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WHICH INTEGRABLE FUNCTIONS FAIL TO BE ABSOLUTELY INTEGRABLE?

In this note we deal with a general integration theory for scalar functions of one real variable which can be the improper Riemann theory, or more generally the Denjoy-Perron (or Henstock-Kurzweil) theory. Actually, the main property of our integration theory we will use is the following: *if f is an integrable function defined on an interval $[a,b]$ then its indefinite integral defined by*

$$F(x) = \int_a^x f \qquad \textit{for all} \quad x \in [a,b]$$

is continuous, differentiable almost everywhere (a.e.) and

$$F'(x) = f(x) \qquad \textit{for almost all} \quad x \in [a,b]$$

The typical example of a function which is integrable but fails to be absolutely integrable is the function $\frac{\sin(x)}{x}$ on the interval $[0,+\infty)$. Strongly inspired by this example, there is a very natural and well known method that one may use to construct further examples in a given nontrivial interval [a,b]. It can be seen, for instance, in [1], Chapter 10, Example 10.2.2, or in [4], Chapter 7, Exercise (5). The idea is to take a conditionally convergent series of real numbers (i. e., a convergent series which is not absolutely convergent) $\sum c_n$,

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and a strictly monotone sequence (t_n) in $[a,b]$, and then build a function f on $[a,b]$ in such a way that

$$c_n = \int_{t_{n-1}}^{t_n} f \quad \text{for all } n. \tag{1}$$

It is natural to ask: *is it possible to construct further examples of integrable non absolutely integrable functions?* I believe that at first one thinks that the answer should be affirmative. We will see that, in some way, the answer is "no".

Notice that the method mentioned above tells us that given a conditionally convergent series $\sum c_n$, and a strictly monotone sequence (t_n) in $[a,b]$, we can build an integrable but nonabsolutely integrable function f on $[a,b]$ such that (1) holds. We have that *conversely*, given an integrable but nonabsolutely integrable function f on $[a,b]$ we can find a conditionally convergent series $\sum c_n$, and a strictly monotone sequence (t_n) in $[a,b]$ such that (1) holds:

Theorem. *Assume that $f : [a,b] \to \mathbb{R}$ is an integrable but nonabsolutely integrable function, then there exists a strictly monotone sequence (t_n) in $[a,b]$ such that the series*

$$\sum_{n=1}^{\infty} \int_{t_{n-1}}^{t_n} f$$

is conditionally convergent.

Of course, I am sure that for some people the result would be "well known," but I could not find it in the literature. Let us see the proof. We need first the following lemma:

Lemma. *Let $F : [a,b] \to \mathbb{R}$ be a function which does not have bounded variation (in short, (BV)) on $[a,b]$, then at least one of the following two conditions holds:*

(L) There exists $c \in (a,b]$ such that for each $\alpha \in [a,c)$ F is not (BV) on $[\alpha,c]$.

(R) There exists $c \in [a,b)$ such that for each $\beta \in (c,b]$ F is not (BV) on $[c,\beta]$.

Proof of the Lemma. Let $J = \{t \in [a,b] : F \text{ is (BV) on } [a,t]\}$. It is clear that $a \in J$, $b \notin J$ and that if $\beta \in J$ then $[a,\beta] \subset J$. Therefore, J is an interval. Either $J = [a,c)$ or $J = [a,c]$ for some $c \in [a,b]$.

Case 1: $J = [a, c)$. Since $a \in J$, it follows that $c > a$ and then $c \in (a, b]$. Let us see that (L) holds. If $\alpha \in [a, c) = J$, then $\alpha \in J$. Since $c \notin J$ it follows that F is not (BV) on $[\alpha, c]$.

Case 2: $J = [a, c]$. Since $b \notin J$, it follows that $c < b$ and then $c \in [a, b)$. Let us see that (R) holds. If $\beta \in (c, b]$, then $\beta \notin J$. Since $c \in J$ it follows that F is not (BV) on $[c, \beta]$. □

PROOF OF THE THEOREM. Assume that $f : [a, b] \to \mathbb{R}$ is an integrable non-absolutely integrable function and let F be its indefinite integral:

$$F(x) = \int_a^x f \qquad \text{for all} \quad x \in [a, b]$$

F is continuous, differentiable a.e. and

$$F'(x) = f(x)$$

a.e. in $[a, b]$. Let M be an upper bound of $|F|$ in $[a, b]$. Since f is not Lebesgue integrable, by a classical result of Lebesgue (see Chapter 7, Exercise 13 (e) of [7], or Theorem 8.19 in the former editions of [7]), we deduce that F is not (BV) on $[a, b]$. Therefore, we can apply the preceding lemma. Let us suppose, for instance, that (L) holds (we could proceed in an analogous way if (R) holds). There exists $c \in (a, b]$ such that for each $\alpha \in [a, c)$ F is not (BV) on $[\alpha, c]$. Since F is not (BV) on $[a, c]$ (with $\alpha = a$), there exists a partition $a = t_0 < t_1 \cdots < t_m = c$ of $[a, c]$ such that

$$\sum_{i=1}^{m} |F(t_i) - F(t_{i-1})| \geq 2M + 1$$

Therefore,

$$\sum_{i=1}^{m-1} |F(t_i) - F(t_{i-1})| + 2M \geq \sum_{i=1}^{m} |F(t_i) - F(t_{i-1})| \geq 2M + 1$$

and it follows

$$\sum_{i=1}^{m-1} |F(t_i) - F(t_{i-1})| \geq 1 \tag{2}$$

Notice that, if we denote $a_1 = t_{m-1}$, we have that $a = t_0 < t_1 \cdots < t_{m-1} = a_1$ is a partition of $[a, a_1]$ satisfying (2) and F is not (BV) on $[a_1, c]$, thanks to (L) (with $\alpha = a_1$). Of course, we can proceed in the same way in $[a_1, c]$. Therefore, by induction, we find a strictly increasing sequence (t_n) in $[a, c]$ such that

$$\sum_{n=1}^{\infty}(F(t_n) - F(t_{n-1})) = \sum_{n=1}^{\infty}\int_{t_{n-1}}^{t_n} f$$

is not absolutely convergent. However, if we denote $L = \sup\{t_n : n \in \mathbb{N}\} = \lim_m t_m$, we have

$$\sum_{n=1}^{\infty}\int_{t_{n-1}}^{t_n} f = \lim_m \sum_{n=1}^{m}\int_{t_{n-1}}^{t_n} f = \lim_m \int_{t_0}^{t_m} f = \int_{t_0}^{L} f$$

□

Remark. *One can easily verify that the statement in the theorem also holds for unbounded intervals.*

There is a large literature on the relations of Riemann types integral of functions taking values in general Banach spaces and series (see, for instance, [2, 3, 5, 6, 8]).

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