = S^{n+1} \cap (\mathbb{R}^{n+1} \times \{1\})$. Let $S^{n+1}_{P,+}$ be the upper hemisphere $\{X \in S^{n+1} \mid P \cdot X > 0\}$. Let $\alpha_P : S^{n+1}_{P,+} \to \mathbb{R}^{n+1} \times \{1\}$ be the map defined by $\alpha_P((X_1, \ldots, X_{n+2})) = \left(\frac{X_1}{X_{n+2}}, \ldots, \frac{X_{n+1}}{X_{n+2}}, 1\right)$ for any $X = (X_1, \ldots, X_{n+2}) \in S^{n+1}_{P,+}$. The map α_P is called the *central projection*. Then, by putting $\mathbf{r} = \alpha_P^{-1} \circ \widetilde{\mathbf{r}}$ we obtain a C^∞ immersion $\mathbf{r} : N \to S^{n+1}$ and thus we can apply Theorem 2.4 to a C^∞ immersion $\mathbf{r} : N \to S^{n+1}$. The restriction of

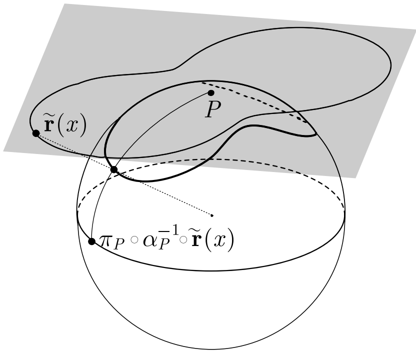

Figure 4. Perspective projection of $\widetilde{\mathbf{r}}(N)$ maps $\widetilde{\mathbf{r}}(x)$ to $\pi_P \circ \alpha_P^{-1} \circ \widetilde{\mathbf{r}}(x)$.

the composition $\pi_P \circ \alpha_P^{-1}|_{\widetilde{\mathbf{r}}(N)} : \widetilde{\mathbf{r}}(N) \to E_P$ is called the *perspective projection* of $\widetilde{\mathbf{r}}(N)$ from the *perspective point* P. Our definition of the perspective projection is a higher dimensional generalization of the usual perspective projection via an immersion $\mathbf{r}$ (for the usual perspective projection, for instance see [4]). By Lemma 2.1 we see that the point $x \in N$ is a singular point of $\pi_P \circ \mathbf{r}$ if and only if the hyperplane $\{\widetilde{\mathbf{r}}(x) + X \mid X \in d\widetilde{\mathbf{r}}_x(T_x(N))\}$ contains the point P. For any point $\widetilde{Q} \in \mathbb{R}^{n+1} \times \{1\}$ we put

$$\widetilde{E}_{\widetilde{Q},P} = \big\{X \in \mathbb{R}^{n+1} \times \{1\} \mid \widetilde{Q} \cdot X = 0\big\}.$$

Then, $\widetilde{E}_{\widetilde{Q},P}$ is a hyperplane of $\mathbb{R}^{n+1} \times \{1\}$ (resp. the empty set) if $P \neq \widetilde{Q}$ (resp. $P = \widetilde{Q}$). Furthermore, we see easily that for any $Q \in S^{n+1}_{P,+}$ the equality $\alpha_P(S^{n+1}_{P,+} \cap E_Q) = \widetilde{E}_{\alpha_P(Q),P}$ holds. The envelope of the hyperplane family $\{\widetilde{E}_{\widetilde{\mathbf{r}}(x),P} \mid x \in N\}$ is called the *dual hypersurface of* $\widetilde{\mathbf{r}}$ *relative to* the point P. Note that the dual hypersurface of $\widetilde{\mathbf{r}}$ relative to P is nothing but the intersection $[\mathbf{n}](N) \cap (\mathbb{R}^{n+1} \times \{1\})$, where $[\mathbf{n}]$ is the projective dual of $\mathbf{r} = \alpha_P^{-1} \circ \widetilde{\mathbf{r}}$ and the set $[\mathbf{n}](N)$ is regarded as the union of 1-dimensional linear subspaces of $\mathbb{R}^{n+2}$ parametrized by N. As a corollary of Theorem 2.4 we obtain the following:

Theorem 2.6 *Let N, $\widetilde{\mathbf{r}} : N \to \mathbb{R}^{n+1} \times \{1\}$ and P be a compact connected n-dimensional C^∞ manifold, a C^∞ immersion and the point $(0, \ldots, 0, 1) \in \mathbb{R}^{n+1} \times \{1\}$ respectively. Then, the following* **(a)** *and* **(b)** *are equivalent. Moreover, if $n \geq 2$, then the following* **(c)***, too, is an equivalent condition.*

(a) *The silhouette of* $\alpha_P^{-1} \circ \widetilde{\mathbf{r}}(N)$ *relative to* π_P *is the empty set.*
(b) *The dual hypersurface of* $\widetilde{\mathbf{r}}$ *relative to* P *is compact.*
(c) *The map* $\pi_P \circ \alpha_P^{-1} \circ \widetilde{\mathbf{r}}$ *is a* C^∞ *diffeomorphism.*

3. Hedgehogs

Definition 1 Let $h : S^n \to \mathbb{R}$ be a C^∞ function. For any $\theta \in S^n \subset \mathbb{R}^{n+1}$ we let the hyperplane $\{X \in \mathbb{R}^{n+1} \times \{1\} \mid X \cdot (\theta, 0) = h(\theta)\}$ be denoted by $\Pi_{h,\theta}$. The envelope constructed by the family $\{\Pi_{h,\theta} \mid \theta \in S^n\}$ is called the *hedgehog defined by the support function* h and is denoted by $\mathcal{H}_h$.

For details of hedgehogs, see for instance [8], [9]. As shown by the subtitle of [8], a hedgehog is the envelope parametrized by its Gauss maps. Thus, hedgehogs seem to have closely related with our research in Section 2. We clarify the relation in this section. Put $P = (0, \ldots, 0, 1) \in S^{n+1}$ and let the cylinder $\{(\theta, \rho) \mid \theta \in S^n, \rho \in \mathbb{R}\}$ be denoted by C_P. Furthermore, let $\beta_P : S^{n+1} - \{\pm P\} \to C_P$ be the central cylindrical projection (see Fig. 5).

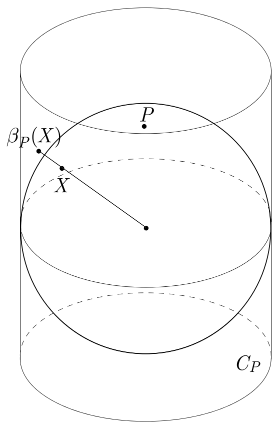

Figure 5. Central cylindrical projection β_P.

Theorem 3.1 *Any hedgehog* $\mathcal{H}_h$ *can be realized as the set* $\alpha_P([\mathbf{n}](S^n) \cap S^{n+1}_{P,+})$*, where* $P = (0, \ldots, 0, 1) \in S^{n+1}$ *and* $[\mathbf{n}]$ *is the projective dual of the embedding* $\mathbf{r} : S^n \to S^{n+1}$ *defined by* $\mathbf{r}(\theta) = \beta_P^{-1}(\theta, -h(\theta))$.

Note that any hedgehog $\mathcal{H}_h$ must be compact by the construction of $\mathbf{r}$ in Theorem 3.1.

Proof of Theorem 3.1. Define the C^∞ map $\mathbf{r} : S^n \to S^{n+1}$ as $\mathbf{r}(\theta) = \beta_P^{-1}(\theta, -h(\theta))$. From the construction of $\mathbf{r}$, we see that the composition $\pi_P \circ \mathbf{r}$ is a C^∞ diffeomorphism. Thus, by Theorem 2.4 we see that P is a hemispherical point for the projective dual of $\mathbf{r}$. Since the projective dual of $\mathbf{r}$ is the envelope of the family $\{E_{\mathbf{r}(\theta)} \mid \theta \in S^n\}$, in order to show that the hedgehog $\mathcal{H}_h$ is equal to the set $\alpha_P([\mathbf{n}](S^n) \cap S^{n+1}_{P,+})$, it is sufficient to show that $\Pi_{h,\theta} = \alpha_P(E_{\mathbf{r}(\theta)} \cap S^{n+1}_{P,+})$ holds for any $\theta \in S^n$.

Let $\Psi_P : S^{n+1} - \{\pm P\} \to S^{n+1}$ be the map given by

$$\Psi_P(X) = \frac{1}{\sqrt{1 - (P \cdot X)^2}}(P - (P \cdot X)X).$$

The map Ψ_P, which has been introduced in [11] for the study of spherical pedal curves (see also [7] where the hyperbolic version of Ψ_P has been introduced and studied), has the following interesting properties:

1. $X \cdot \Psi_P(X) = 0$ for any $X \in S^{n+1} - \{\pm P\}$.
2. $\Psi_P(X) \in \mathbb{R}P + \mathbb{R}X$ for any $X \in S^{n+1} - \{\pm P\}$.
3. $P \cdot \Psi_P(X) > 0$ for any $X \in S^{n+1} - \{\pm P\}$.

By the above property 3, $\alpha_P \circ \Psi_P \circ \mathbf{r}(\theta)$ is well-defined for any $\theta \in S^n$. By the above properties 1 and 2, we have the following (see Fig. 6):

$$h(\theta) = (\alpha_P \circ \Psi_P \circ \mathbf{r}(\theta)) \cdot (\theta, 0) \quad (\forall \theta \in S^n).$$

Then, by considering the geometric meaning of the central projection α_P and the above property 1, we see easily the following equivalence holds for any $\theta \in S^n$.

$$X \in \Pi_{h,\theta} \Leftrightarrow \alpha_P^{-1}(X) \in E_{\mathbf{r}(\theta)}. \qquad \square$$

Theorem 3.2 *Let $\mathbf{r} : S^1 \to S^2$ be a C^∞ embedding such that all inflection points of $\mathbf{r}$ are ordinary inflection points and $\mathbf{r}$ has at least one inflection point. Suppose that the projective dual $[\mathbf{n}] : S^1 \to P^2$ is injective. Then, $\mathbf{r}$ is not hemispherical.*

Proof of Theorem 3.2. Suppose that there exists a point $P \in S^{n+1}$ such that P is a hemispherical point for $\mathbf{r}$. Without loss of generality, we may assume that $\mathbf{r}(S^1) \subset S^2_{P,+}$. Then, $\alpha_P \circ \mathbf{r}$ is a plane curve. Note that by

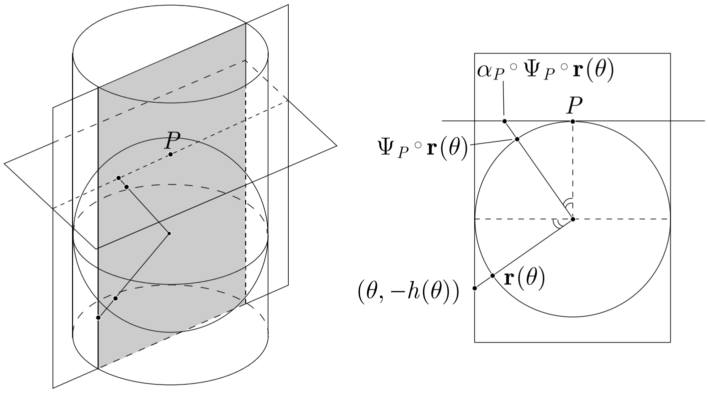

Figure 6. $h(\theta) = (\alpha_P \circ \Psi_P \circ \mathbf{r}(\theta)) \cdot (\theta, 0)$.

the central projection α_P an inflection point of a spherical curve is mapped to an inflection point of a plane curve and an ordinary inflection point of a spherical curve is mapped to an ordinary inflection point of a plane curve. Since we have assumed that the projective dual of $\mathbf{r}$ is injective, there are no lines which tangent to $\alpha_P \circ \mathbf{r}(S^1)$ at more than one points. Then, by the celebrated formula due to Fabricius-Bjerre ([6]) we see that the plane curve $\alpha_P \circ \mathbf{r}$ does not have any inflection points. However, since we have assumed that there exists at least one ordinary inflection point for $\mathbf{r}$, this yields a contradiction. $\square$

Theorem 3.3 *Let $\mathbf{r} : S^n \to S^{n+1}$ be a C^∞ embedding. For a point $P \in S^{n+1}$ the following* **(a)**, **(b)**, **(c)** *and* **(d)** *are equivalent.*

- **(a)** *The silhouette of $\mathbf{r}(S^n)$ relative to π_P is the empty set.*
- **(b)** *The point P is a hemispherical point for the projective dual $[\mathbf{n}]$.*
- **(c)** *The map $\pi_P \circ \mathbf{r}$ is a C^∞ diffeomorphism.*
- **(d)** *The set $\alpha_P([\mathbf{n}](S^n) \cap S^{n+1}_{P,+})$ is a hedgehog $\mathcal{H}_h$ in the hyperplane $\{X \in \mathbb{R}^{n+2} \mid P \cdot X = 1\}$, where the support function h is given by $\beta_P \circ \mathbf{r}(\theta) = (\theta, -h(\theta))$.*

Proof of Theorem 3.3. First we show that **(d)** implies **(b)**. Suppose that there exists a point P such that $\alpha_P([\mathbf{n}](S^n) \cap S^{n+1}_{P,+})$ is a hedgehog. By rotating $\mathbb{R}^{n+2}$ if necessary we may assume that $P = (0, \ldots, 0, 1)$. Then, by Theorem 3.1 $\alpha_P([\mathbf{n}](S^n) \cap S^{n+1}_{P,+})$ is a compact subset of $\mathbb{R}^{n+1} \times \{1\}$ and hence P is a hemispherical point for the projective dual.

Next, we show that **(b)** implies **(c)**. Suppose that P is a hemispherical point for the projective dual. Then, in the case that $n \geq 2$, $\pi_P \circ \mathbf{r}$ is a C^∞ diffeomorphism by Theorem 2.4. Since the given $\mathbf{r}$ is an embedding by the assumption, the set $\mathbf{r}(S^n)$ is an embedded hypersurface. Thus, by the proof of Theorem 2.4 we see that $\pi_P \circ \mathbf{r}$ is a C^∞ diffeomorphism also in the case that $n = 1$,

Since we have already shown that **(c)** implies **(d)** in the proof of Theorem 3.1, we have that **(a)** $\Leftrightarrow$ **(b)** $\Leftrightarrow$ **(c)** $\Leftrightarrow$ **(d)** by Theorem 2.4. $\square$

Similarly as in Section 2, we have the following:

Theorem 3.4 *Let* $\widetilde{\mathbf{r}} : S^n \to \mathbb{R}^{n+1} \times \{1\}$, P *be a* C^∞ *embedding and the point* $(0, \ldots, 0, 1) \in \mathbb{R}^{n+1} \times \{1\}$ *respectively. Then, the following* **(a)**, **(b)**, **(c)** *and* **(d)** *are equivalent.*

(a) *The silhouette of* $\alpha_P^{-1} \circ \widetilde{\mathbf{r}}(S^n)$ *relative to* π_P *is the empty set.*
(b) *The dual hypersurface of* $\widetilde{\mathbf{r}}$ *relative to* P *is compact.*
(c) *The map* $\pi_P \circ \alpha_P^{-1} \circ \widetilde{\mathbf{r}}$ *is a* C^∞ *diffeomorphism.*
(d) *The dual hypersurface of* $\widetilde{\mathbf{r}}$ *relative to* P *is a hedgehog.*

Theorem 3.4 is more significant than Theorem 2.6 since by Theorem 3.4 we can apply properties of hedgehogs to our research in Section 2. However, note that not all hedgehogs can be realized as dual hypersurfaces of C^∞ embeddings. For instance, by Theorem 3.2 the astroid in the right side of Fig. 7, which is a hedgehog since it can be constructed in the same way as in Theorem 3.1, never appear under the situation of Theorem 3.4 while by Theorem 3.1 it can appear under the situation of Theorem 3.3.

Two curves C_1, C_2 in Fig. 8 are the same curves as C_1, C_2 in Fig. 2 respectively. In Fig. 8 dual curves, too, are drawn. By the equivalence of **(a)** and **(d)** in Theorem 3.4, a point P is outside the union of tangent lines for the given plane curve if and only if the dual curve of the given curve relative to the point P is a hedgehog. Thus, we can say that Theorem 3.4 clarifies the hidden meaning of the complement of the union of tangent lines in Fig. 2 from the view point of hedgehogs.

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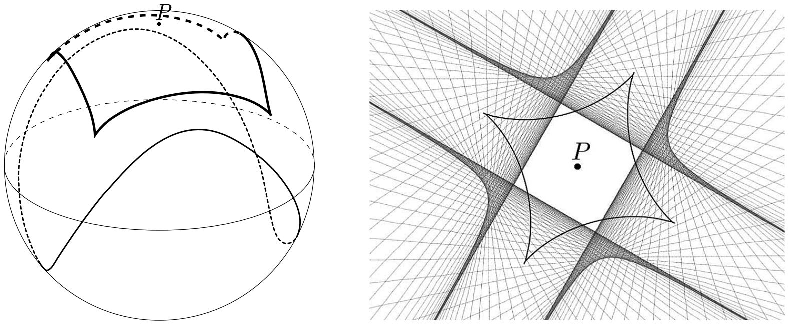

Figure 7. Left: The image of an embedding $\mathbf{r} : S^1 \to S^2$ and the intersection $[\mathbf{n}](S^1) \cap S^2_{P,+}$ which has four singular points. Right: The astroid $\alpha_P([\mathbf{n}](S^1) \cap S^2_{P,+})$ and the image $\alpha_P(\mathbf{r}(S^1) \cap S^2_{P,+})$.

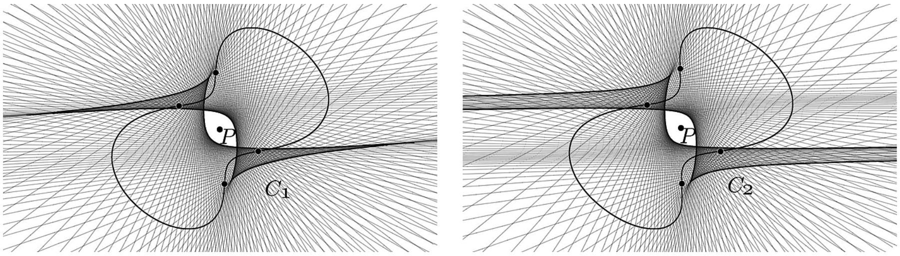

Figure 8. Left: The dual curve of C_1 relative to P is a hedgehog. Right: The dual curve of C_2 relative to P is not a hedgehog.

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