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# Quantum Schubert Cells via Representation Theory and Ring Theory

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## 1. Introduction

The study of the spectra of quantum groups for generic deformation parameters was initiated twenty years ago by Joseph [20; 21] and Hodges–Levasseur–Toro [18] who obtained a number of important results on them. One of the long-term goals was to understand these spectra geometrically in terms of symplectic foliations in an attempt to extend the orbit method [9] to more general classes of algebras and Poisson manifolds. This grew into a very active area of studying the ring theoretic properties of quantum analogues of universal enveloping algebras of solvable Lie algebras. The quantum Schubert cell algebras, defined by De Concini–Kac–Procesi [8] and Lusztig [25], comprise one of the major families of algebras in this area. There is one such algebra $\mathcal{U}^-[w]$ for every simple Lie algebra $\mathfrak{g}$ and an element $w$ of the Weyl group $W$ of $\mathfrak{g}$. It is a subalgebra of the quantized universal enveloping algebra $\mathcal{U}_q(\mathfrak{g})$ and a deformation of the universal enveloping algebra $\mathcal{U}(\mathfrak{n}_- \cap w(\mathfrak{n}_+))$, where $\mathfrak{n}_\pm$ are the nilradicals of a pair of opposite Borel subalgebras $\mathfrak{b}_\pm$ of $\mathfrak{g}$. From another perspective, the algebra $\mathcal{U}^-[w]$ is a deformation of the coordinate ring of the Schubert cell corresponding to $w$ of the full flag variety of $\mathfrak{g}$, equipped with the standard Poisson structure [14]. These algebras played important roles in many different contexts in recent years such as the study of coideal subalgebras of $\mathcal{U}_q(\mathfrak{b}_-)$ and $\mathcal{U}_q(\mathfrak{g})$ [17; 16] and quantum cluster algebras [10].

There are two very different approaches to the study of the spectra of $\mathcal{U}^-[w]$. One is purely ring theoretic and is based on the Cauchon procedure of deleting derivations [6]. The second is a representation theoretic one and builds on the above mentioned methods of Joseph, Hodges, Levasseur, and Toro [21; 18]. Each of these methods has a number of advantages over the other, and relating them is an important open problem with many potential applications. Previously there were no connections between them even for special cases of the algebras $\mathcal{U}^-[w]$, such as the algebras of quantum matrices.

In this paper we unify the ring theoretic and the representation theoretic approaches to the study of $\operatorname{Spec}\mathcal{U}^-[w]$. Furthermore, we resolve several other open problems on the deleting derivation procedure and the spectra of $\mathcal{U}^-[w]$, the two being questions posed by Cauchon and Mériaux [27]. Before we proceed with the statements of these results, we need to introduce some additional background.

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There is a canonical action of the torus $\mathbb{T}^r = (\mathbb{K}^*)^{\times r}$ on $\mathcal{U}^-[w]$ by algebra automorphisms, where $\mathbb{K}$ is the base field and $r$ is the rank of $\mathfrak{g}$. By a general stratification result of Goodearl and Letzter [13], one has a partition

$$\operatorname{Spec}\mathcal{U}^-[w] = \bigsqcup_{I \in \mathbb{T}^{\mathrm{r}}\text{-}\operatorname{Spec}\mathcal{U}^-[w]} \operatorname{Spec}_I \mathcal{U}^-[w].$$

Here $\mathbb{T}^{\mathrm{r}}$-Spec$\mathcal{U}^-[w]$ denotes the set of $\mathbb{T}^r$-invariant prime ideals. By two general results of [13], $\mathbb{T}^{\mathrm{r}}$-Spec$\mathcal{U}^-[w]$ is finite and each stratum

$$\operatorname{Spec}_I \mathcal{U}^-[w] = \left\{ L \in \operatorname{Spec}\mathcal{U}^-[w] \;\middle|\; \bigcap_{t \in \mathbb{T}^r} t \cdot L = I \right\}$$

is homeomorphic to the spectrum of a (commutative) Laurent polynomial ring. The problem of the description of the Zariski topology of Spec$\mathcal{U}^-[w]$, however, is wide open.

The Cauchon method of deleting derivations is a multistage recursive procedure [6] beginning with an iterated Ore extension $A$ of length $l$ (of a certain general type) equipped with a compatible $\mathbb{T}^r$-action and ending with a quantum affine space algebra $\overline{A}$ with a $\mathbb{T}^r$-action. Cauchon constructed in [6] a set-theoretic embedding of Spec $A$ into Spec $\overline{A}$. It induces a set-theoretic embedding $\mathbb{T}^{\mathrm{r}}$-Spec $A \hookrightarrow \mathbb{T}^{\mathrm{r}}$-Spec $\overline{A}$. The $\mathbb{T}^r$-invariant prime ideals of $A$ are then parametrized by some of the subsets of $[1, l]$, called Cauchon diagrams. The $\mathbb{T}^r$-prime ideal of $A$ corresponding to a Cauchon diagram $D \subseteq [1, l]$ will be denoted by $J_D$. The problem of determining which subsets of $[1, l]$ arise in this way (i.e., are Cauchon diagrams) is the essence of the method and is very difficult for each particular class of algebras. It was solved for the algebras of quantum matrices by Cauchon [6] and for all algebras $\mathcal{U}^-[w]$ by Cauchon and Mériaux [27]. To state the latter result, we denote the set of simple roots of $\mathfrak{g}$ by $\Pi$ and the corresponding simple reflections of $W$ by $s_\alpha$, $\alpha \in \Pi$. A word $\mathbf{i} = (\alpha_1, \ldots, \alpha_l)$ in the alphabet $\Pi$ will be called a reduced word for $w$ if $s_{\alpha_1} \cdots s_{\alpha_l}$ is a reduced expression of $w$. Each reduced word $\mathbf{i}$ for $w$ gives rise to a presentation of $\mathcal{U}^-[w]$ as an iterated Ore extension of length $l$. The subsets of $[1, l]$ are index sets for the subwords of $\mathbf{i}$ by the assignment $\{j_1 < \cdots < j_n\} \mapsto (\alpha_{j_1}, \ldots, \alpha_{j_n})$. We will denote by $\leq$ the (strong) Bruhat order on $W$ and set $W^{\leq w} = \{y \in W \mid y \leq w\}$. For each $y \in W^{\leq w}$, there exists a unique left positive subword of $\mathbf{i}$ corresponding to $y$ (see §2.2 for its definition and details on Weyl group combinatorics). Its index set will be denoted by $\mathcal{LP}_{\mathbf{i}}(y)$. The Cauchon–Mériaux classification theorem states the following:

*For all Weyl group elements $w \in W$ and reduced words $\mathbf{i}$ for $w$, consider the presentation of $\mathcal{U}^-[w]$ as an iterated Ore extension corresponding to $\mathbf{i}$. The Cauchon diagrams of the $\mathbb{T}^r$-prime ideals of $\mathcal{U}^-[w]$ are precisely the index sets $\mathcal{LP}_{\mathbf{i}}(y)$ for $y \in W^{\leq w}$.*

The representation theoretic approach [28] to the spectra Spec$\mathcal{U}^-[w]$ relies on a family of surjective $\mathbb{T}^r$-equivariant antihomomorphisms $\phi_w : R_0^w \to \mathcal{U}^-[w]$, where $R_0^w$ are certain quotients of subalgebras of the quantum groups $R_q[G]$. The algebras $R_0^w$ were introduced by Joseph [21] as quantizations of the coordinate rings of $w$-translates of the open Schubert cell of the flag variety of $\mathfrak{g}$, see §2.3 for details. Via these maps one can transfer back and forward questions on the spectra of $\mathcal{U}^-[w]$ to questions on the spectra of quantum function algebras. The latter can be approached via representation theoretic methods, building on the works of Joseph [20; 21], Gorelik [15], and Hodges–Levasseur–Toro [18]. This leads to an explicit picture for $\mathbb{T}^r$-Spec$\,\mathcal{U}^-[w]$. First, the $\mathbb{T}^r$-invariant prime ideals of $\mathcal{U}^-[w]$ are parametrized by $W^{\leq w}$, and the ideal $I_w(y)$ corresponding to $y \in W^{\leq w}$ is explicitly given in terms of Demazure modules using the maps $\phi_w$, see Theorem 2.2 for a precise statement. Second, each of the strata $\mathrm{Spec}_{I_w(y)}\,\mathcal{U}^-[w]$ consists of ideals constructed by contractions from localizations of $\mathcal{U}^-[w]/I_w(y)$ by explicit small multiplicative sets of normal elements.

Each of the above two methods has many advantages over the other. Using the representation theoretic approach, it was proved that all ideals $I_w(y)$ are polynormal, it was established that $\mathcal{U}^-[w]$ are catenary and satisfy Tauvel's height formula, the containment problem for $\mathbb{T}^r$-Spec$\,\mathcal{U}^-[w] = \{I_w(y) \mid y \in W^{\leq w}\}$ was solved, and theorems for separation of variables for $\mathcal{U}^-[w]$ were established (see [28; 30; 31]). In the special case of the algebras of quantum matrices, catenarity and ideal containment were proved earlier [7; 23] within the framework of the ring theoretic approach (though with more complicated arguments), but there was no progress on polynormality or proofs of these results for more general $\mathcal{U}^-[w]$ algebras. On the other hand, using the ring theoretic approach, it was proved that for all $\mathbb{T}^r$-primes $J_D$ of $\mathcal{U}^-[w]$ the factor $\mathcal{U}^-[w]/J_D$ always has a localization that is a quantum torus, its center (which is closely related to the structure of the stratum $\mathrm{Spec}_{J_D}\,\mathcal{U}^-[w]$) was described, and in the case of quantum matrices, $\mathbb{T}^r$-primes were related to total positivity (see [6; 2; 11]).

Our first result resolves Question 5.3.3 of Cauchon and Mériaux [27] and unifies the two approaches to $\mathbb{T}^r$-Spec$\,\mathcal{U}^-[w]$.

Theorem 1.1. *Let $\mathbb{K}$ be an arbitrary base field, $q \in \mathbb{K}^*$ be a non-root of unity, $\mathfrak{g}$ be a simple Lie algebra, $w$ be an element of the Weyl group of $\mathfrak{g}$, and $\mathbf{i}$ be a reduced word for $w$. Consider the presentation of the quantum Schubert cell algebra $\mathcal{U}^-[w]$ as an iterated Ore extension corresponding to $\mathbf{i}$.*

*Then, for all Weyl group elements $y \leq w$, the Cauchon diagram of the $\mathbb{T}^r$-prime ideal $I_w(y)$ of $\mathcal{U}^-[w]$ (from the representation theoretic approach from Theorem 2.2 (i)) is equal to $\mathcal{LP}_{\mathbf{i}}(y)$, the index set of the left positive subword of $\mathbf{i}$ whose total product is $y$.*

*Thus the $\mathbb{T}^r$-prime ideals of $\mathcal{U}^-[w]$ from the representation theoretic approach are related to the ideals $J_D$ from the ring theoretic approach via*

$$I_w(y) = J_{\mathcal{LP}_{\mathbf{i}}(y)}.$$

Furthermore, we prove a theorem that explicitly describes the behavior of the representation theoretic ideals $I_w(y)$ of $\mathcal{U}^-[w]$ in each stage of the Cauchon deleting derivation procedure. This appears in Theorem 4.5 and will not be stated in the introduction since it requires additional background.

With the help of Theorem 1.1, one can now combine the strengths of the two approaches to the spectra of the quantum Schubert cell algebras. We expect that the combination of the two methods will lead to substantial progress in the study of the topology of Spec$\mathcal{U}^-[w]$. We use Theorem 1.1 and previous results of the second author to resolve Question 5.3.2 of Cauchon and Mériaux [27], thereby solving the containment problem for the ideals

$$\{J_{\mathcal{L}\mathcal{P}_{\mathbf{i}}(y)} \mid y \in W^{\leq w}\}$$

of the classification of [27].

THEOREM 1.2. *In the setting of Theorem 1.1, the map*

$$W^{\leq w} \to \mathbb{T}^{\mathrm{r}}\text{-Spec}\,\mathcal{U}^-[w] \quad \textit{given by} \quad y \mapsto J_{\mathcal{L}\mathcal{P}_{\mathbf{i}}(y)}$$

*is an isomorphism of posets with respect to the (strong) Bruhat order and the inclusion order on ideals.*

Finally, Theorem 1.1 also gives a new, independent proof of the Cauchon–Mériaux classification [27] described above. (The proof of Theorem 1.1 does not use results from [27].)

Let us return to the general case of Cauchon's method of deleting derivations. It relates the prime ideals of an initial iterated Ore extension $A$ to the prime ideals of the final algebra $\overline{A}$, the Cauchon quantum affine space algebra associated to $A$. In order to study these ideals, one needs an explicit description of $\overline{A}$ as a subalgebra of the ring of fractions Fract$(A)$. We obtain such a description for all algebras $\mathcal{U}^-[w]$, establishing yet another relationship between the two approaches to the structure of the algebras $\mathcal{U}^-[w]$. Given a reduced word $\mathbf{i} = (\alpha_1, \dots, \alpha_l)$ for $w$, define a successor function $\kappa : [1,l] \to [1,l] \sqcup \{\infty\}$ by

$$\kappa(j) = \min\{k \mid k > j, \alpha_k = \alpha_j\} \quad \text{if } \exists k > j \text{ such that } \alpha_k = \alpha_j,$$
$$\kappa(j) = \infty \quad \text{otherwise.}$$

For $j \in [1,l]$, denote by $\Delta_{\mathbf{i},j} \in \mathcal{U}^-[w]$ the element obtained by evaluating the quantum minor corresponding to the fundamental weight $\varpi_{\alpha_j}$ and the Weyl group elements $s_{\alpha_1} \cdots s_{\alpha_{j-1}}$, $w \in W$ on the $R$-matrix $\mathcal{R}^w$ corresponding to $w$. We refer to §2.3 and §3.1 for details and the description of these elements in the framework of the antiisomorphisms $\phi_w : R_0^w \to \mathcal{U}^-[w]$.

THEOREM 1.3. *In the setting of Theorem 1.1, for all Weyl group elements $w$ and reduced words $\mathbf{i} = (\alpha_1, \dots, \alpha_l)$ for $w$, the generators $\overline{x}_1, \dots, \overline{x}_l$ of the corresponding Cauchon quantum affine space algebras are given by*

$$\overline{x}_j = \begin{cases} (q_{\alpha_j}^{-1} - q_{\alpha_j})^{-1} \Delta_{\mathbf{i},\kappa(j)}^{-1} \Delta_{\mathbf{i},j} & \textit{if } \kappa(j) \neq \infty, \\ (q_{\alpha_j}^{-1} - q_{\alpha_j})^{-1} \Delta_{\mathbf{i},j} & \textit{if } \kappa(j) = \infty \end{cases}$$

*for the standard powers $q_{\alpha_j} \in \mathbb{K}^*$ of $q$, see § 2.1.*

This theorem establishes a connection between the initial cluster for the cluster algebra structure on $\mathcal{U}^-[w]$ of Geiß–Leclerc–Schröer and Cauchon's method of deleting derivations. We will present a deeper study of this in a forthcoming publication. Theorem 1.3 is also an important ingredient in a very recent proof [32] of the second author of the Andruskiewitsch–Dumas conjecture [1].

The paper is organized as follows. Section 2 contains background on the quantum Schubert cell algebras and the representation theoretic and ring theoretic approaches to the study of their spectra. Section 3 contains the proof of Theorem 1.3. Theorems 1.1 and 1.2 are proved in Section 4, where we also establish a theorem describing the behavior of the ideals $I_w(y)$ under the iterations of the deleting derivation procedure.

We will use the following notation throughout the paper. Given a $\mathbb{K}$-algebra $A$, we will denote its center by $Z(A)$. For a $\mathbb{K}$-subspace $V$ of $A$ and $a, b \in A$, we will write $a = b \mod V$ if $a - b \in V$. Set $\mathbb{N} := \{0, 1, \dots\}$ and $\mathbb{Z}_+ := \{1, 2, \dots\}$. For $m, n \in \mathbb{Z}$ set $[m, n] = \{m, \dots, n\}$ if $m \leq n$ and $[m, n] = \emptyset$ otherwise.

## 2. Quantum Schubert Cells

### 2.1. Quantized Universal Enveloping Algebras

We will mostly follow the notation of Jantzen's book [19]. Let $\mathfrak{g}$ be a complex simple Lie algebra with the root system $\Phi$ and the Weyl group $W$. Choose a basis $\Pi$ of $\Phi$. Let $\langle \cdot, \cdot \rangle$ be the invariant bilinear form on $\mathbb{R}\Pi$ normalized by $\langle \alpha, \alpha \rangle = 2$ for short roots $\alpha \in \Phi$. For $\alpha \in \Phi$, denote by $\alpha^\vee$ and $s_\alpha \in W$ the corresponding coroot and reflection. Let $\{\varpi_\alpha \mid \alpha \in \Pi\}$ be the fundamental weights of $\mathfrak{g}$. Denote the root lattice of $\mathfrak{g}$ by $\mathcal{Q} = \mathbb{Z}\Phi$ and set $\mathcal{Q}^+ = \mathbb{N}\Phi$. Let $\mathcal{P}$ be the weight lattice of $\mathfrak{g}$ and $\mathcal{P}^+ = \mathbb{N}\{\varpi_\alpha \mid \alpha \in \Pi\}$ be the set of dominant integral weights of $\mathfrak{g}$. For a subset $I \subseteq \Pi$, set $\mathcal{Q}_I = \mathbb{Z}I$. Recall the standard partial order on $\mathcal{P}$: for $\nu_1, \nu_2 \in \mathcal{P}$, set $\nu_1 \geq \nu_2$ if $\nu_2 = \nu_1 - \gamma$ for some $\gamma \in \mathcal{Q}^+$. Let $\nu_1 > \nu_2$ if $\nu_1 \geq \nu_2$ and $\nu_1 \neq \nu_2$.

Throughout the paper $\mathbb{K}$ will denote a base field (of arbitrary characteristic) and $q \in \mathbb{K}^*$ will denote an element which is not a root of unity. Denote by $\mathcal{U}_q(\mathfrak{g})$ the quantized universal enveloping algebra of $\mathfrak{g}$ over $\mathbb{K}$ with deformation parameter $q$. It has generators $K_\alpha^{\pm 1}$, $E_\alpha$, $F_\alpha$, $\alpha \in \Pi$ and relations [19, §4.3]. The algebra $\mathcal{U}_q(\mathfrak{g})$ has a unique Hopf algebra structure with comultiplication, antipode, and counit satisfying

$$\Delta(K_\alpha) = K_\alpha \otimes K_\alpha, \quad \Delta(E_\alpha) = E_\alpha \otimes 1 + K_\alpha \otimes E_\alpha,$$
$$\Delta(F_\alpha) = F_\alpha \otimes K_\alpha^{-1} + 1 \otimes F_\alpha$$

and

$$S(K_\alpha) = K_\alpha^{-1}, \quad S(E_\alpha) = -K_\alpha^{-1} E_\alpha, \quad S(F_\alpha) = -F_\alpha K_\alpha,$$
$$\varepsilon(K_\alpha) = 1, \quad \varepsilon(E_\alpha) = \varepsilon(F_\alpha) = 0.$$

The subalgebras of $\mathcal{U}_q(\mathfrak{g})$ generated by $\{E_\alpha \mid \alpha \in \Pi\}$, $\{F_\alpha \mid \alpha \in \Pi\}$, and $\{K_\alpha^{\pm 1} \mid \alpha \in \Pi\}$ will be denoted by $\mathcal{U}^+, \mathcal{U}^-$, and $\mathcal{U}^0$, respectively.

Denote by $\leq$ the (strong) Bruhat order on $W$ and by $\ell : W \to \mathbb{N}$ the standard length function. For $w \in W$, set $W^{\leq w} = \{y \in W \mid y \leq w\}$. Let $\mathcal{B}_\mathfrak{g}$ be the braid group of $\mathfrak{g}$ and $\{T_\alpha \mid \alpha \in \Pi\}$ be its standard generating set. We will use Lusztig's action of $\mathcal{B}_{\mathfrak{g}}$ on $\mathcal{U}_q(\mathfrak{g})$ by algebra automorphisms in the version given in [19, §8.14] by eqs. 8.14 (2), (3), (7), and (8).

We will use the following notation for $q$-integers and factorials:

$$[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}}, \qquad [n]_q! := [1]_q \cdots [n]_q, \quad n \in \mathbb{N}.$$

For $\alpha \in \Pi$, define $[n]_\alpha := [n]_{q_\alpha}$ and $[n]_\alpha! := [n]_{q_\alpha}!$, where $q_\alpha := q^{\langle \alpha, \alpha \rangle /2}$.

### *2.2. Weyl Group Combinatorics and Quantum Schubert Cell Algebras*

Fix $w \in W$. A word $\mathbf{i} = (\alpha_1, \ldots, \alpha_l)$ in the alphabet $\Pi$ is called a reduced word for $w$ if $s_{\alpha_1} \cdots s_{\alpha_l}$ is a reduced expression of $w$ (in particular, $\ell(w) = l$). Given a reduced word $\mathbf{i} = (\alpha_1, \ldots, \alpha_l)$ for $w$, define

$$w(\mathbf{i})_{\leq j} := s_{\alpha_1} \cdots s_{\alpha_j} \quad \text{and} \quad w(\mathbf{i})_{> j} := s_{\alpha_{j+1}} \cdots s_{\alpha_l} \quad \text{for } j \in [0, l]. \tag{2.1}$$

Thus $w(\mathbf{i})_{\leq 0} = 1$ and $w(\mathbf{i})_{\leq l} = w$. There is a bijection between the set of subwords of $\mathbf{i}$ and the subsets of $[1, l]$, which associates to a subword $(\alpha_{j_1}, \ldots, \alpha_{j_n})$ of $\mathbf{i}$ its index set $\{j_1 < \cdots < j_n\} \subseteq [1, l]$. Given $D \subseteq [1, l]$, for $j \in [1, l]$ set $s_j^D = s_{\alpha_j}$ if $j \in D$, and $s_j^D = 1$ otherwise. Define

$$w(\mathbf{i})_{\leq j}^D := s_1^D \cdots s_j^D \quad \text{and} \quad w(\mathbf{i})_{> j}^D := s_{j+1}^D \cdots s_l^D \quad \text{for } j \in [1, l]. \tag{2.2}$$

Let

$$w(\mathbf{i})^D := w(\mathbf{i})_{\leq l}^D = s_1^D \cdots s_l^D.$$

Following [26] we call a subword of $\mathbf{i}$ *(right) positive* if its index set $D \subseteq [1, l]$ has the property that

$$w(\mathbf{i})_{\leq j}^D s_{\alpha_{j+1}} > w(\mathbf{i})_{\leq j}^D \quad \text{for all } j \in [1, l-1].$$

A subword of $\mathbf{i}$ will be called *left positive* if its index set $D \subseteq [1, l]$ has the property that

$$s_{\alpha_j} w(\mathbf{i})_{> j}^D > w(\mathbf{i})_{> j}^D \quad \text{for all } j \in [1, l-1]. \tag{2.3}$$

Some authors refer to the left positive subwords of $\mathbf{i}$ as Cauchon diagrams associated to $\mathbf{i}$. However, we will use the term Cauchon diagrams for the general Cauchon procedure of deleting derivations in iterated Ore extensions (see §2.4), and using the same term for different notions will easily lead to confusions.

The map $(\alpha_{j_1}, \ldots, \alpha_{j_n}) \mapsto (\alpha_{j_n}, \ldots, \alpha_{j_1})$ establishes a bijection between the left positive subwords of $\mathbf{i}$ and the right positive subwords of the reduced word $(\alpha_l, \ldots, \alpha_1)$ of $w^{-1}$. Since the map $y \mapsto y^{-1}$ is a bijection between $W^{\leq w}$ and $W^{\leq w^{-1}}$, Lemma 3.5 of Marsh–Rietsch [26] gives that for each $y \in W^{\leq w}$ there exists a unique left positive subword of $\mathbf{i}$ such that its index set $D \subseteq [1, l]$ satisfies $w(\mathbf{i})^D = y$. Denote this index set $D$ by $\mathcal{LP}_{\mathbf{i}}(y)$.

The support of $w \in W$ is defined by

$$\mathcal{S}(w) := \{\alpha \in \Pi \mid s_\alpha \leq w\}. \tag{2.4}$$

Its complement is given by

$$\Pi \setminus \mathcal{S}(w) = \{\alpha \in \Pi \mid w\varpi_\alpha = \varpi_\alpha\}, \tag{2.5}$$

see [29, Lemma 3.2 and eq. (3.2)].

The quantum Schubert cell algebras $\mathcal{U}^{\pm}[w]$, $w \in W$ were defined by De Concini, Kac, and Procesi [8], and Lusztig [25, §40.2] as follows. Given a reduced word $\mathbf{i} = (\alpha_1, \ldots, \alpha_l)$ for $w$, define the roots

$$\beta_j := w(\mathbf{i})_{\leq (j-1)} \alpha_j, \quad j \in [1, l] \tag{2.6}$$

and the Lusztig root vectors

$$E_{\beta_j} := T_{\alpha_1} \cdots T_{\alpha_{j-1}}(E_{\alpha_j}), \qquad F_{\beta_j} := T_{\alpha_1} \cdots T_{\alpha_{j-1}}(F_{\alpha_j}), \quad j \in [1, l], \tag{2.7}$$

see [25, §39.3]. By [8, Proposition 2.2] and [25, Proposition 40.2.1] the subalgebras $\mathcal{U}^{\pm}[w]$ of $\mathcal{U}^{\pm}$ generated by $E_{\beta_j}$, $j \in [1, l]$ and $F_{\beta_j}$, $j \in [1, l]$ do not depend on the choice of a reduced word $\mathbf{i}$ for $w$ and have the PBW bases

$$\begin{aligned} &\{(E_{\beta_l})^{n_l} \cdots (E_{\beta_1})^{n_1} \mid n_1, \ldots, n_l \in \mathbb{N}\} \quad \text{and} \\ &\{(F_{\beta_l})^{n_l} \cdots (F_{\beta_1})^{n_1} \mid n_1, \ldots, n_l \in \mathbb{N}\}, \end{aligned} \tag{2.8}$$

respectively.

The algebra $\mathcal{U}_q(\mathfrak{g})$ is $\mathcal{Q}$-graded by $\deg K_\alpha = 0$, $\deg E_\alpha = \alpha$, $\deg F_\alpha = -\alpha$, $\forall \alpha \in \Pi$. This induces a $\mathcal{Q}$-grading on $\mathcal{U}^{\pm}[w]$. The corresponding graded components will be denoted by $(\mathcal{U}_q(\mathfrak{g}))_\gamma$ and $(\mathcal{U}^{\pm}[w])_\gamma$. One has

$$\mathbb{Z}\{\gamma \in \mathcal{Q} \mid (\mathcal{U}^{\pm}[w])_\gamma \neq 0\} = \mathcal{Q}_{\mathcal{S}(w)}, \tag{2.9}$$

see for example [29, eq. (2.44) and Lemma 3.2 (ii)].

Recall that there is a unique algebra automorphism $\omega$ of $\mathcal{U}_q(\mathfrak{g})$ such that

$$\omega(E_\alpha) = F_\alpha, \quad \omega(F_\alpha) = E_\alpha, \quad \omega(K_\alpha) = K_\alpha^{-1}, \quad \forall \alpha \in \Pi.$$

It satisfies $\omega(T_\alpha(u)) = (-1)^{\langle \alpha^\vee, \gamma \rangle} q^{-\langle \alpha, \gamma \rangle} T_\alpha(\omega(u))$ for all $\gamma \in \mathcal{Q}$, $u \in (\mathcal{U}_q(\mathfrak{g}))_\gamma$, see [19, eq. 8.14 (9)]. In other words, if $\rho$ is the sum of all fundamental weights of $\mathfrak{g}$ and $\rho^\vee$ is the sum of all fundamental coweights of $\mathfrak{g}$, then $\omega(T_\alpha(u)) = (-1)^{\langle s_\alpha(\gamma) - \gamma, \rho^\vee \rangle} q^{-\langle s_\alpha(\gamma) - \gamma, \rho \rangle} T_\alpha(\omega(u))$ for $u \in (\mathcal{U}_q(\mathfrak{g}))_\gamma$. Thus,

$$\begin{gathered} \omega(T_y(u)) = (-1)^{\langle y(\gamma) - \gamma, \rho^\vee \rangle} q^{-\langle y(\gamma) - \gamma, \rho \rangle} T_y(\omega(u)) \\ \text{for all } y \in W, \gamma \in \mathcal{Q}, u \in (\mathcal{U}_q(\mathfrak{g}))_\gamma, \end{gathered}$$

see [19, eq. 8.18 (5)] for an equivalent formulation of this fact. In particular, the restrictions of $\omega$ induce the isomorphisms

$$\begin{aligned} &\omega : \mathcal{U}^+[w] \xrightarrow{\cong} \mathcal{U}^-[w], \qquad \omega(E_{\beta_j}) = (-1)^{\langle \beta_j - \alpha_j, \rho^\vee \rangle} q^{-\langle \beta_j - \alpha_j, \rho \rangle} F_{\beta_j}, \\ &\quad \forall j \in [1, \ell(w)]. \end{aligned} \tag{2.10}$$

To each $\gamma \in \mathcal{Q}$ associate the character of $\mathbb{T}^r = (\mathbb{K}^*)^{\times r}$

$$t \mapsto t^\gamma := \prod_{\alpha \in \Pi} t_\alpha^{\langle \gamma, \varpi_\alpha \rangle}, \quad t = (t_\alpha)_{\alpha \in \Pi} \in \mathbb{T}^r. \tag{2.11}$$

Define the rational $\mathbb{T}^r$-action on $\mathcal{U}_q(\mathfrak{g})$ by algebra automorphisms

$$t \cdot x = t^\gamma x, \quad x \in (\mathcal{U}_q(\mathfrak{g}))_\gamma. \tag{2.12}$$

It preserves the subalgebras $\mathcal{U}^{\pm}[w]$. We will denote by $\mathbb{T}^{\mathrm{r}}$-Spec$\mathcal{U}^-[w]$ the space of $\mathbb{T}^r$-prime ideals of $\mathcal{U}^-[w]$.

Fix a reduced word $\mathbf{i}$ for $w$ and consider the roots (2.6). Equation (2.9) implies that for all $j \in [1, \ell(w)]$ there exists a unique $t_j = (t_{j,\alpha})_{\alpha\in\Pi} \in \mathbb{T}^r$ such that

$$t_j^{\beta_k} = q^{\langle \beta_k, \beta_j\rangle} \quad \forall k \le j \quad \text{and} \quad t_{j,\alpha} = 1 \quad \forall \alpha \in \Pi \setminus \mathcal{S}(w(\mathbf{i})_{\le j}), \tag{2.13}$$

recall (2.11). The Levendorskii–Soibelman straightening law is the following commutation relation in $\mathcal{U}^-[w]$:

$$F_{\beta_j} F_{\beta_k} - q^{-\langle \beta_k, \beta_j\rangle} F_{\beta_k} F_{\beta_j} = \sum_{\mathbf{n}=(n_{k+1},\dots,n_{j-1})\in\mathbb{N}^{\times(j-k-2)}} p_{\mathbf{n}} (F_{\beta_{j-1}})^{n_{j-1}} \cdots (F_{\beta_{k+1}})^{n_{k+1}}, \quad p_{\mathbf{n}} \in \mathbb{K}, \tag{2.14}$$

for all $k < j$, see for example [5, Proposition I.6.10]. The following lemma is a direct consequence of (2.8), (2.13), and (2.14).

LEMMA 2.1. *For all base fields $\mathbb{K}$, $q \in \mathbb{K}^*$ that is not a root of unity, Weyl group elements $w \in W$ of length $l$, reduced words $\mathbf{i} = (\alpha_1, \dots, \alpha_l)$ for $w$, and $j \in [1, l]$, we have*:

(i) *The subalgebra of $\mathcal{U}^-[w]$ generated by $F_{\beta_1}, \dots, F_{\beta_j}$ is equal to $\mathcal{U}^-[w(\mathbf{i})_{\le j}]$.*

(ii) *The algebra $\mathcal{U}^-[w(\mathbf{i})_{\le j}]$ is isomorphic to the Ore extension $\mathcal{U}^-[w(\mathbf{i})_{\le (j-1)}][x_j, \sigma_j, \delta_j]$, where $\sigma_j = (t_j \cdot) \in \mathrm{Aut}(\mathcal{U}^-[w(\mathbf{i})_{\le(j-1)}])$ and $\delta_j$ is a locally nilpotent (left) $\sigma_j$-skew derivation of $\mathcal{U}^-[w(\mathbf{i})_{\le(j-1)}]$ satisfying $\sigma_j\delta_j = q_{\alpha_j}^{-2}\delta_j\sigma_j$. This isomorphism is given by the identity map on $\mathcal{U}^-[w(\mathbf{i})_{\le(j-1)}]$ and $F_{\beta_j} \mapsto x_j$. Furthermore, $\mathcal{U}^-[w(\mathbf{i})_{\le 0}] = \mathcal{U}^-[1] \cong \mathbb{K}$, $\sigma_1 = \mathrm{id}$, and $\delta_1 = 0$.*

(iii) *The eigenvalues $t_j \cdot F_{\beta_j} = q_{\alpha_j}^{-2} F_{\beta_j}$ are not roots of unity.*

The $\sigma_j$-skew derivation $\delta_j$ of $\mathcal{U}^-[w(\mathbf{i})_{\le(j-1)}]$ in part (ii) of the lemma is explicitly given by

$$\delta_j(x) := F_{\beta_j} x - q^{\langle \beta_j, \gamma\rangle} x F_{\beta_j} \quad \text{for } x \in (\mathcal{U}^-[w(\mathbf{i})_{\le(j-1)}])_\gamma, \gamma \in \mathcal{Q} \tag{2.15}$$

and is computed using (2.14).

The isomorphisms from part (ii) give rise to the Ore extension presentations

$$\mathcal{U}^-[w(\mathbf{i})_{\le j}] = \mathcal{U}^-[w(\mathbf{i})_{\le(j-1)}][F_{\beta_j}, \sigma_j, \delta_j], \quad 1 \le j \le l.$$

When those are iterated, for each reduced word $\mathbf{i}$ for $w$, one obtains a presentation of $\mathcal{U}^-[w]$ as an iterated Ore extension

$$\mathcal{U}^-[w] = \mathbb{K}[F_{\beta_1}][F_{\beta_2}; \sigma_2, \delta_2] \cdots [F_{\beta_l}; \sigma_l, \delta_l]. \tag{2.16}$$

### *2.3. The Prime Spectrum of $\mathcal{U}^-[w]$ via Demazure Modules*

We proceed with the realization of the algebras $\mathcal{U}^-[w]$ in terms of quantum function algebras and the description of the spectra of $\mathcal{U}^-[w]$ via Demazure modules from [28].

The $q$-weight spaces of a $\mathcal{U}_q(\mathfrak{g})$-module $V$ are defined by

$$V_\nu := \{v \in V \mid K_\alpha v = q^{\langle \nu, \alpha\rangle} v, \forall \alpha \in \Pi\}, \quad \nu \in \mathcal{P}.$$

A $\mathcal{U}_q(\mathfrak{g})$-module is called a type one module if $V = \bigoplus_{\nu \in \mathcal{P}} V_\nu$. The category of (left) finite-dimensional type one $\mathcal{U}_q(\mathfrak{g})$-modules is semisimple (see [19, Theorem 5.17] and the remark on p. 85 of [19]). It is closed under taking tensor products and duals (defined as left modules using the antipode of $\mathcal{U}_q(\mathfrak{g})$). Denote by $V(\lambda)$ the irreducible type one $\mathcal{U}_q(\mathfrak{g})$-module of highest weight $\lambda \in \mathcal{P}^+$. Those exhaust all irreducible finite-dimensional type one modules, see [19, Theorem 5.10].

For algebraically closed fields $\mathbb{K}$ of characteristic 0, we will denote by $G$ the connected, simply connected algebraic group with a Lie algebra $\mathfrak{g}$. For all base fields $\mathbb{K}$ and deformation parameters $q \in \mathbb{K}^*$ that are not roots of unity, the quantum group $R_q[G]$ is defined as the Hopf subalgebra of the restricted dual $(\mathcal{U}_q(\mathfrak{g}))^\circ$, spanned by the matrix coefficients of the modules $V(\lambda)$, $\lambda \in \mathcal{P}^+$. The latter are given by

$$c^\lambda_{\xi,v} \in (\mathcal{U}_q(\mathfrak{g}))^\circ, \qquad c^\lambda_{\xi,v}(x) := \xi(xv), \quad v \in V(\lambda), \xi \in V(\lambda)^*, x \in \mathcal{U}_q(\mathfrak{g}). \tag{2.17}$$

Because we work with arbitrary base fields, in the notation $R_q[G]$, $G$ is just a symbol.

For each $\lambda \in \mathcal{P}^+$, fix a highest weight vector $v_\lambda$ of $V(\lambda)$. Set for brevity

$$c^\lambda_\xi := c^\lambda_{\xi, v_\lambda}, \quad \lambda \in \mathcal{P}^+, \xi \in V(\lambda)^*.$$

Define the subalgebra

$$R^+ := \mathrm{Span}\{c^\lambda_\xi \mid \lambda \in \mathcal{P}^+, \xi \in V(\lambda)^*\}$$

of $R_q[G]$.

The braid group $\mathcal{B}_\mathfrak{g}$ acts on the finite-dimensional type one $\mathcal{U}_q(\mathfrak{g})$-modules $V$ by

$$T_\alpha(v) := \sum_{l,m,n} \frac{(-1)^m q_\alpha^{m-ln}}{[l]_\alpha! [m]_\alpha! [n]_\alpha!} E_\alpha^l F_\alpha^m E_\alpha^n v, \quad v \in V_\mu, \mu \in \mathcal{P}, \tag{2.18}$$

where the sum is over $l, m, n \in \mathbb{N}$ such that $-l + m - n = \langle \mu, \alpha^\vee \rangle$, cf. [19, §8.6] and [25, §5.2]. This action and the $\mathcal{B}_\mathfrak{g}$-action on $\mathcal{U}_q(\mathfrak{g})$ are compatible by

$$T_w(x \cdot v) := (T_w x) \cdot (T_w v), \quad \forall w \in W, x \in \mathcal{U}_q(\mathfrak{g}), v \in V(\lambda), \lambda \in \mathcal{P}^+, \tag{2.19}$$

see [19, eq. 8.14 (1)]. Moreover, $T_w(V(\lambda)_\mu) = V(\lambda)_{w\mu}$, $\forall w \in W, \lambda \in \mathcal{P}^+, \mu \in \mathcal{P}$. In particular, $\dim V(\lambda)_{w\lambda} = 1$, $\forall w \in W$, $\lambda \in \mathcal{P}^+$.

For $\alpha \in \Pi$, denote by $\mathcal{U}^\alpha$ the subalgebra of $\mathcal{U}_q(\mathfrak{g})$ generated by $E_\alpha$, $F_\alpha$, and $K_\alpha^{\pm 1}$:

$$\mathcal{U}^\alpha = \mathbb{K}\langle E_\alpha, F_\alpha, K_\alpha^{\pm 1}\rangle. \tag{2.20}$$

It is canonically isomorphic to $\mathcal{U}_{q_\alpha}(\mathfrak{sl}_2)$. We will later need the following formulas for the irreducible type one finite-dimensional $\mathcal{U}^\alpha$-modules. For all $m, N \in \mathbb{N}$, $m \leq N$, we have

$$T_\alpha v_{N\varpi_\alpha} = \frac{(-q_\alpha)^N}{[N]_\alpha!} F_\alpha^N v_{N\varpi_\alpha}, \qquad T_\alpha^{-1} v_{N\varpi_\alpha} = \frac{1}{[N]_\alpha!} F_\alpha^N v_{N\varpi_\alpha}, \tag{2.21}$$

and

$$E_\alpha^m F_\alpha^m v_{N\varpi_\alpha} = \frac{[m]_\alpha! \, [N]_\alpha!}{[N-m]_\alpha!} v_{N\varpi_\alpha}, \tag{2.22}$$

by [19, eqs. 8.6 (6), (7), and Lemma 1.7].

For $\lambda \in \mathcal{P}^+$ and $w \in W$, let $\xi_{w,\lambda} \in (V(\lambda)^*)_{-w\lambda}$ be the unique vector such that

$$\langle \xi_{w,\lambda}, T_{w^{-1}}^{-1} v_\lambda \rangle = 1. \tag{2.23}$$

For $y, w \in W$ and $\lambda \in \mathcal{P}^+$, define the quantum minors

$$e_{y,w}^\lambda := c^\lambda_{\xi_{y,\lambda}, T_{w^{-1}}^{-1} v_\lambda} \in R_q[G] \quad \text{and} \quad e_w^\lambda := e_{1,w}^\lambda = c^\lambda_{\xi_{w,\lambda}} \in R^+. \tag{2.24}$$

Using the second equality in (2.21) one easily shows that they coincide with the quantum minors of Berenstein and Zelevinsky from [4, eq. (9.10)]. If one works with $T_w$ instead of $T_{w^{-1}}^{-1}$, then additional scalars arise from the first equality in (2.21). This is why we use the latter throughout the paper.

We have

$$e_w^{\lambda_1} e_w^{\lambda_2} = e_w^{\lambda_1+\lambda_2} = e_w^{\lambda_2} e_w^{\lambda_1}, \quad \forall \lambda_1, \lambda_2 \in \mathcal{P}^+, w \in W, \tag{2.25}$$

which is proved analogously to [29, eq. (2.18)] using one more time the second equality in (2.21). Joseph proved that the multiplicative sets $E_w = \{e_w^\lambda \mid \lambda \in \mathcal{P}^+\} \subset R^+$ are Ore sets, see [21, Lemma 9.1.10]. Joseph's proof works for all base fields $\mathbb{K}$ and $q \in \mathbb{K}^*$ that is not a root of unity, see [31, §2.2]. Define the quotient algebras

$$R^w := R^+[E_w^{-1}], \quad w \in W$$

and their subalgebras

$$R_0^w := \{c_\xi^\lambda (e_w^\lambda)^{-1} \mid \lambda \in \mathcal{P}^+, \xi \in V(\lambda)^*\} \tag{2.26}$$

introduced by Joseph [21, §10.4.8]. One does not need to take a span in the right-hand side of the above formula, cf. [21, §10.4.8] or [30, eq. (2.18)]. The algebra $R_0^w$ is $\mathcal{Q}$-graded by

$$(R_0^w)_\gamma := \{c_\xi^\lambda (e_w^\lambda)^{-1} \mid \lambda \in \mathcal{P}^+, \xi \in (V(\lambda)^*)_{\gamma + w(\lambda)}\}, \quad \gamma \in \mathcal{Q}.$$

For $\mu = \lambda_1 - \lambda_2 \in \mathcal{P}$, $\lambda_1, \lambda_2 \in \mathcal{P}^+$, set

$$e_w^\mu := e_w^{\lambda_1} (e_w^{\lambda_2})^{-1} \in R_0^w. \tag{2.27}$$

It follows from (2.25) that this does not depend on the choice of $\lambda_1$, $\lambda_2$ and that $e_w^{\mu_1} e_w^{\mu_2} = e_w^{\mu_1+\mu_2}$ for all $\mu_1, \mu_2 \in \mathcal{P}$.

The $\mathcal{U}^\pm \mathcal{U}^0$-submodules $\mathcal{U}^\pm V(\lambda)_{y\lambda} = \mathcal{U}^\pm T_y v_\lambda$ of $V(\lambda)$, where $y \in W$, are called Demazure modules. They give rise to the quantum Schubert cell ideals of $R^+$

$$Q(y)^\pm := \mathrm{Span}\{c_\xi^\lambda \mid \lambda \in \mathcal{P}^+, \xi \in V(\lambda)^*, \xi \perp \mathcal{U}^\pm T_y v_\lambda\}, \quad y \in W.$$

Their counterparts in the algebras $R_w^0$ are the ideals

$$\begin{aligned} Q(y)_w^\pm &:= \{c_\xi^\lambda e_w^{-\lambda} \mid \lambda \in \mathcal{P}^+, \xi \in V(\lambda)^*, \xi \perp \mathcal{U}^\pm T_y v_\lambda\} \\ &= Q(y)^\pm E_w^{-1} \cap R_0^w. \end{aligned} \tag{2.28}$$

Analogously to (2.26), one does not need to take a span in (2.28), see [15; 28]. For $\gamma \in \mathcal{Q}^+ \setminus \{0\}$, define $m_w(\gamma) = \dim(\mathcal{U}^+[w])_\gamma = \dim(\mathcal{U}^-[w])_{-\gamma}$. Let $\{u_{\gamma,i}\}_{i=1}^{m_w(\gamma)}$ and $\{u_{-\gamma,i}\}_{i=1}^{m_w(\gamma)}$ be dual bases of $(\mathcal{U}^+[w])_\gamma$ and $(\mathcal{U}^-[w])_{-\gamma}$ with respect to the Rosso–Tanisaki form, see [19, Ch. 6]. The quantum $R$ matrix corresponding to $w$ is given by

$$\mathcal{R}^w := 1 \otimes 1 + \sum_{\gamma \in \mathcal{Q}^+, \gamma \neq 0} \sum_{i=1}^{m_w(\gamma)} u_{\gamma,i} \otimes u_{-\gamma,i} \in \mathcal{U}^+ \widehat{\otimes} \mathcal{U}^-, \tag{2.29}$$

where $\mathcal{U}^+ \widehat{\otimes} \mathcal{U}^-$ is the completion of $\mathcal{U}^+ \otimes \mathcal{U}^-$ with respect to the descending filtration [25, §4.1.1]. Finally, we recall that there is a unique graded algebra antiautomorphism $\tau$ of $\mathcal{U}_q(\mathfrak{g})$ defined by

$$\tau(E_\alpha) = E_\alpha, \quad \tau(F_\alpha) = F_\alpha, \quad \tau(K_\alpha) = K_\alpha^{-1}, \quad \alpha \in \Pi, \tag{2.30}$$

see [19, Lemma 4.6 (b)]. It satisfies

$$\tau(T_w x) = T_{w^{-1}}^{-1}(\tau(x)), \quad \forall x \in \mathcal{U}_q(\mathfrak{g}), w \in W, \tag{2.31}$$

see [19, eq. 8.18 (6)].

The next theorem summarizes the representation theoretic approach to $\operatorname{Spec}\mathcal{U}^-[w]$ via quantum function algebras and Demazure modules.

Theorem 2.2. *For all base fields $\mathbb{K}$, $q \in \mathbb{K}^*$ that is not a root of unity, simple Lie algebras $\mathfrak{g}$, and Weyl group elements $w \in W$, the following hold*:

(i) *The maps*

$$\begin{aligned} &\phi_w : R_0^w \to \mathcal{U}^-[w], \qquad \phi_w(c_\xi^\lambda e_w^{-\lambda}) := (c^\lambda_{\xi, T_{w^{-1}}^{-1} v_\lambda} \otimes \mathrm{id})(\tau \otimes \mathrm{id})\mathcal{R}^w, \\ &\lambda \in \mathcal{P}^+, \xi \in V(\lambda)^* \end{aligned} \tag{2.32}$$

*are well defined surjective $\mathcal{Q}$-graded algebra antihomomorphisms with kernels* $\ker \phi_w = Q(w)_w^+$.

(ii) *For $y \in W^{\leq w}$, the ideals*

$$I_w(y) = \phi_w(Q(w)_w^+ + Q(y)_w^-) = \phi_w(Q(y)_w^-)$$

*are distinct, $\mathbb{T}^r$-invariant, completely prime ideals of $\mathcal{U}^-[w]$. All $\mathbb{T}^r$-prime ideals of $\mathcal{U}^-[w]$ are of this form.*

(iii) *The map $y \in W^{\leq w} \mapsto I_w(y) \in \mathbb{T}^r$-$\operatorname{Spec}\mathcal{U}^-[w]$ is an isomorphism of posets with respect to the Bruhat order on $W^{\leq w}$ and the inclusion order on* $\mathbb{T}^r$-$\operatorname{Spec}\mathcal{U}^-[w]$.

Part (i) is [29, Theorem 2.6]. It was first proved in [28] for another version of the Hopf algebra $\mathcal{U}_q(\mathfrak{g})$ equipped with the opposite coproduct, a different braid group action and Lusztig's root vectors. Theorem 2.6 in [29] used $T_w$ in place of $T_{w^{-1}}^{-1}$ in equations (2.23) and (2.32). The two formulations are equivalent since $\dim V(\lambda)_{w\lambda} = 1$ and $T_w(V(\lambda)_\mu) = V(\lambda)_{w\mu}$ for all $w \in W$, $\lambda \in \mathcal{P}^+$, $\mu \in \mathcal{P}$. Parts (ii)–(iii) of Theorem 2.2 are proved in [31, Theorem 3.1 (a)] relying on results of Gorelik [15] and Joseph [20]. These statements were earlier proved in [28, Theorem 1.1 (a)–(b)] under slightly stronger conditions on $\mathbb{K}$ and $q$.

Recall (2.24). The elements

$$b^{\lambda}_{y,w} := \phi_w(e^{\lambda}_y e^{-\lambda}_w) = (e^{\lambda}_{y,w}\tau \otimes \mathrm{id})\mathcal{R}^w, \quad \lambda \in \mathcal{P}^+$$

are nonzero normal elements of $\mathcal{U}^-[w]/I_w[y]$:

$$b^{\lambda}_{y,w} x = q^{-\langle (w+y)\lambda, \gamma\rangle} x b^{\lambda}_{y,w}, \quad \forall \lambda \in \mathcal{P}^+, \gamma \in \mathcal{Q}_{\mathcal{S}(w)}, x \in (\mathcal{U}^-[w]/I_w(y))_\gamma \tag{2.33}$$

by [30, Theorem 3.1 (b) and eq. (3.1)]. Here and below we denote by the same symbols the images of elements of $\mathcal{U}^-[w]$ and $R_q[G]$ in their factors, which is a standard notational convention. The $R$-matrix commutation relations in $R^+$ (see e.g. [5, Theorem I.8.15]) and (2.25) imply that for all $\lambda_1, \lambda_2 \in \mathcal{P}^+$, $b^{\lambda_1}_{y,w} b^{\lambda_2}_{y,w} = q^{-\langle \lambda_1, \lambda_2 - y^{-1}w\lambda_2\rangle} b^{\lambda_1+\lambda_2}_{y,w}$. Thus,

$$B_{y,w} := \mathbb{K}^* \{ b^{\lambda}_{y,w} \mid \lambda \in \mathcal{P}^+ \}$$

is a multiplicative subset of $\mathcal{U}^-[w]/I_w(y)$ consisting of normal elements. The quotient ring $R_{y,w} := (\mathcal{U}^-[w]/I_w(y))[B^{-1}_{y,w}]$ is $\mathbb{T}^r$-simple. Its center is a Laurent polynomial ring of dimension $\dim \ker(w+y)$. The prime spectrum of $\mathcal{U}^-[w]$ is partitioned into

$$\mathrm{Spec}\, \mathcal{U}^-[w] = \bigsqcup_{y \in W^{\leq w}} \mathrm{Spec}_{I_w(y)} \mathcal{U}^-[w],$$

where

$$\mathrm{Spec}_{I_w(y)} \mathcal{U}^-[w] := \{ J \in \mathrm{Spec}\, \mathcal{U}^-[w] \mid J \supseteq I_w(y) \text{ and } J \cap B_{y,w} = \emptyset \}.$$

Moreover, extension and contraction establish the homeomorphisms

$$\mathrm{Spec}\, Z(R_{y,w}) \xrightarrow{\cong} \mathrm{Spec}\, R_{y,w} \xrightarrow{\cong} \mathrm{Spec}_{I_w(y)} \mathcal{U}^-[w],$$

and the centers $Z(R_{y,w})$ are Laurent polynomial rings. We refer to [30, Theorem 3.1 and Proposition 4.1] for details and proofs of the above statements. The dimensions of the Laurent polynomial rings $Z(R_{y,w})$ were explicitly determined in [2; 31]. The above results fit to the general framework of Goodearl and Letzter [13] for reconstruction of the spectra of algebras from their torus invariant prime spectra. Compared to [13], the above framework for $\mathrm{Spec}\, \mathcal{U}^-[w]$ is much more explicit. It deals with explicit $\mathbb{T}^r$-prime ideals and localizations by small sets of normal elements.

The antihomomorphisms $\phi_w : R^w_0 \to \mathcal{U}^-[w]$ are explicitly given by

$$\phi_w(c^{\lambda}_{\xi} e^{-\lambda}_w) = \sum_{m_1,\ldots,m_l \in \mathbb{N}} \left( \prod_{j=1}^{l} \frac{(q^{-1}_{\alpha_j} - q_{\alpha_j})^{m_j}}{q^{m_j(m_j-1)/2}_{\alpha_j} [m_j]_{\alpha_j}!} \right) \times \langle \xi, (\tau E_{\beta_1})^{m_1} \cdots (\tau E_{\beta_l})^{m_l} T^{-1}_{w^{-1}} v_\lambda \rangle F^{m_l}_{\beta_l} \cdots F^{m_1}_{\beta_1} \tag{2.34}$$

for all $\lambda \in \mathcal{P}^+$, $\xi \in V(\lambda)^*$. This follows from (2.32) and the standard formula [19, eqs. 8.30 (1) and (2)] for the inner product of the pairs of monomials (2.8) with respect to the Rosso–Tanisaki form.

### 2.4. *Cauchon's Method of Deleting Derivations*

We continue by outlining Cauchon's ring theoretic approach to the study of Spec$\mathcal{U}^-[w]$ via the method of deleting derivations. We follow [6; 27] and the review in [3, Section 2].

Fix an iterated Ore extension

$$A := \mathbb{K}[x_1][x_2;\sigma_2,\delta_2]\cdots[x_l;\sigma_l,\delta_l]. \tag{2.35}$$

In particular, for $j \in [2,l]$, $\sigma_j$ is an automorphism and $\delta_j$ is a (left) $\sigma_j$-skew derivation of the $(j-1)$th algebra $A_{j-1} := \mathbb{K}[x_1][x_2;\sigma_2,\delta_2]\cdots[x_{j-1};\sigma_{j-1}, \delta_{j-1}]$ in the above chain.

DEFINITION 2.3. An iterated Ore extension $A$ as in (2.35) is called a Cauchon–Goodearl–Letzter (CGL) extension if it is equipped with a rational action of the torus $\mathbb{T}^r = (\mathbb{K}^*)^{\times r}$, $r \in \mathbb{Z}_+$ by algebra automorphisms satisfying the following conditions:

(i) The elements $x_1,\ldots,x_l$ are $\mathbb{T}^r$-eigenvectors.
(ii) For every $j \in [2,l]$, $\delta_j$ is a locally nilpotent $\sigma_j$-derivation of $A_{j-1}$.
(iii) For every $j \in [1,l]$, there exists $t_j \in \mathbb{T}^r$ such that $\sigma_j = (t_j\cdot)$ as elements of Aut$(A_{j-1})$ and the $t_j$-eigenvalue of $x_j$, to be denoted by $q_j$, is not a root of unity.

One easily deduces that for all CGL extensions, $\sigma_j\delta_j = q_j\delta_j\sigma_j$, $\forall j \in [2,l]$. For $1 \le i < j \le l$, define the eigenvalues

$$t_j \cdot x_i = q_{j,i}x_i.$$

Given a CGL extension $A$ as in (2.35), for $j = l+1, l, \ldots, 2$, Cauchon iteratively constructed in [6] $l$-tuples of nonzero elements

$$(x_1^{(j)},\ldots,x_l^{(j)})$$

and families of subalgebras

$$A^{(j)} := \mathbb{K}\langle x_1^{(j)},\ldots,x_l^{(j)}\rangle$$

of the division ring of fractions Fract$(A)$ of $A$. First, set

$$(x_1^{(l+1)},\ldots,x_l^{(l+1)}) := (x_1,\ldots,x_l) \quad \text{and} \quad A^{(l+1)} = A.$$

For $j = l,\ldots,2$, the $l$-tuple $(x_1^{(j)},\ldots,x_l^{(j)})$ is determined from $(x_1^{(j+1)},\ldots, x_l^{(j+1)})$ by

$$x_i^{(j)} := \begin{cases} x_i^{(j+1)} & \text{if } i \ge j, \\ \sum_{m=0}^{\infty} \frac{(1-q_j)^{-m}}{(m)_{q_j}!}[\delta_j^m\sigma_j^{-m}(x_i^{(j+1)})](x_j^{(j+1)})^{-m} & \text{if } i < j. \end{cases} \tag{2.36}$$

Here $(0)_q = 1$, $(m)_q = (1-q^m)/(1-q)$ for $m > 0$, and $(m)_q! = (0)_q \cdots (m)_q$ for $m \in \mathbb{N}$. For $j \in [2, l+1]$, Cauchon constructed an algebra isomorphism

$$A^{(j)} \xrightarrow{\cong} \mathbb{K}[y_1]\cdots[y_{j-1}; \sigma_{j-1}, \delta_{j-1}][y_j; \tau_j]\cdots[y_l; \tau_l], \tag{2.37}$$

where $\tau_k$ denotes the automorphism of $\mathbb{K}[y_1]\cdots[y_{j-1}; \sigma_{j-1}, \delta_{j-1}][y_j; \tau_j]\cdots[y_{k-1}; \tau_{k-1}]$ such that $\tau_k(y_i) = q_{k,i} y_i$ for all $i \in [1, k-1]$. This isomorphism is given by $x_i^{(j)} \mapsto y_i$, $i = 1, \ldots, l$. Define

$$S_j := \{(x_j^{(j+1)})^m \mid m \in \mathbb{N}\}, \quad j \in [2, l].$$

Then $S_j$ is an Ore subset of $A^{(j)}$ and $A^{(j+1)}$. Cauchon proved that $A^{(j)}[S_j^{-1}] = A^{(j+1)}[S_j^{-1}]$.

Set $q_{i,i} = 1$ for $i \in [1, l]$ and $q_{i,j} = q_{j,i}^{-1}$ for $1 \le i < j \le l$. The quantum affine space algebra $R_{\mathbf{q}}[\mathbb{A}^l]$ associated to the matrix $\mathbf{q} := (q_{i,j})_{i,j=1}^l$ is the $\mathbb{K}$-algebra with generators $y_1, \ldots, y_l$ and relations $y_i y_j = q_{i,j} y_j y_i$, $\forall i, j \in [1, l]$. We will call the algebra $A^{(2)}$ obtained at the end of the Cauchon deleting derivation procedure *the Cauchon quantum affine space algebra associated to* $A$ and will denote it by $\overline{A} := A^{(2)}$. Correspondingly, the final $l$-tuple of $x$-elements will be denoted by $(\overline{x}_1, \ldots, \overline{x}_l) = (x_1^{(2)}, \ldots, x_l^{(2)})$. For $j = 2$, (2.37) gives an isomorphism

$$\overline{A} = A^{(2)} \xrightarrow{\cong} R_{\mathbf{q}}[\mathbb{A}^n], \qquad \overline{x}_i = x_i^{(2)} \mapsto y_i, \quad i \in [1, l]. \tag{2.38}$$

Furthermore, Cauchon constructed set-theoretic embeddings

$$\varphi_j : \operatorname{Spec} A^{(j+1)} \hookrightarrow \operatorname{Spec} A^{(j)}, \quad j \in [2, l],$$

which have certain topological properties but are not topological embeddings. They are given by

$$\varphi_j(J_{j+1}) = \begin{cases} J_{j+1} S_j^{-1} \cap A^{(j)} & \text{if } x_j^{(j+1)} \notin J_{j+1}, \\ g_j^{-1}(J_{j+1}/(x_j^{(j+1)})) & \text{if } x_j^{(j+1)} \in J_{j+1}, \end{cases}$$

where $J_{j+1} \in \operatorname{Spec} A^{(j+1)}$. Here $g_j : A^{(j)} \to A^{(j+1)}/(x_j^{(j+1)})$ is the homomorphism given by $g_j(x_i^{(j)}) := x_i^{(j+1)} + (x_j^{(j+1)})$, $i \in [1, l]$. For this construction one needs [6] the additional condition $x_j^{(j+1)} \notin J_{j+1} \Rightarrow J_{j+1} \cap S_{j+1} = \emptyset$. This condition is satisfied for all $J_{j+1} \in \mathbb{T}^r\text{-}\operatorname{Spec} A^{(j+1)}$ since by a result of Goodearl and Letzter [13, Proposition 4.2] all $\mathbb{T}^r$-prime ideals of a CGL extension are completely prime (recall (2.37)). A CGL extension $A$ as in Definition 2.3 is called torsion free if the subgroup of $\mathbb{K}^*$ generated by all $q_{j,i}$, $1 \le i < j \le l$, is torsion free. By another result of Goodearl and Letzter [12, Theorem 2.3], all prime ideals of a torsion-free CGL extension are completely prime. Thus the above mentioned condition is satisfied for all torsion-free CGL extensions $A$ because of (2.37). By Lemma 2.1 all algebras $\mathcal{U}^-[w]$ are torsion-free CGL extensions when $q \in \mathbb{K}^*$ is not a root of unity.

The composition $\varphi := \varphi_2 \cdots \varphi_l : \operatorname{Spec} A \hookrightarrow \operatorname{Spec} \overline{A}$ is a set-theoretic embedding, which induces an embedding $\mathbb{T}^r\text{-}\operatorname{Spec} A \hookrightarrow \mathbb{T}^r\text{-}\operatorname{Spec} \overline{A}$. Since $\overline{A}$ is a quantum affine space algebra, see (2.38), the $\mathbb{T}^r$-prime ideals of $\overline{A} = A^{(2)}$ are the ideals $K_D := \overline{A}\{\overline{x}_i \mid i \in D\}$ for $D \subseteq [1,l]$. The Cauchon diagram of $J \in \mathbb{T}^r\text{-Spec}\, A$ is the unique set $D \subseteq [1,l]$ such that $\varphi(J) = K_D$. We will denote the Cauchon diagram of $J$ by $\mathcal{CD}(J)$. If $D \subseteq [1,l]$ is the Cauchon diagram of a $\mathbb{T}^r$-invariant prime ideal of $A$, then this prime ideal will be denoted by

$$J_D := \varphi^{-1}(K_D). \tag{2.39}$$

Let

$$A' := \mathbb{K}\langle x_1, \dots, x_{l-1}\rangle = \mathbb{K}[x_1][x_2; \sigma_2, \delta_2] \cdots [x_{l-1}; \sigma_{l-1}, \delta_{l-1}]. \tag{2.40}$$

So $A = A'[x_l; \sigma_l, \delta_l]$. Set

$$A'' = \mathbb{K}\langle x_1^{(l)}, \dots, x_{l-1}^{(l)}\rangle.$$

So $A^{(l)} = A''[x_l; \tau_l]$. Note that $A'$ and $A''$ are $\mathbb{T}^r$-stable subalgebras of $A = A^{(l+1)}$ and $A^{(l)}$, respectively. They are isomorphic via the following $\mathbb{T}^r$-equivariant algebra isomorphism (recall (2.36)):

$$\theta\colon A' \xrightarrow{\cong} A'', \qquad \theta(a') = \sum_{m=0}^{\infty} \frac{(1-q_l)^{-m}}{(m)_{q_l}!} [\delta_l^m \sigma_l^{-m}(a')] x_l^{-m}. \tag{2.41}$$

It satisfies $\theta(x_i) = x_i^{(l)}$, $i \in [1, l-1]$. For an ideal $J$ of $A$, define its leading part consisting of the leading terms of the elements of $J$ written as left or right polynomials in $x_l$ with coefficients in $A'$:

$$\begin{aligned} \operatorname{lt}(J) &:= \{a' \in A' \mid \exists a \in J, m \in \mathbb{N} \text{ such that } a - a'x_l^m \in A'x_l^{m-1} + \cdots + A'\} \\ &= \{a' \in A' \mid \exists a \in J, m \in \mathbb{N} \text{ such that} \\ &\quad a - x_l^m a' \in x_l^{m-1}A' + \cdots + A'\}. \end{aligned} \tag{2.42}$$

(The equality holds because $\sigma_l$ is locally finite.)

The proof of the following lemma is analogous to [22, Lemma 4.7] and is left to the reader.

LEMMA 2.4. *Let $x$ be a regular element of the $\mathbb{K}$-algebra $A$ for which there exist two $\mathbb{K}$-linear maps $\sigma, \delta : A \to A$ such that $\sigma$ is locally finite, $\delta$ is locally nilpotent, $\sigma\delta = q\delta\sigma$ for some $q \in \mathbb{K}^*$, and*

$$xa = \sigma(a)x + \delta(a), \quad \forall a \in A.$$

*Then the set $\Omega = \{1, x, x^2, \dots\}$ is an Ore subset of $A$ and*

$$\operatorname{GK}\dim(A[\Omega^{-1}]) = \operatorname{GK}\dim A.$$

We will need the following facts for a recursive computation of Cauchon diagrams and Gelfand–Kirillov dimensions of quotients.

PROPOSITION 2.5. *Assume that $J$ is a $\mathbb{T}^r$-prime ideal of a CGL extension $A$ given by* (2.35).

(i) *If $x_l \notin J$, then*

$$JS_l^{-1} = \bigoplus_{m\in\mathbb{Z}} \theta(\mathrm{lt}(J))x_l^m, \qquad \varphi_l(J) = \bigoplus_{m\in\mathbb{N}} \theta(\mathrm{lt}(J))x_l^m, \tag{2.43}$$

$\mathcal{CD}(J) = \mathcal{CD}(\mathrm{lt}(J))$, *and*

$$\mathrm{GK}\dim\left(\frac{A}{J}\right) = \mathrm{GK}\dim\left(\frac{A'}{\mathrm{lt}(J)}\right) + 1. \tag{2.44}$$

(ii) *If $x_l \in J$, then $\varphi_l(J) = \theta(J\cap A') + A^{(l)}x_l$, $\mathcal{CD}(J) = \mathcal{CD}(J\cap A')\sqcup\{l\}$, and we have the $\mathbb{T}^r$-equivariant algebra isomorphisms $A/J \cong A^{(l)}/\varphi_l(J) \cong A'/(J\cap A') \cong A''/(\varphi_l(J)\cap A'')$. In particular,* $\mathrm{GK}\dim(A/J) = \mathrm{GK}\dim(A'/(J\cap A'))$.

Here the Cauchon diagrams $\mathcal{CD}(\mathrm{lt}(J))$ and $\mathcal{CD}(J\cap A')$ are computed with respect to the presentation (2.40) of $A'$ as a CGL extension.

*Proof.* Part (i): By [24, Lemma 2.2] every $\mathbb{T}^r$-invariant ideal $L$ of $AS_l^{-1} = A''[x_l^{\pm 1}; \tau_l]$ has the form

$$L = \bigoplus_{m\in\mathbb{Z}} L_0 x_l^m \quad \text{for some ideal } L_0 \text{ of } A''. \tag{2.45}$$

If $a = \sum_m a_m x_l^m \in L$, then $t_l^k \cdot (x_l^{-k} a x_l^k) = \sum_m q_l^{km} a_m x_l^m \in L$ for all $k \in \mathbb{N}$, where $t_l \in \mathbb{T}^r$ is the element from Definition 2.3 (iv). Thus $a_m x_l^m \in L$, $\forall m \in \mathbb{Z}$, which proves (2.45).

We apply this to the ideal $L := JS_l^{-1}$. Equation (2.41) implies that for all $a' \in A'$ and $m \in \mathbb{Z}$,

$$\theta(a')x_l^m = a'x_l^m + \sum_{k=n}^{m-1} b'_k x_l^k$$

for some $n < m$, $b'_k \in A'$. Since every nonzero element of $JS_l^{-1}$ has the form $a'x_l^m + \sum_{k=n}^{m-1} a'_k x_l^k$ for some $a' \in \mathrm{lt}(I)\setminus\{0\}$, $n < m \in \mathbb{Z}$, and $a'_k \in A'$, it should also have the form $\theta(a')x_l^m + \sum_{k=n}^{m-1} a''_k x_l^k$ for some $a' \in \mathrm{lt}(I)\setminus\{0\}$, $n < m \in \mathbb{Z}$, and $a''_k \in A''$. Now the two equalities in (2.43) follow from (2.45). The equality $\mathcal{CD}(I) = \mathcal{CD}(\mathrm{lt}(I))$ is a consequence of the definition of Cauchon diagrams. The last statement of part (i) follows from Lemma 2.4 and the fact that $(A/J)[S_l^{-1}] \cong \theta(A'/\mathrm{lt}(J))[x_l^{\pm}, \tau_l]$.

Part (ii): The first two statements follow from the definition of $\varphi_l$. The latter also implies that $g_l$ induces the $\mathbb{T}^r$-equivariant algebra isomorphism $A^{(l)}/\varphi_l(J) \cong A/J$. Since $x_l \in J$ and $x_l \in \varphi_l(J)$, the embeddings $A' \hookrightarrow A$ and $A'' \hookrightarrow A^{(l)}$ induce the $\mathbb{T}^r$-equivariant algebra isomorphisms $A'/(J\cap A') \cong A/J$ and $A''/(\varphi_l(J)\cap A'') \cong A^{(l)}/\varphi_l(J)$. $\square$

By Lemma 2.1, the quantum Schubert cell algebras $\mathcal{U}^-[w]$ are torsion-free CGL extensions for all base fields $\mathbb{K}$ and $q \in \mathbb{K}^*$ not a root of unity. There is one presentation (2.16) of $\mathcal{U}^-[w]$ as a CGL extension for each reduced word $\mathbf{i}$ for $w$. Cauchon and Mériaux established in [27] the following classification result for their $\mathbb{T}^r$-spectra.

THEOREM 2.6 (Cauchon–Mériaux [27]). *For all base fields $\mathbb{K}$, $q \in \mathbb{K}^*$ that is not a root of unity, simple Lie algebras $\mathfrak{g}$, Weyl group elements $w$, and reduced words $\mathbf{i}$ for $w$, consider the presentation (2.16) of $\mathcal{U}^-[w]$ as a torsion-free CGL extension. In this presentation, the $\mathbb{T}^r$-prime ideals of $\mathcal{U}^-[w]$ are the ideals $J_{\mathcal{LP}_{\mathbf{i}}(y)}$ for the elements $y \in W^{\leq w}$ (recall (2.39)), where $\mathcal{LP}_{\mathbf{i}}(y) \subseteq [1,l]$ is the index set of the left positive subword of $\mathbf{i}$ whose total product is $y$, cf. § 2.2.*

In other words the theorem asserts that the Cauchon diagrams of the $\mathbb{T}^r$-invariant prime ideals of $\mathcal{U}^-[w]$ for the presentation (2.16) as an iterated Ore extension are precisely the index sets of all left positive subwords of $\mathbf{i}$. In [27] Theorem 2.6 was formulated for the algebras $\mathcal{U}^+[w]$. The two statements are equivalent because of the isomorphism (2.10).

We give a second, independent proof of this theorem in Section 4.

## 3. Cauchon's Affine Space Algebras Associated to $\mathcal{U}^-[w]$

### 3.1. Statement of the Main Result

For each reduced word $\mathbf{i}$ for a Weyl group element $w \in W$, we have a presentation (2.16) of the quantum Schubert cell algebra $\mathcal{U}^-[w]$ as a torsion-free CGL extension. The Cauchon quantum affine space algebra associated to each of the algebras $\mathcal{U}^-[w]$ and a presentation of $\mathcal{U}^-[w]$ as a CGL extension via a reduced word $\mathbf{i}$ for $w$ is the result of an intricate iterative procedure. In this section we obtain an explicit description of each of these quantum affine space algebras using the antiisomorphisms from Theorem 2.2 (i). This is done in Theorem 3.1. It expresses each of the generators of the Cauchon quantum affine space algebras associated to $\mathcal{U}^-[w]$ and $\mathbf{i}$ as a quantum minor or as a fraction of two quantum minors.

Fix a Weyl group element $w \in W$ and a reduced word $\mathbf{i} = (\alpha_1, \dots, \alpha_l)$ for it where $l = \ell(w)$. Let

$$\overline{F}_{\mathbf{i},1}, \dots, \overline{F}_{\mathbf{i},l}$$

denote the generators $\overline{x}_1, \dots, \overline{x}_l$ of the Cauchon quantum affine space algebra associated to the presentation (2.16) of $\mathcal{U}^-[w]$ as a CGL extension corresponding to the reduced word $\mathbf{i}$ for $w$, recall §2.4. Define a successor function $\kappa : [1,l] \sqcup \{\infty\} \to [1,l] \sqcup \{\infty\}$ associated to $\mathbf{i}$ as follows. Let $j \in [1,l]$. If there exists $k > j$ such that $\alpha_k = \alpha_j$, then we let

$$\kappa(j) = \min\{k \mid k > j, \alpha_k = \alpha_j\}. \tag{3.1}$$

Otherwise, we let $\kappa(j) = \infty$. Set $\kappa(\infty) = \infty$. Let

$$O(j) = \max\{n \in \mathbb{N} \mid \kappa^n(j) \neq \infty\}, \tag{3.2}$$

where as usual $\kappa^0 := \mathrm{id}$. Define the quantum minors

$$\begin{aligned}\Delta_{\mathbf{i},j} &:= b^{\varpi_{\alpha_j}}_{w(\mathbf{i})_{\leq(j-1)},w} = \phi_w(e^{\varpi_{\alpha_j}}_{w(\mathbf{i})_{\leq(j-1)}} e_w^{-\varpi_{\alpha_j}}) \\ &= (e^{\varpi_{\alpha_j}}_{w(\mathbf{i})_{\leq(j-1)},w} \tau \otimes \mathrm{id})\mathcal{R}^w \in \mathcal{U}^-[w], \quad j \in [1,\ell(w)],\end{aligned} \tag{3.3}$$

recall (2.24), (2.29), (2.30), and Theorem 2.2 (i).

THEOREM 3.1. *Assume that $\mathbb{K}$ is an arbitrary base field, $q \in \mathbb{K}^*$ is not a root of unity, $\mathfrak{g}$ is a simple Lie algebra, $w \in W$ is a Weyl group element, and $\mathbf{i}$ is a reduced word for $w$. Then the generators $\overline{F}_{\mathbf{i},1}, \dots, \overline{F}_{\mathbf{i},\ell(w)}$ of the Cauchon quantum affine space algebra associated to the presentation (2.16) of $\mathcal{U}^-[w]$ as a CGL extension corresponding to $\mathbf{i}$ are given by*

$$\overline{F}_{\mathbf{i},j} = (q_{\alpha_j}^{-1} - q_{\alpha_j})^{-1} \Delta_{\mathbf{i},\kappa(j)}^{-1} \Delta_{\mathbf{i},j} \quad \textit{if } \kappa(j) \neq \infty$$

*and*

$$\overline{F}_{\mathbf{i},j} = (q_{\alpha_j}^{-1} - q_{\alpha_j})^{-1} \Delta_{\mathbf{i},j} \quad \textit{if } \kappa(j) = \infty.$$

Theorem 3.1 is equivalent to the following theorem which will be proved in §3.3.

THEOREM 3.2. *In the setting of Theorem 3.1, the quantum minors (3.3) are expressed in terms of the generators $\overline{F}_{\mathbf{i},1}, \dots, \overline{F}_{\mathbf{i},\ell(w)}$ of the Cauchon quantum affine space algebra associated to the presentation (2.16) of $\mathcal{U}^-[w]$ as a CGL extension corresponding to $\mathbf{i}$ by*

$$\Delta_{\mathbf{i},j} = (q_{\alpha_j}^{-1} - q_{\alpha_j})^{O(j)} \overline{F}_{\mathbf{i},\kappa^{O(j)}(j)} \cdots \overline{F}_{\mathbf{i},j}, \quad j \in [1, \ell(w)]. \tag{3.4}$$

The special case of this theorem for the algebras of quantum matrices $R_q[M_{m,n}]$ is due to Cauchon [7]. Given $m, n \in \mathbb{Z}_+$, let $\mathfrak{g} := \mathfrak{sl}_{m+n}$ and $w := w_{m,n} \in S_{m+n}$ for $w_{m,n} = c^m$ and $c := (1 \ 2 \ \cdots \ m+n)$. The algebra $R_q[M_{m,n}]$ is isomorphic to the algebras $\mathcal{U}^{\pm}[w_{m,n}]$ by [27, Proposition 2.1.1] and [31, Lemma 4.1]. In this case by [31, Lemma 4.3] the elements $b^{\varpi_\alpha}_{y,w_{m,n}} \in \mathcal{U}^{\pm}[w_{m,n}]$ correspond (under this isomorphism) to scalar multiples of quantum minors of $R_q[M_{m,n}]$ for all $\alpha \in \Pi$, $y \in S_{m+n}^{\leq w_{m,n}}$. In particular, the elements $\Delta_{\mathbf{i},1}, \dots, \Delta_{\mathbf{i},mn} \in \mathcal{U}^{\pm}[w_{m,n}]$ correspond to scalar multiples of quantum minors of $R_q[M_{m,n}]$ for all reduced words $\mathbf{i}$ of $w_{m,n}$.

### 3.2. *Leading Terms of Quantum Minors*

There are several different ways to construct iterated Ore extensions associated to the algebras $\mathcal{U}^-[w]$ by adjoining root vectors in different order. Passing from one to the other will play a major role in our proof of Theorem 3.2 in §3.3. In §3.2–3.3 we examine these iterated Ore extensions and prove a leading term result for the elements $\Delta_{\mathbf{i},j}$.

For a reduced word $\mathbf{i} = (\alpha_1, \dots, \alpha_l)$ for $w \in W$ and $j, k \in [1, l]$ denote by

$$\mathcal{U}^-[w]_{\mathbf{i},[j,k]} \text{ the subalgebra of } \mathcal{U}^-[w] \text{ generated by } F_{\beta_m} \text{ for } j \leq m \leq k$$

in terms of the notation from equation (2.7). One easily shows that

$$\mathcal{U}^-[w]_{\mathbf{i},[j,k]} = T_{w(\mathbf{i})_{\leq(j-1)}}(\mathcal{U}^-[(w(\mathbf{i})_{\leq(j-1)})^{-1} w(\mathbf{i})_{\leq k}])$$

for $j \leq k$, but we will not need this.

PROPOSITION 3.3. *For all base fields* $\mathbb{K}$, $q \in \mathbb{K}^*$ *that is not a root of unity, simple Lie algebras* $\mathfrak{g}$, $w \in W$ *of length* $l$, *reduced words* $\mathbf{i}$ *for* $w$, *and* $j \in [1, l]$, *we have*

$$\Delta_{\mathbf{i},j} = (q_{\alpha_j}^{-1} - q_{\alpha_j})\Delta_{\mathbf{i},\kappa(j)} F_{\beta_j} \mod \mathcal{U}^-[w]_{\mathbf{i},[j+1,l]} \quad \text{if } \kappa(j) \neq \infty \tag{3.5}$$

*and*

$$\Delta_{\mathbf{i},j} = (q_{\alpha_j}^{-1} - q_{\alpha_j}) F_{\beta_j} \mod \mathcal{U}^-[w]_{\mathbf{i},[j+1,l]} \quad \text{if } \kappa(j) = \infty. \tag{3.6}$$

*Proof.* We fix a reduced expression $\mathbf{i} = (\alpha_1, \dots, \alpha_l)$ of $w$ and define $w_{\leq k} := w(\mathbf{i})_{\leq k}$, $k \in [0, l]$, cf. (2.1). Recall that $\tau$, given by (2.30), is an algebra antiautomorphism of $\mathcal{U}_q(\mathfrak{g})$ and $T_w$ is an algebra automorphism of $\mathcal{U}_q(\mathfrak{g})$ for all $w \in W$. The algebra $\tau T_{w_{\leq(k-1)}}(\mathcal{U}^{\alpha_k})$ is (anti)isomorphic to $\mathcal{U}_{q_{\alpha_k}}(\mathfrak{sl}_2)$ for all $k \in [1, l]$, see (2.20).

Let $1 \leq k < j$. Consider the $\tau T_{w_{\leq(k-1)}}(\mathcal{U}^{\alpha_k})$-submodule of $V(\varpi_{\alpha_j})$ generated by

$$V(\varpi_{\alpha_j})_{w_{\leq(j-1)}\varpi_{\alpha_j}} = \mathbb{K} T_{w_{\leq(j-1)}} v_{\varpi_{\alpha_j}} = \mathbb{K} T^{-1}_{w^{-1}_{\leq(j-1)}} v_{\varpi_{\alpha_j}}.$$

It is irreducible since

$$\begin{aligned}
(\tau F_{\beta_k})(T^{-1}_{w^{-1}_{\leq(j-1)}} v_{\varpi_{\alpha_j}}) &= (\tau(T_{w_{\leq(k-1)}} F_{\alpha_k}))(T^{-1}_{w^{-1}_{\leq(j-1)}} v_{\varpi_{\alpha_j}}) \\
&= (T^{-1}_{w^{-1}_{\leq(k-1)}} F_{\alpha_k})(T^{-1}_{w^{-1}_{\leq(j-1)}} v_{\varpi_{\alpha_j}}) \\
&= T^{-1}_{w^{-1}_{\leq(j-1)}}((T_{\alpha_{j-1}} \cdots T_{\alpha_k}(F_{\alpha_k}))v_{\varpi_{\alpha_j}}) = 0,
\end{aligned} \tag{3.7}$$

cf. (2.19) and (2.31). In the last equation we used that $-s_{\alpha_{j-1}} \cdots s_{\alpha_k}(\alpha_k) \in \mathcal{Q}^+$ and $T_{\alpha_{j-1}} \cdots T_{\alpha_k}(F_{\alpha_k}) \in \mathcal{U}_q(\mathfrak{g})_{s_{\alpha_{j-1}} \cdots s_{\alpha_k}(\alpha_k)}$. Therefore there exists a splitting of $\tau T_{w_{\leq(k-1)}}(\mathcal{U}^{\alpha_k})$-modules

$$V(\varpi_{\alpha_j}) = (\tau T_{w_{\leq(k-1)}}(\mathcal{U}^{\alpha_k}))V(\varpi_{\alpha_j})_{w_{\leq(j-1)}\varpi_{\alpha_j}} \oplus V_k$$

such that $V_k$ is also $\mathcal{U}^0$-stable. From this and equation (3.7) it follows that

$$\langle \xi_{w_{\leq(j-1)}, \varpi_{\alpha_j}}, (\tau E_{\beta_k})v \rangle = 0, \quad \forall v \in V(\varpi_{\alpha_j}), 1 \leq k < j, \tag{3.8}$$

recall (2.23).

Next, we consider the $\tau T_{w_{\leq(j-1)}}(\mathcal{U}^{\alpha_j})$-submodule of $V(\varpi_{\alpha_j})$ generated by $T^{-1}_{w^{-1}_{\leq(j-1)}} v_{\varpi_{\alpha_j}}$. Using (2.21)–(2.22), we obtain

$$\begin{aligned}
(\tau E_{\beta_j})(T^{-1}_{w^{-1}_{\leq j}} v_{\varpi_{\alpha_j}}) &= (\tau(T_{w_{\leq(j-1)}} E_{\alpha_j}))(T^{-1}_{w^{-1}_{\leq j}} v_{\varpi_{\alpha_j}}) \\
&= (T^{-1}_{w^{-1}_{\leq(j-1)}} E_{\alpha_j})(T^{-1}_{w^{-1}_{\leq j}} v_{\varpi_{\alpha_j}}) \\
&= T^{-1}_{w^{-1}_{\leq(j-1)}}(E_{\alpha_j} T^{-1}_{\alpha_j} v_{\varpi_{\alpha_j}}) = T^{-1}_{w^{-1}_{\leq(j-1)}} v_{\varpi_{\alpha_j}}.
\end{aligned} \tag{3.9}$$

Analogously, one shows that

$$(\tau E_{\beta_j})(T^{-1}_{w^{-1}_{\leq(j-1)}} v_{\varpi_{\alpha_j}}) = 0 \quad \text{and}$$

$$(\tau(T_{w_{\leq(j-1)}} K_{\alpha_j}))(T^{-1}_{w^{-1}_{\leq(j-1)}} v_{\varpi_{\alpha_j}}) = q_{\alpha_j}^{-1}(T^{-1}_{w^{-1}_{\leq(j-1)}} v_{\varpi_{\alpha_j}}).$$

Therefore

$$(\tau T_{w_{\leq(j-1)}}(\mathcal{U}^{\alpha_j}))T^{-1}_{w^{-1}_{\leq(j-1)}} v_{\varpi_{\alpha_j}} = \mathbb{K}T^{-1}_{w^{-1}_{\leq(j-1)}} v_{\varpi_{\alpha_j}} \oplus \mathbb{K}T^{-1}_{w^{-1}_{\leq j}} v_{\varpi_{\alpha_j}}.$$

Using this, the complete reducibility of finite-dimensional type one $\mathcal{U}^{\alpha}$-modules, and equation (3.9), we obtain

$$\begin{gathered} \langle \xi_{w_{\leq(j-1)},\varpi_{\alpha_j}}, (\tau E_{\beta_j})v\rangle = \langle \xi_{w_{\leq j},\varpi_{\alpha_j}}, v\rangle \quad \text{and} \\ \langle \xi_{w_{\leq(j-1)},\varpi_{\alpha_j}}, (\tau E_{\beta_j})^m v\rangle = 0, \quad \forall v \in V(\varpi_{\alpha_j}), m > 1. \end{gathered} \tag{3.10}$$

In a similar way one proves that for all $j < k \leq \min\{l, \kappa(j)-1\}$,

$$\begin{gathered} (\tau T_{w_{\leq(k-1)}}(\mathcal{U}^{\alpha_k}))T^{-1}_{w^{-1}_{\leq(k-1)}} v_{\varpi_{\alpha_j}} = \mathbb{K}T^{-1}_{w^{-1}_{\leq(k-1)}} v_{\varpi_{\alpha_j}} \quad \text{and} \\ T^{-1}_{w^{-1}_{\leq(k-1)}} v_{\varpi_{\alpha_j}} = T^{-1}_{w^{-1}_{\leq k}} v_{\varpi_{\alpha_j}}. \end{gathered}$$

From this one obtains that

$$\begin{gathered} \langle \xi_{w_{\leq(k-1)},\varpi_{\alpha_j}}, (\tau E_{\beta_k})v\rangle = 0 \quad \text{and} \quad \xi_{w_{\leq(k-1)},\varpi_{\alpha_j}} = \xi_{w_{\leq k},\varpi_{\alpha_j}}, \\ \forall v \in V(\varpi_{\alpha_j}),\ j < k \leq \min\{l, \kappa(j)-1\}. \end{gathered} \tag{3.11}$$

Equation (3.5) is deduced from (3.8), (3.10), and (3.11) as follows. Denote for brevity

$$p_{k,m} := \frac{(q_{\alpha_k}^{-1} - q_{\alpha_k})^m}{q_{\alpha_k}^{m(m-1)/2}[m]_{\alpha_k}!}, \quad k \in [1,l], m \in \mathbb{N}.$$

Using (2.34), (3.8), and (3.10), we obtain

$$\begin{aligned} \Delta_{\mathbf{i},j} = & \sum_{m_j,\ldots,m_l \in \mathbb{N}} \left(\prod_{k=j}^{l} p_{k,m_k}\right) \langle \xi_{w_{\leq(j-1)},\varpi_{\alpha_j}}, (\tau E_{\beta_j})^{m_j} \cdots (\tau E_{\beta_l})^{m_l} T^{-1}_{w^{-1}} v_\lambda\rangle \\ & \times F_{\beta_l}^{m_l} \cdots F_{\beta_j}^{m_j} \\ = & (q_{\alpha_j}^{-1} - q_{\alpha_j}) \sum_{m_{j+1},\ldots,m_l \in \mathbb{N}} \left(\prod_{k=j+1}^{l} p_{k,m_k}\right) \\ & \times \langle \xi_{w_{\leq j},\varpi_{\alpha_j}}, (\tau E_{\beta_{j+1}})^{m_{j+1}} \cdots (\tau E_{\beta_l})^{m_l} T^{-1}_{w^{-1}} v_\lambda\rangle \\ & \times F_{\beta_l}^{m_l} \cdots F_{\beta_{j+1}}^{m_{j+1}} F_{\beta_j} \quad \bmod \mathcal{U}^-[w]_{\mathbf{i},[j+1,l]}. \end{aligned}$$

If $\kappa(j) \leq l$, it follows from (3.11) that the right-hand side of the last congruence is equal to

$$\begin{aligned} & (q_{\alpha_j}^{-1} - q_{\alpha_j}) \sum_{m_{\kappa(j)},\ldots,m_l \in \mathbb{N}} \left(\prod_{k=\kappa(j)}^{l} p_{k,m_k}\right) \\ & \times \langle \xi_{w_{\leq(\kappa(j)-1)},\varpi_{\alpha_j}}, (\tau E_{\beta_{\kappa(j)}})^{m_{\kappa(j)}} \cdots (\tau E_{\beta_l})^{m_l} T^{-1}_{w^{-1}} v_\lambda\rangle \\ & \times F_{\beta_l}^{m_l} \cdots F_{\beta_{\kappa(j)}}^{m_{\kappa(j)}} F_{\beta_j} = (q_{\alpha_j}^{-1} - q_{\alpha_j})\Delta_{\mathbf{i},\kappa(j)} F_{\beta_j}. \end{aligned}$$

This proves (3.5). The proof of (3.6) is analogous, requiring only a small modification of the last argument. It is left to the reader. □

Starting from a reduced word $\mathbf{i} = (\alpha_1, \ldots, \alpha_l)$ for $w \in W$, one can construct a presentation of $\mathcal{U}^-[w]$ as an iterated Ore extension by adjoining the elements $F_{\beta_1}, \ldots, F_{\beta_l}$ (recall (2.7)) in the opposite order. For all $j \in [1, l]$, we have the Ore extension presentation

$$\mathcal{U}^-[w]_{\mathbf{i},[j,l]} = \mathcal{U}^-[w]_{\mathbf{i},[j+1,l]}[F_{\beta_j}; \sigma'_j, \delta'_j], \tag{3.12}$$

where $\sigma'_j$ and $\delta'_j$ are defined as follows. Let $t'_j$ be an element of $\mathbb{T}^r$ such that

$$(t'_j)^{\beta_k} = q^{-\langle \beta_k, \beta_j \rangle}, \quad \forall k \geq j$$

(cf. (2.11) and (2.13)) and $\sigma'_j := (t'_j \cdot)$ in terms of the restriction of the $\mathbb{T}^r$-action (2.12) to $\mathcal{U}^-[w]_{\mathbf{i},[j+1,l]}$. The skew derivation $\delta'_j$ of $\mathcal{U}^-[w]_{\mathbf{i},[j+1,l]}$ is defined by

$$\delta'_j(x) := F_{\beta_j} x - q^{-\langle \beta_j, \gamma \rangle} x F_{\beta_j}, \quad x \in (\mathcal{U}^-[w]_{\mathbf{i},[j+1,l]})_\gamma, \gamma \in \mathcal{Q},$$

cf. (2.15). (It follows from the Levendorskii–Soibelman straightening law (2.14) that $\delta'_j$ preserves $\mathcal{U}^-[w]_{\mathbf{i},[j+1,l]}$, $\sigma'_l = \mathrm{id}$, and $\delta'_l = 0$.) Equations (2.8) and (2.14) imply (3.12). Iterating (3.12) and taking into account $\mathcal{U}^-[w]_{\mathbf{i},[l+1,l]} = \mathbb{K}$ leads to the iterated Ore extension presentation

$$\mathcal{U}^-[w] = \mathbb{K}[F_{\beta_l}][F_{\beta_{l-1}}; \sigma'_{l-1}, \delta'_{l-1}] \cdots [F_{\beta_1}; \sigma'_1, \delta'_1],$$

which is reverse to the presentation (2.16). It is straightforward to show that this presentation of $\mathcal{U}^-[w]$ is a torsion-free CGL extension for the action (2.12).

In this framework, Proposition 3.3 proves that $\Delta_{\mathbf{i},j} \in \mathcal{U}^-[w]_{\mathbf{i},[j,l]}$ and computes its leading term as a left polynomial with respect to the Ore extension (3.12) for all $j \in [1, l]$, cf. §2.4.

### 3.3. *Proof of Theorem 3.2*

We keep the notation for $\mathbf{i}$, $w$, and $l$ from the previous two subsections. For $j \in [1, l]$, consider the chain of extensions

$$\mathbb{K} \subset \mathcal{U}^-[w]_{\mathbf{i},[j,j]} \subset \mathcal{U}^-[w]_{\mathbf{i},[j,j+1]} \subset \cdots \subset \mathcal{U}^-[w]_{\mathbf{i},[j,l]}.$$

It follows from the Levendorskii–Soibelman straightening law (2.14) and the definition of the $\mathbb{T}^r$-action (2.12) that the maps $\delta_k$ and $\sigma_k$ from Lemma 2.1 (ii) preserve the subalgebra $\mathcal{U}^-[w]_{\mathbf{i},[j,k-1]}$ of $\mathcal{U}^-[w(\mathbf{i})_{\leq(k-1)}] = \mathcal{U}^-[w]_{\mathbf{i},[1,k-1]}$ for all $1 \leq j \leq k \leq l$. Define the restrictions

$$\delta_{j,k} = \delta_k|_{\mathcal{U}^-[w]_{\mathbf{i},[j,k-1]}} \quad \text{and} \quad \sigma_{j,k} = \sigma_k|_{\mathcal{U}^-[w]_{\mathbf{i},[j,k-1]}} \quad \text{for } 1 \leq j \leq k \leq l.$$

Lemma 2.1 (ii) implies that we have the Ore extension presentation

$$\mathcal{U}^-[w]_{\mathbf{i},[j,k]} = \mathcal{U}^-[w]_{\mathbf{i},[j,k-1]}[F_{\beta_k}; \sigma_{j,k}, \delta_{j,k}] \quad \text{for } 1 \leq j \leq k \leq l.$$

Iterating those and using that $\mathcal{U}^-[w]_{\mathbf{i},[j,j-1]} = \mathbb{K}$, $\sigma_{j,j} = \mathrm{id}$, and $\delta_{j,j} = 0$ leads to the iterated Ore extension presentation of $\mathcal{U}^-[w]_{\mathbf{i},[j,k]}$:

$$\mathcal{U}^-[w]_{\mathbf{i},[j,l]} = \mathbb{K}[F_{\beta_j}][F_{\beta_{j+1}}; \sigma_{j,j+1}, \delta_{j,j+1}] \cdots [F_{\beta_l}; \sigma_{j,l}, \delta_{j,l}]. \tag{3.13}$$

It follows now from Lemma 2.1 that $\mathcal{U}^-[w]_{\mathbf{i},[j,k]}$ is a CGL extension with respect to the $\mathbb{T}^r$-action (2.12). Since $\{0\}$ is a $\mathbb{T}^r$-prime ideal of $\mathcal{U}^-[w]_{\mathbf{i},[j,k]}$, we can apply a theorem of Goodearl [5, Theorem II.6.4] to obtain that it is a strongly rational ideal, that is,

$$Z(\mathrm{Fract}(\mathcal{U}^-[w]_{\mathbf{i},[j,l]}))^{\mathbb{T}^r} = \mathbb{K}. \tag{3.14}$$

Recall that $Z(A)$ stands for the center of an algebra $A$. As in §2.4, $\mathrm{Fract}(A)$ denotes the division ring of fractions of a domain $A$. Furthermore, $(\cdot)^{\mathbb{T}^r}$ refers to the fixed point subalgebra with respect to the action (2.12).

Denote by $\mathcal{T}_{\mathbf{i}}$ the quantum torus algebra generated by $\overline{F}_{\mathbf{i},1}^{\pm 1}, \dots, \overline{F}_{\mathbf{i},l}^{\pm}$. Equations (2.14) and (2.38) imply that

$$\overline{F}_{\mathbf{i},j}\overline{F}_{\mathbf{i},k} = q^{\langle \beta_j,\beta_k\rangle}\overline{F}_{\mathbf{i},k}\overline{F}_{\mathbf{i},j}, \quad \forall 1 \le j < k \le l. \tag{3.15}$$

For $j,k \in [1,l]$, denote by $\mathcal{T}_{\mathbf{i},[j,k]}$ the quantum subtorus of $\mathcal{T}_{\mathbf{i}}$ generated by $\overline{F}_{\mathbf{i},m}^{\pm 1}$ for $j \le m \le k$.

Using that

$$\delta_k(F_{\beta_j}) \in \mathcal{U}^-[w]_{\mathbf{i},[k+1,j-1]},$$

by a simple induction argument, one proves the following lemma.

LEMMA 3.4. *In the above setting, the following hold for all* $j \in [1,l]$:

(i) $F_{\beta_j} - \overline{F}_{\mathbf{i},j} \in \mathcal{T}_{\mathbf{i},[j+1,l]}$.

(ii) *The generators for the Cauchon quantum affine space algebra associated to the iterated Ore extension presentation* (3.13) *of* $\mathcal{U}^-[w]_{\mathbf{i},[j,l]}$ *are precisely the elements* $\overline{F}_{\mathbf{i},j}, \dots, \overline{F}_{\mathbf{i},l}$, *recall* § 2.4.

The lemma implies that $\mathcal{U}^-[w]_{\mathbf{i},[j,l]} \subset \mathcal{T}_{\mathbf{i},[j,l]} \subset \mathrm{Fract}(\mathcal{U}^-[w]_{\mathbf{i},[j,l]})$. Therefore, the strong rationality result (3.14) gives that

$$Z(\mathcal{T}_{\mathbf{i},[j,l]})_0 = \mathbb{K}, \tag{3.16}$$

where $(\cdot)_0$ refers to the 0-component with respect to the $\mathcal{Q}$-grading induced from the grading of $\mathcal{U}_q(\mathfrak{g})$.

Next we apply a theorem of Berenstein and Zelevinsky [4, Theorem 10.1] to obtain that there exist integers $n_{jk} \in \mathbb{Z}$ $(1 \le j < k \le n)$ such that

$$e^{\varpi_{\alpha_j}}_{w(\mathbf{i})_{\le(j-1)}} e^{\varpi_{\alpha_k}}_{w(\mathbf{i})_{\le(k-1)}} = q^{n_{jk}} e^{\varpi_{\alpha_k}}_{w(\mathbf{i})_{\le(k-1)}} e^{\varpi_{\alpha_j}}_{w(\mathbf{i})_{\le(j-1)}}, \quad \forall 1 \le j < k \le l.$$

(The setting of [4] is for $\mathbb{K} = \mathbb{Q}(q)$, but the proof of Theorem 10.1 in [4] only uses the $R$-matrix commutation relations in $R_q[G]$ and the left and right actions of $\mathcal{U}_q(\mathfrak{g})$ on $R_q[G]$, which work for all fields $\mathbb{K}$ and $q \in \mathbb{K}^*$ that is not a root of unity.) Moreover, the $R$-matrix commutation relations in $R_q[G]$ (see e.g. [5, Theorem I.8.15]) imply that

$$e_w^{\lambda} c_{\xi'}^{\lambda'} = q^{-\langle \lambda, \lambda' + w^{-1}\mu'\rangle} c_{\xi'}^{\lambda'} e_w^{\lambda} \mod Q(w)^+, \quad \forall \lambda, \lambda' \in \mathcal{P}^+, \mu \in \mathcal{P}, \xi' \in V(\lambda')_{\mu'}.$$

Using (3.3) and the fact that the maps $\phi_w : R_w^0 \to \mathcal{U}^-[w]$ are antihomomorphisms by Theorem 2.2 (i), we obtain

$$\Delta_{\mathbf{i},j}\Delta_{\mathbf{i},k} = q^{n'_{jk}}\Delta_{\mathbf{i},k}\Delta_{\mathbf{i},j}, \quad \forall 1 \le j < k \le l \tag{3.17}$$

for some $n'_{jk} \in \mathbb{Z}$.

*Proof of Theorem 3.2*. By Lemma 3.4 (ii)

$$\mathcal{U}^-[w]_{\mathbf{i},[j,l]} \subseteq \mathcal{T}_{\mathbf{i},[j,l]}, \quad \forall j \in [1,l].$$

Combining this, Proposition 3.3, and Lemma 3.4 (i), we obtain

$$\Delta_{\mathbf{i},j} = (q_{\alpha_j}^{-1} - q_{\alpha_j})\Delta_{\mathbf{i},\kappa(j)}\overline{F}_{\mathbf{i},j} \mod \mathcal{T}_{\mathbf{i},[j+1,l]} \quad \text{if } \kappa(j) \le l \tag{3.18}$$

and

$$\Delta_{\mathbf{i},j} = (q_{\alpha_j}^{-1} - q_{\alpha_j})\overline{F}_{\mathbf{i},j} \mod \mathcal{T}_{\mathbf{i},[j+1,l]} \quad \text{if } \kappa(j) = \infty. \tag{3.19}$$

We prove equation (3.4) by induction on $j$, from $l$ to 1. By (3.19), $\Delta_{\mathbf{i},l} - (q_{\alpha_l}^{-1} - q_{\alpha_l})\overline{F}_{\mathbf{i},l} \in \mathbb{K}$. Since $\Delta_{\mathbf{i},l}$ is a homogeneous element of nonzero degree (equal to $\beta_l$), this implies (3.4) for $j = l$.

Now assume that for some $j \in [1, l-1]$,

$$\Delta_{\mathbf{i},k} = (q_{\alpha_k}^{-1} - q_{\alpha_k})^{O(k)}\overline{F}_{\mathbf{i},\kappa^{O(k)}(k)} \cdots \overline{F}_{\mathbf{i},k} \quad \text{for all } k \in [j+1, l]. \tag{3.20}$$

If

$$\Delta_{\mathbf{i},j} = (q_{\alpha_j}^{-1} - q_{\alpha_j})^{O(j)}\overline{F}_{\mathbf{i},\kappa^{O(j)}(j)} \cdots \overline{F}_{\mathbf{i},j}, \tag{3.21}$$

then we are done with the inductive step. Assume the opposite, that (3.21) is not satisfied. Combining the inductive hypothesis with (3.18) and (3.19) (whichever applies to the particular $j$), we get that

$$\Delta_{\mathbf{i},j} - (q_{\alpha_j}^{-1} - q_{\alpha_j})^{O(j)}\overline{F}_{\mathbf{i},\kappa^{O(j)}(j)} \cdots \overline{F}_{\mathbf{i},j} \in \mathcal{T}_{\mathbf{i},[j+1,l]}. \tag{3.22}$$

It follows from (3.15), (3.17), and (3.20) that

$$\Delta_{\mathbf{i},j}\overline{F}_{\mathbf{i},k} = q^{m_k}\overline{F}_{\mathbf{i},k}\Delta_{\mathbf{i},j}, \quad \forall k = j+1, \dots, l$$

for some $m_{j+1}, \dots, m_l \in \mathbb{Z}$. Quantum tori have bases consisting of Laurent monomials in their generators. By comparing the coefficients of $\overline{F}_{\mathbf{i},\kappa^{O(j)}(j)} \cdots \overline{F}_{\mathbf{i},j}\overline{F}_{\mathbf{i},k}$ in the two sides of the above equality and using (3.22), we get that

$$(\overline{F}_{\mathbf{i},\kappa^{O(j)}(j)} \cdots \overline{F}_{\mathbf{i},j})\overline{F}_{\mathbf{i},k} = q^{m_k}\overline{F}_{\mathbf{i},k}(\overline{F}_{\mathbf{i},\kappa^{O(j)}(j)} \cdots \overline{F}_{\mathbf{i},j}), \quad \forall k = j+1, \dots, l$$

for the same collection of integers $m_{j+1}, \dots, m_l$. From the last two equalities, it follows that

$$y := (\overline{F}_{\mathbf{i},\kappa^{O(j)}(j)} \cdots \overline{F}_{\mathbf{i},j})^{-1}\Delta_{\mathbf{i},j}$$

commutes with $\overline{F}_{\mathbf{i},j+1}, \dots, \overline{F}_{\mathbf{i},l}$:

$$y\overline{F}_{\mathbf{i},k} = \overline{F}_{\mathbf{i},k}y, \quad \forall k = j+1, \dots, l. \tag{3.23}$$

Since (3.21) is not satisfied, (3.22) implies that

$$y = (q_{\alpha_j}^{-1} - q_{\alpha_j}) + y'\overline{F}_{\mathbf{i},j}^{-1} \quad \text{for some } y' \in \mathcal{T}_{\mathbf{i},[j+1,l]} \setminus \{0\}. \tag{3.24}$$

But $y$ commutes with itself and by (3.23) it commutes with $y' \neq 0$. Thus $y$ also commutes with $\overline{F}_{\mathbf{i},j}$. Combining this with (3.23) leads to the fact that $y$ belongs to the center of $\mathcal{T}_{\mathbf{i},[j,l]}$. Since $\Delta_{\mathbf{i},j}$ is a homogeneous element of $\mathcal{U}^-[w]$ with respect to its $\mathcal{Q}$-grading, (3.22) implies

$$y \in Z(\mathcal{T}_{\mathbf{i},[j,l]})_0.$$

At the same time $y \notin \mathbb{K}$ by (3.24), which contradicts the strong rationality result (3.16). Thus (3.21) holds. This completes the proofs of the inductive step and the theorem. □

## 4. Unification of the Two Approaches to $\mathbb{T}^{\mathrm{r}}$-Spec$\,\mathcal{U}^-[w]$

### *4.1. Solutions of Two Questions of Cauchon and Mériaux*

In this section we establish a relationship between the representation theoretic and ring theoretic approaches to the prime spectra of the quantum Schubert cell algebras $\mathcal{U}^-[w]$, see §2.3 and §2.4. Theorem 4.5 explicitly describes the behavior of all $\mathbb{T}^r$-prime ideals $I_w(y)$ of the algebras $\mathcal{U}^-[w]$ from Theorem 2.2 under the iterations of Cauchon's deleting derivation construction, recall Proposition 2.5. In Theorem 4.1 we describe explicitly the Cauchon diagrams of all ideals $I_w(y)$ and use this to resolve [27, Question 5.3.3] of Cauchon and Mériaux. We use the combination of Theorems 2.2 and 4.1 to give a new, independent proof of the classification result in Theorem 2.6 of Cauchon and Mériaux. Finally, we also settle [27, Question 5.3.2] of Cauchon and Mériaux, solving the containment problem for the ideals in the classification of Theorem 2.6, see Theorem 4.4.

THEOREM 4.1. *Assume that $\mathbb{K}$ is an arbitrary base field, $q \in \mathbb{K}^*$ is not a root of unity, $\mathfrak{g}$ is a simple Lie algebra, $w$ is a Weyl group element, and $\mathbf{i}$ is a reduced word for $w$. Then, for all Weyl group elements $y \le w$, the Cauchon diagram of the $\mathbb{T}^r$-prime ideal $I_w(y)$ (see Theorem 2.2 (ii)) for the presentation (2.16) of $\mathcal{U}^-[w]$ is precisely the index set of the left positive subword of $\mathbf{i}$ whose total product is $y$*

$$\mathcal{CD}(I_w(y)) = \mathcal{LP}_{\mathbf{i}}(y),$$

*recall § 2.2 and 2.4 for definitions.*

REMARK 4.2. Theorem 4.1 gives a new, independent proof of Theorem 2.6 of Cauchon and Mériaux [27]. By Theorem 2.2 (ii)

$$\mathbb{T}^{\mathrm{r}}\text{-Spec}\,\mathcal{U}^-[w] = \{I_w(y) \mid y \in W^{\le w}\}.$$

Since $\mathcal{CD}(I_w(y)) = \mathcal{LP}_{\mathbf{i}}(y)$, by Theorem 4.1 we have

$$\mathbb{T}^{\mathrm{r}}\text{-Spec}\,\mathcal{U}^-[w] = \{J_{\mathcal{LP}_{\mathbf{i}}(y)} \mid y \in W^{\le w}\},$$

which is the statement of Theorem 2.6, recall (2.39).

The following theorem is an immediate consequence of Theorem 4.1. It settles Question 5.3.3 of Cauchon and Mériaux [27].

THEOREM 4.3. *For all base fields $\mathbb{K}$, $q \in \mathbb{K}^*$ that is not a root of unity, simple Lie algebras $\mathfrak{g}$, Weyl group elements $w$, and reduced words $\mathbf{i}$ for $w$,*

$$I_w(y) = J_{\mathcal{LP}_{\mathbf{i}}(y)}, \quad \forall y \in W^{\le w} \tag{4.1}$$

(*recall* (2.39)), *that is, the classifications of $\mathbb{T}^{\mathrm{r}}$-Spec$\,\mathcal{U}^-[w]$ of Cauchon–Mériaux* [27] *from Theorem 2.6 and that of Yakimov* [28] *from Theorem 2.2 coincide.*

Finally, the next theorem answers Question 5.3.2 of Cauchon and Mériaux [27].

THEOREM 4.4. *For all base fields* $\mathbb{K}$, $q \in \mathbb{K}^*$ *that is not a root of unity, simple Lie algebras* $\mathfrak{g}$, *Weyl group elements* $w$, *and reduced words* $\mathbf{i}$ *for* $w$, *the map*

$$y \in W^{\leq w} \mapsto J_{\mathcal{LP}_{\mathbf{i}}(y)} \in \mathbb{T}^{\mathrm{r}}\text{-Spec}\,\mathcal{U}^-[w], \quad y \in W^{\leq w},$$

*is an isomorphism of posets with respect to the Bruhat order and inclusion of ideals.*

*Proof.* Theorem 4.4 follows from Theorem 2.2 (iii) and equation (4.1). □

Our proof of Theorem 4.1 is based on a result, which gives a full picture of the behavior of the ideals $I_w(y)$ from Theorem 2.2 (i) under the deleting derivation procedure from §2.4. Recall the definition (2.42) of leading part $\mathrm{lt}(J)$ of an ideal of an Ore extension. According to Proposition 2.5, Cauchon's method relies on taking leading parts or contractions of ideals in CGL extensions. Assume that $\mathbf{i} = (\alpha_1, \ldots, \alpha_l)$ is a reduced word for $w \in W$. Then

$$w(\mathbf{i})_{\leq(l-1)} = ws_{\alpha_l}. \tag{4.2}$$

Lemma 2.1 (i)–(ii) implies that

$$\begin{aligned} \mathcal{U}^-[ws_{\alpha_l}] = \mathcal{U}^-[w(\mathbf{i})_{\leq(l-1)}] \subset \mathcal{U}^-[w] \quad \text{and} \\ \mathcal{U}^-[w] = \mathcal{U}^-[ws_{\alpha_l}][F_{\beta_l}; \sigma_l, \delta_l], \end{aligned} \tag{4.3}$$

where $\sigma_l$ and $\delta_l$ are the automorphism and left $\sigma_l$-skew derivation of $\mathcal{U}^-[w(\mathbf{i})_{\leq(l-1)}]$ from Lemma 2.1 (ii). We have the following.

THEOREM 4.5. *Assume that* $\mathbb{K}$ *is an arbitrary base field*, $q \in \mathbb{K}^*$ *is not a root of unity*, $\mathfrak{g}$ *is a simple Lie algebra*, $w \in W$ *is a Weyl group element of length* $l$, *and* $\mathbf{i} = (\alpha_1, \ldots, \alpha_l)$ *is a reduced word for* $w$. *Then the following hold for all* $y \in W^{\leq w}$:

(i) *If* $l \notin \mathcal{LP}_{\mathbf{i}}(y)$, *then* $\mathrm{lt}(I_w(y)) = I_{ws_{\alpha_l}}(y)$, *where the leading part of* $I_w(y)$ *(cf.* (2.42)*) is computed with respect to the Ore extension* $\mathcal{U}^-[w] = \mathcal{U}^-[ws_{\alpha_l}][F_{\beta_l}; \sigma_l, \delta_l]$, *cf.* (4.3).

(ii) *If* $l \in \mathcal{LP}_{\mathbf{i}}(y)$, *then* $I_w(y) \cap \mathcal{U}^-[ws_{\alpha_l}] = I_{ws_{\alpha_l}}(ys_{\alpha_l})$.

We prove Theorem 4.1 using Theorem 4.5 in this subsection. We establish Theorem 4.5 in §4.2–4.3. Before we proceed with the proof of Theorem 4.1, we prove an auxiliary lemma.

LEMMA 4.6. *If, in the setting of Theorem 4.5,* $y \in W^{\leq w}$ *is such that* $l \in \mathcal{LP}_{\mathbf{i}}(y)$, *then*

$$T_{ws_{\alpha_l}} v_{\varpi_{\alpha_l}} \notin \mathcal{U}^- T_y v_{\varpi_{\alpha_l}}. \tag{4.4}$$

*Proof.* A similar statement that $T_{ws_{\alpha_l}} v_\lambda \notin \mathcal{U}^- T_y v_\lambda$ for $\lambda \in \sum_{\alpha \in \Pi} \mathbb{Z}_+ \varpi_\alpha$ follows from [21, Lemma 4.4.5] and the fact that $y \not\leq ws_{\alpha_l}$, which is easy to show. The last lemma is not applicable in our case, but we use some ideas of its proof.

We argue by induction on $l = \ell(w)$. If $l = 1$, then $T_{ws_{\alpha_l}} v_{\varpi_{\alpha_l}} = v_{\varpi_{\alpha_l}}$ and the statement is true since $y(\varpi_{\alpha_l}) < \varpi_{\alpha_l}$. Assume the validity of the lemma for length $l-1$.

Let $y \leq w \in W$ and $\mathbf{i}$ be as in the statement of the lemma. Assume that (4.4) does not hold, that is,

$$T_{ws_{\alpha_l}} v_{\varpi_{\alpha_l}} \in \mathcal{U}^- T_y v_{\varpi_{\alpha_l}}. \tag{4.5}$$

We consider two cases: (A) $1 \in \mathcal{LP}_{\mathbf{i}}(y)$ and (B) $1 \notin \mathcal{LP}_{\mathbf{i}}(y)$. Note that

$$\mathbf{i}'' := (\alpha_2, \dots, \alpha_l) \text{ is a reduced word for } s_{\alpha_1} w.$$

Case (A) $1 \in \mathcal{LP}_{\mathbf{i}}(y)$. Using the left positivity of the index set $\mathcal{LP}_{\mathbf{i}}(y)$, we obtain

$$y = s_{\alpha_1} w(\mathbf{i})_{>1}^{\mathcal{LP}_{\mathbf{i}}(y)} > w(\mathbf{i})_{>1}^{\mathcal{LP}_{\mathbf{i}}(y)} = s_{\alpha_1} y. \tag{4.6}$$

Moreover, we have $s_{\alpha_1} y \leq s_{\alpha_1} w$ and $\mathcal{LP}_{\mathbf{i}''}(s_{\alpha_1} y) = \mathcal{LP}_{\mathbf{i}}(y) \setminus \{1\}$. Recall the definition (2.20) of the subalgebras $\mathcal{U}^{\alpha}$ of $\mathcal{U}_q(\mathfrak{g})$, $\alpha \in \Pi$. Equations (4.5), (4.6) and [21, Lemma 4.4.3 (iii)–(iv)] imply

$$T_{s_{\alpha_1} ws_{\alpha_l}} v_{\varpi_{\alpha_l}} \in \mathcal{U}^{\alpha_1} T_{ws_{\alpha_l}} v_{\varpi_{\alpha_l}} \subseteq \mathcal{U}^{\alpha_1} \mathcal{U}^- T_y v_{\varpi_{\alpha_l}} = \mathcal{U}^- \mathcal{U}^{\alpha_1} T_y v_{\varpi_{\alpha_l}} = \mathcal{U}^- T_{s_{\alpha_1} y} v_{\varpi_{\alpha_l}},$$

which contradicts the induction assumption for the triple $(s_{\alpha_1} y, s_{\alpha_1} w, \mathbf{i}'')$.

Case (B) $1 \notin \mathcal{LP}_{\mathbf{i}}(y)$. The argument in this case is similar to the previous one. From the left positivity of the index set $\mathcal{LP}_{\mathbf{i}}(y)$, we have

$$s_{\alpha_1} y = s_{\alpha_1} w(\mathbf{i})_{>1}^{\mathcal{LP}_{\mathbf{i}}(y)} > w(\mathbf{i})_{>1}^{\mathcal{LP}_{\mathbf{i}}(y)} = y. \tag{4.7}$$

Furthermore, $y < s_{\alpha_1} w$ and $\mathcal{LP}_{\mathbf{i}''}(y) = \mathcal{LP}_{\mathbf{i}}(y)$. Equations (4.5), (4.7) and [21, Lemma 4.4.3 (iii)–(iv)] imply

$$T_{s_{\alpha_1} ws_{\alpha_l}} v_{\varpi_{\alpha_l}} \in \mathcal{U}^{\alpha_1} T_{ws_{\alpha_l}} v_{\varpi_{\alpha_l}} \subseteq \mathcal{U}^{\alpha_1} \mathcal{U}^- T_y v_{\varpi_{\alpha_l}} = \mathcal{U}^- \mathcal{U}^{\alpha_1} T_y v_{\varpi_{\alpha_l}} = \mathcal{U}^- T_y v_{\varpi_{\alpha_l}}.$$

This contradicts the induction assumption for the triple $(y, s_{\alpha_1} w, \mathbf{i}'')$.

We reached a contradiction in both cases. Thus (4.5) is incorrect, which completes the proof of the lemma. □

*Proof of Theorem 4.1.* We prove Theorem 4.1 by induction on the length $l = \ell(w)$. The case $\ell(w) = 0$ is trivial. Assume the validity of the statement of the theorem for length $l-1$.

Fix $w \in W$ and a reduced word $\mathbf{i} = (\alpha_1, \dots, \alpha_l)$ for it. Define the reduced word

$$\mathbf{i}' := (\alpha_1, \dots, \alpha_{l-1})$$

for $ws_{\alpha_l}$. In the setting of §2.4, $\overline{x}_l = x_l$. Theorem 3.1 implies that

$$F_{\beta_l} = p_l \Delta_{\mathbf{i}, l} = p_l b_{ws_{\alpha_l}, w}^{\varpi_{\alpha_l}}$$

for some $p_l \in \mathbb{K}^*$. Let $y \in W^{\leq w}$. We have two cases: (1) $l \notin \mathcal{LP}_{\mathbf{i}}(y)$ and (2) $l \in \mathcal{LP}_{\mathbf{i}}(y)$. For brevity, in this proof we set

$$D := \mathcal{LP}_{\mathbf{i}}(y).$$

Case (1) $l \notin D$. In this case $w(\mathbf{i})_{>j}^{D} = (ws_{\alpha_l})(\mathbf{i}')_{>j}^{D}$ for all $j \in [0, l-1]$. Taking into account (2.3), one sees that $D \subseteq [1, l-1]$ is the index set of a left positive subword of $\mathbf{i}'$. Therefore $y = (ws_{\alpha_l})^D < ws_{\alpha_l}$ and $\mathcal{LP}_y(\mathbf{i}') = D$. The inductive assumption applied to $y \leq ws_{\alpha_l}$ implies

$$\mathcal{CD}(I_{ws_{\alpha_l}}(y)) = D. \tag{4.8}$$

Recall from §2.3 that $b^{\varpi_{\alpha_l}}_{ws_{\alpha_l},w} \notin I_w(ws_{\alpha_l})$, see [31, Theorem 3.1 (b)] for a proof. Thus $F_{\beta_l} = p_l b^{\varpi_{\alpha_l}}_{ws_{\alpha_l},w} \notin I_w(ws_{\alpha_l})$ because $p_l \in \mathbb{K}^*$. Theorem 2.2 (ii) implies that $I_w(y) \subseteq I_w(ws_{\alpha_l})$. Therefore $F_{\beta_l} \notin I_w(y)$. Now we are in the situation of part (i) of Proposition 2.5 with respect to the iterated Ore extension from (2.16) and the ideal $J = I_w(y)$. By Theorem 4.5 (i), $\mathrm{lt}(I_w(y)) = I_{ws_{\alpha_l}}(y)$ and from Proposition 2.5 (i), we obtain that $\mathcal{CD}(I_w(y)) = \mathcal{CD}(I_{ws_{\alpha_l}}(y))$. It follows from this and (4.8) that in the first case $\mathcal{CD}(I_w(y)) = D = \mathcal{LP}_{\mathbf{i}}(y)$.

Case (2) $l \in D$. Define $D' = D \setminus \{l\}$. Since $D = \mathcal{LP}_{\mathbf{i}}(y)$ we have $s_{\alpha_j} w(\mathbf{i})^D_{>j} > w(\mathbf{i})^D_{>j}$, $\forall j \in [1, l]$. Moreover, $w(\mathbf{i})^D_{>j} = (ws_{\alpha_l})(\mathbf{i}')^{D'}_{>j} s_{\alpha_l}$ and $\ell(w(\mathbf{i})^D_{>j}) = \ell((ws_{\alpha_l})(\mathbf{i}')^{D'}_{>j}) + 1$. This implies that $s_{\alpha_j}((ws_{\alpha_l})(\mathbf{i}')^{D'}_{>j}) > (ws_{\alpha_l})(\mathbf{i}')^{D'}_{>j}$, $\forall j \in [1, l-1]$. Therefore $D'$ is the index set of a left positive subword of $\mathbf{i}'$. Because $y = w(\mathbf{i})^D = (ws_{\alpha_l})(\mathbf{i}')^{D'} s_{\alpha_l}$, we have $D' = \mathcal{LP}_{\mathbf{i}'}(ys_{\alpha_l})$. The inductive assumption, applied to $ys_{\alpha_l} \leq ws_{\alpha_l}$, implies

$$\mathcal{CD}(I_{ws_{\alpha_l}}(ys_{\alpha_l})) = D' = D \setminus \{l\}. \tag{4.9}$$

Lemma 4.6 asserts that $T_{ws_{\alpha_l}} v_{\varpi_{\alpha_l}} \notin \mathcal{U}^- T_y v_{\varpi_{\alpha_l}}$, so $\xi_{ws_{\alpha_l},\varpi_{\alpha_l}} \in (\mathcal{U}^- T_y v_{\varpi_{\alpha_l}})^\perp$ and $F_{\beta_l} = p_l b^{\varpi_{\alpha_l}}_{ws_{\alpha_l},w} \in I_w(y)$. We are in the situation of part (ii) of Proposition 2.5 with respect to the iterated Ore extension from (2.16) and the ideal $J = I_w(y)$. Theorem 4.5 (ii) implies $I_w(y) \cap \mathcal{U}^-[ws_{\alpha_l}] = I_{ws_{\alpha_l}}(ys_{\alpha_l})$. It follows from Proposition 2.5 (i) and equation (4.9) that $\mathcal{CD}(I_w(y)) = \mathcal{CD}(I_{ws_{\alpha_l}}(ys_{\alpha_l})) \sqcup \{l\} = D' \sqcup \{l\} = \mathcal{LP}_{\mathbf{i}}(y)$. $\square$

## *4.2. Proof of the First Part of Theorem 4.5*

Recall that in the setting of Theorem 4.5 we have the Ore extension $\mathcal{U}^-[w] = \mathcal{U}^-[ws_{\alpha_l}][F_{\beta_l}; \sigma_l, \delta_l]$ from (4.3). We will prove the first part of Theorem 4.5 by showing that the leading part $\mathrm{lt}(I_w(y))$ of the ideal $I_w(y)$ with respect to this Ore extension contains the ideal $I_{ws_{\alpha_l}}(y)$. We will then compare the Gelfand–Kirillov dimensions of the quotients $\mathcal{U}^-[w]/I_w(y)$ and $\mathcal{U}^-[ws_{\alpha_l}]/\,\mathrm{lt}(I_w(y))$ using results of [30] and Proposition 2.5 (i) to show that the leading part $\mathrm{lt}(I_w(y))$ is precisely $I_{ws_{\alpha_l}}(y)$. The first part of this argument is based on the following proposition.

PROPOSITION 4.7. *For all base fields $\mathbb{K}$, $q \in \mathbb{K}^*$ that is not a root of unity, Weyl group elements $w \in W$, reduced words $\mathbf{i} = (\alpha_1, \ldots, \alpha_l)$ for $w$, $\lambda \in \mathcal{P}^+$, and $\xi \in V(\lambda)^*$, we have*

$$\phi_w(c^\lambda_\xi e^{-\lambda}_w) - (q^{-1}_{\alpha_l} - q_{\alpha_l})^N q^{-N(N-1)/2}_{\alpha_l} F^N_{\beta_l} \phi_{ws_{\alpha_l}}(c^\lambda_\xi e^{-\lambda}_{ws_{\alpha_l}}) \in \sum_{m=0}^{N-1} F^m_{\beta_l} \mathcal{U}^-[ws_{\alpha_l}],$$

*where $N := \langle \lambda, \alpha_l^\vee \rangle$ (recall (2.7), (2.32), and (4.3)).*

Proposition 4.7 computes the leading term of $\phi_w(c^\lambda_\xi e^{-\lambda}_w)$ written as a right polynomial in $F_{\beta_l}$ with coefficients in $\mathcal{U}^-[ws_{\alpha_l}]$ (with respect to the Ore extension (4.3)) if this polynomial has degree equal to $\langle\lambda, \alpha_l^\vee\rangle$, which is the highest expected degree. This proposition can be viewed as a dual result to Proposition 3.3.

*Proof of Proposition 4.7.* Set

$$w' := ws_{\alpha_l} = w(\mathbf{i})_{\le(l-1)}.$$

Recall (2.20). The vector $v_\lambda$ is a highest weight vector for $\mathcal{U}^{\alpha_l}$ of highest weight $N\varpi_{\alpha_l}$. Equations (2.21) and (2.22) imply

$$E^N_{\alpha_l} T^{-1}_\alpha v_\lambda = \frac{1}{[N]_{\alpha_l}!} E^N_{\alpha_l} F^N_{\alpha_l} v_\lambda = [N]_{\alpha_l}! v_\alpha \quad \text{and} \quad E^m_{\alpha_l} T^{-1}_\alpha v_\lambda = 0, \quad \forall m > N.$$

Therefore

$$\begin{aligned}(\tau E_{\beta_l})^N T^{-1}_{w^{-1}} v_\lambda &= (T^{-1}_{(w')^{-1}}(E^N_{\alpha_l}))(T^{-1}_{(w')^{-1}} T^{-1}_\alpha v_\lambda)\\ &= T^{-1}_{(w')^{-1}}(E^N_{\alpha_l} T^{-1}_\alpha v_\lambda) = [N]_{\alpha_l}! \, T^{-1}_{(w')^{-1}} v_\lambda\end{aligned}$$

and similarly

$$(\tau E_{\beta_l})^m T^{-1}_{w^{-1}} v_\lambda = 0, \quad \forall m > N,$$

recall (2.30) and (2.31). Using the formula (2.34) for the antihomomorphism $\phi_w : R^w_0 \to \mathcal{U}^-[w]$, we obtain that for all $\lambda \in \mathcal{P}^+$, $\xi \in V(\lambda)^*$,

$$\begin{aligned}\phi_w(c^\lambda_\xi e^{-\lambda}_w) &= \frac{(q^{-1}_{\alpha_l} - q_{\alpha_l})^N}{q^{N(N-1)/2}_{\alpha_l}} \sum_{m_1,\dots,m_{l-1}\in\mathbb{N}} \left( \prod_{j=1}^{l-1} \frac{(q^{-1}_{\alpha_j} - q_{\alpha_j})^{m_j}}{q^{m_j(m_j-1)/2}_{\alpha_j} [m_j]_{\alpha_j}!} \right)\\ &\quad \times \langle \xi, (\tau E_{\beta_1})^{m_1} \cdots (\tau E_{\beta_{l-1}})^{m_{l-1}} T^{-1}_{(w')^{-1}} v_\lambda \rangle F^N_{\beta_l} F^{m_{l-1}}_{\beta_{l-1}} \cdots F^{m_1}_{\beta_1}\\ &= \frac{(q^{-1}_{\alpha_l} - q_{\alpha_l})^N}{q^{N(N-1)/2}_{\alpha_l}} F^N_{\beta_l} \phi_{w'}(c^\lambda_\xi e^{-\lambda}_{w'}) \mod \sum_{m=0}^{N-1} F^m_{\beta_l} \mathcal{U}^-[w'],\end{aligned}$$

which completes the proof of the proposition. $\square$

*Proof of Theorem 4.5 (i).* In the proof of Theorem 4.1 we showed that $l \notin \mathcal{LP}_{\mathbf{i}}(y)$ implies $F_{\beta_l} \notin I_w(y)$. We apply Proposition 2.5 (i) to the iterated Ore extension (2.16) and $J = I_w(y)$. Since $I_w(y)$ is a $\mathbb{T}^r$-invariant completely prime ideal of $\mathcal{U}^-[w]$, $\mathrm{lt}(I_w(y))$ is a $\mathbb{T}^r$-invariant completely prime ideal of $\mathcal{U}^-[ws_{\alpha_l}]$. By Theorem 2.2 (i),

$$\mathrm{lt}(I_w(y)) = I_{ws_{\alpha_l}}(y')$$

for some $y' \in W^{\le ws_\alpha}$. Let $\lambda \in \mathcal{P}^+$ and $\xi \in (\mathcal{U}^- T_y v_\lambda)^\perp \subset (V(\lambda))^*$. Then $\phi_w(c^\lambda_\xi e^{-\lambda}_w) \in I_w(y)$ and by Proposition 4.7, $\phi_{ws_{\alpha_l}}(c^\lambda_\xi e^{-\lambda}_{ws_{\alpha_l}}) \in \mathrm{lt}(I_w(y))$. Therefore $\mathrm{lt}(I_w(y)) \supseteq I_{ws_{\alpha_l}}(y)$. Applying Theorem 2.2 (ii), we obtain that $y' \ge y$. By (2.44),

$$\mathrm{GK}\dim\left(\frac{\mathcal{U}^-[w]}{I_w(y)}\right) = \mathrm{GK}\dim\left(\frac{\mathcal{U}^-[ws_{\alpha_l}]}{\mathrm{lt}(I_w(y))}\right) + 1 = \mathrm{GK}\dim\left(\frac{\mathcal{U}^-[ws_{\alpha_l}]}{I_w(y')}\right) + 1.$$

It follows from [30, Theorem 5.8] that

$$\operatorname{GK}\dim\left(\frac{\mathcal{U}^-[w]}{I_w(y)}\right)=l-\ell(y) \quad \text{and} \quad \operatorname{GK}\dim\left(\frac{\mathcal{U}^-[ws_{\alpha_l}]}{I_w(y')}\right)=l-1-\ell(y').$$

Therefore $\ell(y')=\ell(y)$. Since $y'\geq y$, this is only possible if $y'=y$, that is,

$$\operatorname{lt}(I_w(y))=I_{ws_{\alpha_l}}(y). \qquad \square$$

### 4.3. *Proof of the Second Part of Theorem 4.5*

A straightforward computation of the contraction $I_w(y)\cap\mathcal{U}^-[ws_{\alpha_l}]$ in the Ore extension (4.3) is very involved and impractical. We investigate this contraction in a roundabout way by comparing monoids of normal elements. We apply Proposition 2.5 (ii) to deduce that

$$\frac{\mathcal{U}^-[w]}{I_w(y)}\cong\frac{\mathcal{U}^-[ws_{\alpha_l}]}{I_w(y)\cap\mathcal{U}^-[ws_{\alpha_l}]} \tag{4.10}$$

and Theorem 2.2 (i) to deduce that $I_w(y)\cap\mathcal{U}^-[ws_{\alpha_l}]=I_{ws_{\alpha_l}}(y')$ for some $y'\in W^{\leq ws_{\alpha_l}}$. From (2.33) we have a supply of nonzero normal elements of the algebras $\mathcal{U}^-[w]/I_w(y)$. We prove a characterization of certain (equivariantly) normal elements of $\mathcal{U}^-[w]/I_w(y)$. With its help we compare the monoids of these equivariantly normal elements of the two sides of (4.10) and deduce that $y'=ys_{\alpha_l}$.

The weight lattice $\mathcal{P}$ of $\mathfrak{g}$ is embedded in $\mathbb{T}^r$ via $\mu\mapsto(q^{\langle\mu,\alpha^\vee\rangle})_{\alpha\in\Pi}$. The $\mathbb{T}^r$-action (2.12) gives rise to an action of $\mathcal{P}$ on $\mathcal{U}_q(\mathfrak{g})$, $\mathcal{U}^-[w]$, and $\mathcal{U}^-[w]/I_w(y)$, given by

$$\mu\cdot x=q^{\langle\mu,\gamma\rangle}x, \quad \gamma\in\mathcal{Q}, x\in(\mathcal{U}_q(\mathfrak{g}))_\gamma.$$

If a group $M$ acts on a ring $R$ by ring automorphisms, an element $u$ of $R$ is called an $M$-normal element if there exists $\mu\in M$ such that

$$ux=(\mu\cdot x)u, \quad \forall x\in R.$$

(In relation to equivariant polynormality, in the definition of $M$-normal element one sometimes requires that $u$ be an $M$-eigenvector, see [30]. For the sake of clarity, we will use the extra term homogeneous to emphasize this.) Here and below, the term homogeneous will refer to the $\mathcal{Q}$-gradings of $\mathcal{U}_q(\mathfrak{g})$, $\mathcal{U}^-[w]$, and $\mathcal{U}^-[w]/I_w(y)$.

By (2.33), for all $y\in W^{\leq w}$, the elements $b^\lambda_{y,w}$, $\lambda\in\mathcal{P}$ are nonzero homogeneous $\mathcal{P}$-normal elements of $\mathcal{U}^-[w]/I_w(y)$. The next proposition is a result in the opposite direction concerning the possible weights of all homogeneous $\mathcal{P}$-normal elements of $\mathcal{U}^-[w]/I_w(y)$.

Proposition 4.8. *For all base fields $\mathbb{K}$, $q\in\mathbb{K}^*$ that is not a root of unity, Weyl group elements $y\leq w$, and nonzero homogeneous $\mathcal{P}$-normal elements $u\in\mathcal{U}^-[w]/I_w(y)$, there exists $\mu\in(1/2)\mathcal{P}$ such that*

$$(w-y)\mu\in\mathcal{Q}_{\mathcal{S}(w)}, \quad u\in(\mathcal{U}^-[w]/I_w(y))_{(w-y)\mu}, \quad (w+y)\mu\in\mathcal{P},$$

*and*

$$ux = q^{-\langle (w+y)\mu, \gamma \rangle} xu, \quad \forall \gamma \in \mathcal{Q}, x \in \left( \frac{\mathcal{U}^-[w]}{I_w(y)} \right)_\gamma .$$

*Proof.* Let $u \in (\mathcal{U}^-[w]/I_w(y))_{\gamma'}$, $\gamma' \in \mathcal{Q}_{\mathcal{S}(w)}$ be a homogeneous $\mathcal{P}$-normal element of $\mathcal{U}^-[w]/I_w(y)$ such that

$$ux = q^{\langle \mu', \gamma \rangle} xu, \quad \forall \gamma \in \mathcal{Q}, x \in \left( \frac{\mathcal{U}^-[w]}{I_w(y)} \right)_\gamma \tag{4.11}$$

for some $\mu' \in \mathcal{P}$. Equations (2.33) and (4.11) imply

$$b^\lambda_{y,w} u = q^{-\langle (w+y)\lambda, \gamma' \rangle} u b^\lambda_{y,w} = q^{-\langle (w+y)\lambda, \gamma' \rangle} q^{\langle \mu', (w-y)\lambda \rangle} b^\lambda_{y,w} u$$

for all $\lambda \in \mathcal{P}^+$. Because $q \in \mathbb{K}^*$ is not a root of unity and $\mathcal{U}^-[w]/I_w(y)$ is a domain,

$$-\langle (w+y)\lambda, \gamma' \rangle + \langle \mu', (w-y)\lambda \rangle = 0, \quad \forall \lambda \in \mathcal{P}^+.$$

Therefore

$$\langle w\lambda, (wy^{-1}+1)\gamma' \rangle + \langle w\lambda, (wy^{-1}-1)\mu' \rangle = 0, \quad \forall \lambda \in \mathcal{P}^+,$$

that is,

$$(wy^{-1}+1)\gamma' = (wy^{-1}-1)(-\mu') = 0.$$

Using the standard linear algebra argument for Cayley transforms, we obtain that there exits $\mu \in \mathbb{Q}\Pi$ such that

$$\begin{aligned} \gamma' &= (wy^{-1}-1)y\mu = (w-y)\mu \quad \text{and} \\ -\mu' &= (wy^{-1}+1)y\mu = (w+y)\mu \end{aligned} \tag{4.12}$$

(see for instance the proof of [29, Theorem 3.6]). Adding the two equalities leads to $2w(\mu) = \gamma' - \mu'$, that is, $\mu = (1/2)w^{-1}(\gamma' - \mu') \in (1/2)\mathcal{P}$. Moreover, $(w-y)\mu = \gamma' \in \mathcal{Q}_{\mathcal{S}(w)}$, $u \in (\mathcal{U}^-[w]/I_w(y))_{\gamma'} = (\mathcal{U}^-[w]/I_w(y))_{(w-y)\mu}$, and $(w+y)\mu = -\mu' \in \mathcal{P}$. Finally, substituting (4.12) in (4.11) gives

$$ux = q^{-\langle (w+y)\mu, \gamma \rangle} xu, \quad \forall \gamma \in \mathcal{Q}, x \in \left( \frac{\mathcal{U}^-[w]}{I_w(y)} \right)_\gamma . \qquad \square$$

*Proof of Theorem 4.5 (ii).* It was shown in the proof of Theorem 4.1 that $l \in \mathcal{LP}_{\mathbf{i}}(y)$ implies $F_{\beta_l} \in I_w(y)$. Recall equation (4.2). Since $I_w(y)$ is a $\mathbb{T}^r$-invariant completely prime ideal of $\mathcal{U}^-[w]$, $I_w(y) \cap \mathcal{U}^-[ws_{\alpha_l}]$ is a $\mathbb{T}^r$-invariant completely prime ideal of $\mathcal{U}^-[ws_{\alpha_l}]$. It follows from Theorem 2.2 (i) that

$$I_w(y) \cap \mathcal{U}^-[ws_{\alpha_l}] = I_{ws_{\alpha_l}}(y')$$

for some $y' \in W^{\leq ws_{\alpha_l}}$. By Proposition 2.5 (ii) we have the isomorphism of $\mathcal{Q}$-graded algebras

$$\frac{\mathcal{U}^-[ws_{\alpha_l}]}{I_{ws_{\alpha_l}}(y')} \cong \frac{\mathcal{U}^-[w]}{I_w(y)}$$

because the $\mathbb{T}^r$-eigenvectors of $\mathcal{U}_q(\mathfrak{g})$ with respect to the action (2.12) are precisely the homogeneous vectors of the $\mathcal{Q}$-grading of $\mathcal{U}_q(\mathfrak{g})$. Define the support of the $\mathcal{Q}$-grading of the above algebras as follows:

$$\mathcal{Q}' := \mathbb{Z}\left\{\gamma \in \mathcal{Q} \;\middle|\; \left(\frac{\mathcal{U}^-[ws_{\alpha_l}]}{I_{ws_{\alpha_l}}(y')}\right)_\gamma \neq 0\right\} \subseteq \mathcal{Q}.$$

Let $\lambda \in \mathcal{P}$. Equation (2.33) implies that $b^\lambda_{y,w}$ is a nonzero homogeneous $\mathcal{P}$-normal element of $\mathcal{U}^-[ws_{\alpha_l}]/I_{ws_{\alpha_l}}(y')$ such that

$$\begin{aligned} &b^\lambda_{y,w} \in \left(\frac{\mathcal{U}^-[ws_{\alpha_l}]}{I_{ws_{\alpha_l}}(y')}\right)_{(w-y)\lambda} \quad \text{and} \\ &b^\lambda_{y,w}x = q^{-\langle (w+y)\lambda,\gamma\rangle} x b^\lambda_{y,w}, \quad \forall \gamma \in \mathcal{Q}', x \in \left(\frac{\mathcal{U}^-[ws_{\alpha_l}]}{I_{ws_{\alpha_l}}(y')}\right)_\gamma. \end{aligned} \tag{4.13}$$

We apply Proposition 4.8 to the algebra $\mathcal{U}^-[ws_{\alpha_l}]/I_{ws_{\alpha_l}}(y')$ and the $\mathcal{P}$-normal element $b^\lambda_{y,w}$. This shows that there exists $\mu' \in (1/2)\mathcal{P}$ such that

$$\begin{aligned} &b^\lambda_{y,w} \in \left(\frac{\mathcal{U}^-[ws_{\alpha_l}]}{I_{ws_{\alpha_l}}(y')}\right)_{(ws_{\alpha_l}-y')\mu}, \\ &b^\lambda_{y,w}x = q^{-\langle (ws_{\alpha_l}+y')\mu,\gamma\rangle} x b^\lambda_{y,w}, \quad \forall \gamma \in \mathcal{Q}', x \in \left(\frac{\mathcal{U}^-[ws_{\alpha_l}]}{I_{ws_{\alpha_l}}(y')}\right)_\gamma, \end{aligned} \tag{4.14}$$

and $(ws_{\alpha_l}+y')\mu \in \mathcal{P}$. Combining (4.13) and (4.14) and using the fact that $q \in \mathbb{K}^*$ is not a root of unity and $\mathcal{U}^-[ws_{\alpha_l}]/I_{ws_{\alpha_l}}(y')$ is a domain leads to

$$\begin{aligned} &(w-y)\lambda = (ws_{\alpha_l}-y')\mu \quad \text{and} \quad \langle (w+y)\lambda,\gamma\rangle = \langle (ws_{\alpha_l}+y')\mu,\gamma\rangle, \\ &\quad \forall \gamma \in \mathcal{Q}'. \end{aligned} \tag{4.15}$$

Therefore

$$\begin{aligned} \langle w\lambda,\gamma\rangle &= \langle (w-y)\lambda + (w+y)\lambda,\gamma\rangle \\ &= \langle (ws_{\alpha_l}-y')\mu + (ws_{\alpha_l}+y')\mu,\gamma\rangle \\ &= \langle ws_{\alpha_l}(\mu),\gamma\rangle, \quad \forall \gamma \in \mathcal{Q}'. \end{aligned} \tag{4.16}$$

For all $\nu \in \mathcal{P}^+$ we have $(ws_{\alpha_l}-y')\nu \in \mathcal{Q}'$ because $b^\nu_{ws_{\alpha_l},y'} \in (\mathcal{U}^-[ws_{\alpha_l}]/I_{ws_{\alpha_l}}(y'))_{(ws_{\alpha_l}-y')\nu} \setminus \{0\}$. Hence, by (4.16)

$$\langle ws_\alpha(s_{\alpha_l}\lambda-\mu), (ws_{\alpha_l}-y')\nu\rangle = 0, \quad \forall \nu \in \mathcal{P}^+,$$

that is,

$$\langle (y'-ws_\alpha)(s_{\alpha_l}\lambda-\mu), y'\nu\rangle = 0, \quad \forall \nu \in \mathcal{P}^+.$$

Thus $(y'-ws_\alpha)\mu = (y'-ws_\alpha)s_{\alpha_l}\lambda$. By taking into account the first part of (4.15), we obtain

$$(w-y)\lambda = (ws_\alpha - y')s_{\alpha_l}\lambda.$$

Therefore $y\lambda = y's_{\alpha_l}(\lambda)$ for all $\lambda \in \mathcal{P}^+$. We have $y' = ys_{\alpha_l}$ and hence

$$I_w(y) \cap \mathcal{U}^-[ws_{\alpha_l}] = I_{ws_{\alpha_l}}(ys_{\alpha_l}),$$

which completes the proof of part (ii) of Theorem 4.1. □

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