Michigan Math. J. 62 (2013), 3–37

Algebraic Montgomery–Yang Problem: The Nonrational Surface Case

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Dedicated to Professor I. Dolgachev on the occasion of the IgorFest

1. Introduction

A normal projective surface with the Betti numbers of the projective plane $\mathbb{CP}^2$ is called a *rational homology projective plane* or a $\mathbb{Q}$*-homology projective plane* or a $\mathbb{Q}$*-homology* $\mathbb{CP}^2$. When a normal projective surface S has rational singularities only, S is a $\mathbb{Q}$-homology projective plane if its second Betti number $b_2(S) = 1$. This can be seen easily by considering the Albanese fibration on a resolution of S.

It is known that a $\mathbb{Q}$-homology projective plane with quotient singularities (and no worse singularities) has at most five singular points (cf. [HK1, Cor. 3.4]). The authors have recently classified $\mathbb{Q}$-homology projective planes with five quotient singularities ([HK1]; also see [K2]).

In this paper we continue our study on the algebraic Montgomery–Yang problem, which was formulated by J. Kollár as follows.

Conjecture 1.1 [Kol2] (Algebraic Montgomery–Yang Problem). *Let S be a $\mathbb{Q}$-homology projective plane with quotient singularities. Assume that $S^0 := S \setminus \mathrm{Sing}(S)$ is simply connected. Then S has at most three singular points.*

In [HK2] we confirm the conjecture when S has at least one noncyclic quotient singularity. Thus we may assume that S has cyclic singularities only. In this paper, we verify the conjecture when S is not rational.

Theorem 1.2. *Let S be a $\mathbb{Q}$-homology projective plane with cyclic singularities only. Assume that $H_1(S^0, \mathbb{Z}) = 0$. If S is not rational, then S has at most three singular points.*

Remark 1.3. The condition $H_1(S^0, \mathbb{Z}) = 0$ is weaker than the original condition $\pi(S^0) = \{1\}$, and there are infinitely many examples of $\mathbb{Q}$-homology projective planes with four quotient singularities—not all cyclic—such that $H_1(S^0, \mathbb{Z}) = 0$. Such $\mathbb{Q}$-homology projective planes are completely classified in [HK2]. It turns out that such a surface is a log del Pezzo surface with three cyclic singularities and

Received August 1, 2011. Revision received May 21, 2012.

This research was supported by the National Research Foundation (NRF) of Korea, funded by the Ministry of Education, Science and Technology (2007-C00002). This work was also partially supported by the new faculty research fund of Ajou University.

one noncyclic singularity such that $H_1(S^0,\mathbb{Z})=0$ but $\pi_1(S^0)\cong\mathfrak{A}_5$, the simple group of order 60.

The proof of Theorem 1.2 is given in Section 6 and proceeds as follows. Let S be a $\mathbb{Q}$-homology projective plane with cyclic singularities such that $H_1(S^0,\mathbb{Z})=0$. Then the orders of the local fundamental groups of singular points are pairwise relatively prime (see Lemma 3.6). Also, by the orbifold Bogomolov–Miyaoka–Yau inequality (see Theorems 3.2 and 3.3), S has at most four singular points. Assume that S has four singular points. Then the inequality enables us to enumerate all possible 4-tuples consisting of the orders of the local fundamental groups of singular points:

$$\begin{aligned}
&(2,3,5,q), && q\ge 7, && \gcd(q,30)=1;\\
&(2,3,7,q), && 11\le q\le 41, && \gcd(q,42)=1;\\
&(2,3,11,13).
\end{aligned}$$

Given its minimal resolution $f\colon\ S'\to S$, the exceptional curves and the canonical class $K_{S'}$ span a sublattice $R+\langle K_{S'}\rangle$ of the unimodular lattice

$$H^2(S',\mathbb{Z})_{\mathrm{free}}:=H^2(S',\mathbb{Z})/(\mathrm{torsion}),$$

where R is the sublattice spanned by the exceptional curves. By the condition $H_1(S^0,\mathbb{Z})=0$ we know that K_S is not numerically trivial (see Lemma 3.6); hence $R+\langle K_{S'}\rangle$ is of finite index in the cohomology lattice $H^2(S',\mathbb{Z})_{\mathrm{free}}$. This implies, in particular, that its discriminant

$$D:=|\det(R+\langle K_{S'}\rangle)|$$

is a positive square number (Lemma 3.6). This criterion significantly reduces the infinite list of all possible cases for R. For example, the order-3 singularity of the case $(2,3,5,q)$ must be of type $\frac{1}{3}(1,1)$ (Lemma 5.3). The reduced list is still infinite, and few cases can be ruled out by any further argument from lattice theory—for example, computation of ε-invariants does not work here even though it was effective in the proof of [HK1]. To handle this infinite list, we compute (-1)-curves on the minimal resolution S'.

Assume further that S is not rational. This assumption implies that K_S is ample and S' contains a (-1)-curve E with $E.(f^*K_S/K_S^2)$ small—that is, with (f^*K_S/K_S^2)-degree small (Lemma 4.5). Then we prove that the existence of such a (-1)-curve E leads to a contradiction; toward that end, we use certain expressions of the intersection numbers $EK_{S'}$ and E^2 in terms of the intersection numbers of E with the exceptional curves and f^*K_S (Proposition 4.2). Here we also use the classification result for the case of five singular points [HK1].

The idea of computing (-1)-curves on the minimal resolution was first used in [K1] for S having some fixed types of singularities. In Proposition 4.2, we derive general formulas for an arbitrary and not necessarily effective divisor E on S' for S having arbitrary cyclic singularities. These formulas are useful in proving the nonexistence of a divisor on S' with prescribed intersection numbers with the exceptional curves (see e.g. [K3, Prop. 2.4]).

Throughout this paper, we work over the field $\mathbb{C}$ of complex numbers and employ the following notation.

- $[n_1, n_2, \dots, n_l]$ denotes a Hirzebruch–Jung continued fraction,
$$[n_1, n_2, \dots, n_l] = n_1 - \cfrac{1}{n_2 - \cfrac{1}{\ddots - \cfrac{1}{n_l}}} = \frac{q}{q_1},$$
corresponding to a cyclic singularity of type $\frac{1}{q}(1, q_1)$.
- $|[n_1, n_2, \dots, n_l]| = q$.
- $b_i(X)$ is the ith Betti number of a complex variety X.
- $f\colon S' \to S$ is a minimal resolution of a normal surface S.
- $\mathrm{Sing}(S)$ is the singular locus of S.
- $\mathcal{F} := f^{-1}(\mathrm{Sing}(S))$ is a reduced integral divisor on S'.
- R_p denotes the sublattice of $H^2(S', \mathbb{Z})_{\text{free}}$ spanned by the numerical classes of the components of $f^{-1}(p)$, where $H^2(S', \mathbb{Z})_{\text{free}} = H^2(S', \mathbb{Z})/(\text{torsion})$.
- $R := \bigoplus_{p \in \mathrm{Sing}(S)} R_p$ is the sublattice of $H^2(S', \mathbb{Z})_{\text{free}}$ spanned by the numerical classes of the irreducible exceptional curves of $f\colon S' \to S$.
- $L = L_S := \mathrm{rank}(R)$ is the number of the irreducible components of $\mathcal{F} = f^{-1}(\mathrm{Sing}(S))$ or the number of exceptional curves of $f\colon S' \to S$.

2. Hirzebruch–Jung Continued Fractions

Let $\mathcal{H}$ be the set of all Hirzebruch–Jung continued fractions $[n_1, n_2, \dots, n_l]$:
$$\mathcal{H} = \bigcup_{l \geq 1} \{[n_1, n_2, \dots, n_l] \mid \text{all } n_j \text{ are integers} \geq 2\}.$$

NOTATION 2.1. Fix $w = [n_1, n_2, \dots, n_l] \in \mathcal{H}$.

(1) The *length* of w, denoted by $l(w)$, is the number of entries of w.

(2) The *trace* of w, $\mathrm{tr}(w) = \sum_{j=1}^{l} n_j$, is the sum of entries of w.

(3) $|w| = |[n_1, n_2, \dots, n_l]| := |\det(M(-n_1, \dots, -n_l))|$, where
$$M(-n_1, \dots, -n_l) = \begin{pmatrix} -n_1 & 1 & 0 & \cdots & \cdots & 0 \\ 1 & -n_2 & 1 & \cdots & \cdots & 0 \\ 0 & 1 & -n_3 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -n_{l-1} & 1 \\ 0 & 0 & 0 & \cdots & 1 & -n_l \end{pmatrix}$$
is the intersection matrix of $[n_1, n_2, \dots, n_l]$.

(4) $q := |w|$ = the order of the cyclic singularity corresponding to w; that is, $w = q/q_1$ for some q_1 with $1 \leq q_1 < q$, $\gcd(q, q_1) = 1$. Also,
$$q_{a_1, a_2, \dots, a_m} := |\det(M')| \quad \text{and}$$
$$q_{1,2,\dots,l} := |\det(M(\emptyset))| = 1,$$

where M' is the $(l-m)\times(l-m)$ matrix obtained by deleting $-n_{a_1}, -n_{a_2}, \dots,$ $-n_{a_m}$ from $M(-n_1,\dots,-n_l)$. For example:

$$q_1 = |\det(M(-n_2,\dots,-n_l))| = |[n_2,n_3,\dots,n_l]|,$$
$$q_l = |\det(M(-n_1,\dots,-n_{l-1}))| = |[n_1,n_2,\dots,n_{l-1}]|,$$
$$q_{1,l} = |\det(M(-n_2,\dots,-n_{l-1}))| = |[n_2,n_3,\dots,n_{l-1}]|.$$

Note that

$$[n_l,n_{l-1},\dots,n_1] = \frac{q}{q_l} \quad \text{and}$$
$$q_1q_l = q_{1,l}q+1 \quad \text{if } l \geq 2.$$

We will write simply l and tr for $l(w)$ and $\mathrm{tr}(w)$ if no confusion will result.

The following number-theoretic property of Hirzebruch–Jung continued fractions will play a key role in the proof of Lemma 5.3.

PROPOSITION 2.2. *For $w = [n_1,n_2,\dots,n_l] \in \mathcal{H}$,*

$$q_1 + q_l + \mathrm{tr}\cdot q \not\equiv 0 \textit{ modulo } 3 \iff q \equiv 0 \textit{ modulo } 3.$$

Proof. In the following, $a \equiv b$ means that $a \equiv b$ modulo 3.

($\Leftarrow$) Assume $q \equiv 0$. If $l = 1$ and $w = [n_1]$, then $q_1 = q_l = |\det(M(\emptyset))| = 1$ and $q = \mathrm{tr} = n_1 \equiv 0$; hence

$$q_1 + q_l + \mathrm{tr}\cdot q \equiv 1+1+0 \not\equiv 0.$$

If $l \geq 2$, then we see from the equality $q_1q_l = q_{1,l}q+1$ that $q_1q_l \equiv 1$. Thus $q_1 \equiv q_l \equiv \pm 1$ and

$$q_1 + q_l + \mathrm{tr}\cdot q \equiv \pm 1 \pm 1 + 0 \not\equiv 0.$$

($\Rightarrow$) Assume $q \not\equiv 0$ (i.e., $q \equiv \pm 1$). We will show by induction on l that

$$q_1 + q_l + \mathrm{tr}\cdot q \equiv 0 \tag{2.1}$$

If $l = 1$ and $w = [n_1]$, then $q_1 = q_l = 1$ and $q = \mathrm{tr} = n_1 \equiv \pm 1$; hence

$$q_1 + q_l + \mathrm{tr}\cdot q \equiv 1+1+(\pm 1)^2 \equiv 0.$$

If $l = 2$ and $w = [n_1,n_2]$, then $q = n_1n_2 - 1 \equiv \pm 1$ and so $n_1n_2 \equiv -1$ or 0; therefore, $n_1 \equiv -n_2$ or $n_1 \equiv 0$ or $n_2 \equiv 0$. In any case,

$$q_1 + q_l + \mathrm{tr}\cdot q = n_2 + n_1 + (n_1+n_2)(n_1n_2-1) = n_1n_2(n_1+n_2) \equiv 0.$$

Now assume that $l \geq 3$. We divide the proof into three cases: $q_1 \equiv 1, -1, 0$.

Case 1: $q_1 \equiv 1$. By the induction hypothesis, the congruence (2.1) holds for $[n_2,\dots,n_l]$; that is,

$$q_{1,2} + q_{1,l} + (\mathrm{tr} - n_1)\cdot q_1 \equiv 0.$$

Plugging $q = n_1q_1 - q_{1,2}$ into this congruence, we get

$$q_{1,l} + \mathrm{tr} \cdot q_1 - q \equiv 0.$$

Thus

$$\begin{aligned} q_1 + q_l + \mathrm{tr} \cdot q &\equiv 1 + q_l + \mathrm{tr} \cdot q \\ &\equiv -1 - 1 + 1 \cdot q_l + \mathrm{tr} \cdot q \\ &\equiv -1 - q^2 + q_1 q_l + \mathrm{tr} \cdot q \\ &= q_{1,l} q + \mathrm{tr} \cdot q - q^2 \\ &= (q_{1,l} + \mathrm{tr} - q) q \\ &\equiv (q_{1,l} + \mathrm{tr} \cdot q_1 - q) q \\ &\equiv 0. \end{aligned}$$

Case 2: $q_1 \equiv -1$. As in Case 1, in this case the induction hypothesis also gives $q_{1,l} + \mathrm{tr} \cdot q_1 - q \equiv 0$. Therefore,

$$\begin{aligned} q_1 + q_l + \mathrm{tr} \cdot q &\equiv -1 + q_l + \mathrm{tr} \cdot q \\ &\equiv 1 - q_1 q_l + \mathrm{tr} \cdot q + q^2 \\ &\equiv -q_{1,l} q - \mathrm{tr} \cdot q_1 q + q^2 \\ &= -(q_{1,l} + \mathrm{tr} \cdot q_1 - q) q \\ &\equiv 0. \end{aligned}$$

Case 3: $q_1 \equiv 0$. First note that $q = n_1 q_1 - q_{1,2} \equiv -q_{1,2}$, so $q_{1,2} \equiv -q \not\equiv 0$. Note in addition that $q_{1,l} q = q_1 q_l - 1 \equiv -1$, so $q_{1,l} \equiv -q$. Since $q_{1,2} \not\equiv 0$, we apply the induction hypothesis to $[n_3, \ldots, n_l]$ and obtain

$$q_{1,2,3} + q_{1,2,l} + (\mathrm{tr} - n_1 - n_2) \cdot q_{1,2} \equiv 0.$$

Note that $q_1 = n_2 q_{1,2} - q_{1,2,3}$ and $n_1 q_{1,l} - q_l = q_{1,2,l}$. Since $q_{1,2} \equiv q_{1,l} \equiv -q$, we have

$$\begin{aligned} q_1 + q_l + \mathrm{tr} \cdot q &\equiv q_1 + q_l - \mathrm{tr} \cdot q_{1,2} \\ &\equiv q_1 - (n_1 q_{1,l} - q_l) - \mathrm{tr} \cdot q_{1,2} + n_1 q_{1,2} \\ &= (n_2 q_{1,2} - q_{1,2,3}) - q_{1,2,l} - \mathrm{tr} \cdot q_{1,2} + n_1 q_{1,2} \\ &= -q_{1,2,3} - q_{1,2,l} - (\mathrm{tr} - n_1 - n_2) \cdot q_{1,2} \\ &\equiv 0. \end{aligned}$$

□

We next collect some properties of Hirzebruch–Jung continued fractions that will be frequently used in the subsequent sections.

NOTATION 2.3. For a fixed continued fraction $w = [n_1, n_2, \ldots, n_l] \in \mathcal{H}$ and an integer $0 \le s \le l + 1$, we define

(1) $u_s := q_{s,\ldots,l} = |[n_1, n_2, \ldots, n_{s-1}]|$ for $2 \le s \le l + 1$, where $u_0 = 0$ and $u_1 = 1$;
(2) $v_s := q_{1,\ldots,s} = |[n_{s+1}, n_{s+2}, \ldots, n_l]|$ for $0 \le s \le l - 1$, where $v_l = 1$ and $v_{l+1} = 0$.

We remark that $u_l = q_l$, $u_{l+1} = q$, $v_0 = q$, and $v_1 = q_1$.

LEMMA 2.4. *Let* $w = [n_1, n_2, \ldots, n_l] \in \mathcal{H}$. *Then*:
(1) $u_{j+1} = n_j u_j - u_{j-1}$ *and* $v_{j-1} = n_j v_j - v_{j+1}$;
(2) $v_j u_{j+1} - v_{j+1} u_j = v_{j-1} u_j - v_j u_{j-1} = q$;
(3) $v_j u_j = \frac{1}{n_j}(q + v_{j+1} u_j + v_j u_{j-1})$;
(4) $\frac{u_j + v_j}{q} \leq \frac{2}{n_j}$; *and*
(5) $|[n_1, \ldots, n_{j-1}, n_j + 1, n_{j+1}, \ldots, n_l]| = u_j v_j + |[n_1, n_2, \ldots, n_l]| > q$.

Proof. Part (1) is well known, and (2) is obtained by a direct calculation using (1) as follows:

$$\begin{aligned} v_j u_{j+1} - v_{j+1} u_j &= (n_j u_j - u_{j-1}) v_j - v_{j+1} u_j \\ &= (n_j v_j - v_{j+1}) u_j - v_j u_{j-1} \\ &= v_{j-1} u_j - v_j u_{j-1} \\ &\;\vdots \\ &= v_1 u_2 - v_2 u_1 = q_1 n_1 - q_{1,2} = q. \end{aligned}$$

Part (3) follows from the equality

$$n_j v_j u_j = (v_{j-1} + v_{j+1}) u_j = q + v_j u_{j-1} + v_{j+1} u_j.$$

(4) For every $0 \leq j \leq l$ we have $v_j \geq v_{j+1} + 1$ and $u_{j+1} - 1 \geq u_j$, so

$$q - (v_j + u_j) = v_j(u_{j+1} - 1) - (v_{j+1} + 1) u_j \geq v_j u_j - v_j u_j = 0;$$

hence $v_j + u_j \leq q$. Also note that $v_{l+1} + u_{l+1} = q$. Now, for every $1 \leq j \leq l$,

$$\begin{aligned} n_j(v_j + u_j) &= (v_{j+1} + v_{j-1}) + (u_{j+1} + u_{j-1}) \quad \text{(by (1))} \\ &= (u_{j+1} + v_{j+1}) + (u_{j-1} + v_{j-1}) \leq 2q. \end{aligned}$$

(5) Note that

$$|[n_1, \ldots, n_{j-1}, n_j + 1]| = (n_j + 1) u_j - u_{j-1} = u_j + u_{j+1}.$$

Then, by (2),

$$\begin{aligned} |[n_1, \ldots, n_{j-1}, n_j + 1, n_{j+1}, \ldots, n_l]| &= |[n_1, \ldots, n_{j-1}, n_j + 1]| v_j - u_j v_{j+1} \\ &= u_j v_j + u_{j+1} v_j - u_j v_{j+1} \\ &= u_j v_j + |[n_1, n_2, \ldots, n_l]|. \end{aligned}$$

□

LEMMA 2.5. *Assume* $l \geq 5$. *Then, for arbitrary nonnegative integers* $z_1, \ldots, z_l$:

$$\sum_{j=1}^{l} (u_j + v_j) z_j \leq \begin{cases} \sum_{j=1}^{l} (u_j v_j) z_j^2 & \text{if } \sum_{j=1}^{l} z_j \geq 3, \\ \sum_{j=1}^{l} (u_j v_j) z_j^2 + 2 & \text{if } \sum_{j=1}^{l} z_j = 2, \\ \sum_{j=1}^{l} (u_j v_j) z_j^2 + 1 & \text{if } \sum_{j=1}^{l} z_j = 1. \end{cases}$$

Proof. Note that $(u_1 + v_1) z_1 = (1 + v_1) z_1 \leq v_1 z_1^2 - 2$ if $z_1 \geq 2$ and also that $(u_1 + v_1) z_1 = (1 + v_1) z_1 = v_1 z_1^2 + 1$ if $z_1 = 1$. Similarly, $(u_l + v_l) z_l = (u_l + 1) z_l \leq u_l z_1^2 - 2$ if $z_l \geq 2$ and $(u_l + v_l) z_l = (u_l + 1) z_l = u_l z_l^2 + 1$ if $z_l = 1$. For $2 \leq j \leq l - 1$ we have $u_j \geq 2$, $v_j \geq 2$, and $u_j + v_j \geq 6$ since $l \geq 5$, so $(u_j + v_j) z_j \leq (u_j v_j) z_j \leq (u_j v_j) z_j^2$ and $(u_j + v_j) z_j \leq (u_j v_j) z_j^2 - 2$ if $z_j \geq 1$. □

3. Algebraic Surfaces with Quotient Singularities

3.1

A singularity p of a normal surface S is called a *quotient singularity* if the germ is locally analytically isomorphic to $(\mathbb{C}^2/G, O)$ for some nontrivial finite subgroup G of $\mathrm{GL}_2(\mathbb{C})$ without quasi-reflections. Brieskorn [B] classified all such finite subgroups of $\mathrm{GL}(2, \mathbb{C})$.

Let S be a normal projective surface with quotient singularities, and let

$$f : S' \to S$$

be a minimal resolution of S. It is well known that quotient singularities are log-terminal singularities. Thus one can write

$$K_{S'} \underset{\text{num}}{\equiv} f^*K_S - \sum_{p \in \mathrm{Sing}(S)} \mathcal{D}_p,$$

where $\mathcal{D}_p = \sum(a_j A_j)$ is an effective $\mathbb{Q}$-divisor with $0 \leq a_j < 1$ supported on $f^{-1}(p) = \bigcup A_j$ for each singular point p. Intersecting the formula with $\mathcal{D}_p$ yields

$$\mathcal{D}_p K_{S'} = -\mathcal{D}_p^2,$$

from which it follows that

$$K_S^2 = K_{S'}^2 - \sum_p \mathcal{D}_p^2 = K_{S'}^2 + \sum_p \mathcal{D}_p K_{S'}.$$

For each singular point p, the coefficients of the $\mathbb{Q}$-divisor $\mathcal{D}_p$ can be obtained by solving the equations given by the adjunction formula

$$\mathcal{D}_p A_j = -K_{S'} A_j = 2 + A_j^2$$

for each exceptional curve $A_j \subset f^{-1}(p)$.

When p is a cyclic singularity of order q, the coefficients of $\mathcal{D}_p$ can be expressed in terms of v_j and u_j (see Notation 2.3) as follows.

LEMMA 3.1. *Let p be a cyclic quotient singular point of S. Assume that $f^{-1}(p)$ has l components $A_1, \dots, A_l$, with $A_i^2 = -n_i$ forming a string of smooth rational curves* $\overset{-n_1}{\circ} - \overset{-n_2}{\circ} - \cdots - \overset{-n_l}{\circ}$. *Then*

(1) $\mathcal{D}_p = \sum_{j=1}^{l} \left(1 - \dfrac{v_j + u_j}{q}\right) A_j,$

(2) $\mathcal{D}_p K_{S'} = -\mathcal{D}_p^2 = \sum_{j=1}^{l} \left(1 - \dfrac{v_j + u_j}{q}\right)(n_j - 2),$

(3) $\mathcal{D}_p^2 = 2l - \sum_{j=1}^{l} n_j + 2 - \dfrac{q_1 + q_l + 2}{q}.$

In particular, if $l = 1$ then $\mathcal{D}_p^2 = -\frac{(n_1-2)^2}{n_1}$.

Proof. The equality in (1) is well known (see [Me; HK1, Lemma 2.2]). Part (2) follows from (1) and the adjunction formula. The equality in (3) is also well known (see [LW; HK1, Lemma 3.6]). □

Recall the orbifold Euler characteristic

$$e_{\mathrm{orb}}(S) := e(S) - \sum_{p \in \mathrm{Sing}(S)} \left(1 - \frac{1}{|G_p|}\right),$$

where G_p is the local fundamental group of p.

The following result, known as the orbifold Bogomolov–Miyaoka–Yau inequality, is one of the main ingredients in the proof of our main theorem.

THEOREM 3.2 [KoNS; Me; Mi; S]. *Let S be a normal projective surface with quotient singularities such that K_S is nef. Then*

$$K_S^2 \le 3e_{\mathrm{orb}}(S).$$

In particular,

$$0 \le e_{\mathrm{orb}}(S).$$

The weaker inequality also holds when $-K_S$ is nef.

THEOREM 3.3 [KeM, Cor. 1.8.1]. *Let S be a normal projective surface with quotient singularities such that $-K_S$ is nef. Then*

$$0 \le e_{\mathrm{orb}}(S).$$

3.2

Let S be a normal projective surface with quotient singularities, and let $f: S' \to S$ be a minimal resolution of S. It is well known that the torsion-free part of the second cohomology group,

$$H^2(S', \mathbb{Z})_{\mathrm{free}} := H^2(S', \mathbb{Z})/(\mathrm{torsion}),$$

has a lattice structure that is unimodular. For a quotient singular point $p \in S$, let

$$R_p \subset H^2(S', \mathbb{Z})_{\mathrm{free}}$$

be the sublattice of $H^2(S', \mathbb{Z})_{\mathrm{free}}$ spanned by the numerical classes of the components of $f^{-1}(p)$. It is a negative definite lattice, and its discriminant group

$$\mathrm{disc}(R_p) := \mathrm{Hom}(R_p, \mathbb{Z})/R_p$$

is isomorphic to the abelianization $G_p/[G_p, G_p]$ of the local fundamental group G_p. In particular, the absolute value $|\det(R_p)|$ of the determinant of the intersection matrix of R_p is equal to the order $|G_p/[G_p, G_p]|$. Let

$$R = \bigoplus_{p \in \mathrm{Sing}(S)} R_p \subset H^2(S', \mathbb{Z})_{\mathrm{free}}$$

be the sublattice of $H^2(S', \mathbb{Z})_{\mathrm{free}}$ spanned by the numerical classes of the exceptional curves of $f: S' \to S$. We also consider the sublattice

$$R + \langle K_{S'} \rangle \subset H^2(S', \mathbb{Z})_{\text{free}}$$

spanned by R and the canonical class $K_{S'}$. Note that

$$\text{rank}(R) \le \text{rank}(R + \langle K_{S'} \rangle) \le \text{rank}(R) + 1.$$

LEMMA 3.4 [HK1, Lemma 3.3]. *Let S be a normal projective surface with quotient singularities, and let $f : S' \to S$ be a minimal resolution of S. Then the following statements hold.*

(1) $\text{rank}(R + \langle K_{S'} \rangle) = \text{rank}(R)$ *if and only if K_S is numerically trivial.*
(2) $\det(R + \langle K_{S'} \rangle) = \det(R) \cdot K_S^2$ *if K_S is not numerically trivial.*
(3) *If also $b_2(S) = 1$ and K_S is not numerically trivial, then $R + \langle K_{S'} \rangle$ is a sublattice of finite index in the unimodular lattice $H^2(S', \mathbb{Z})_{\text{free}}$; in particular, $|\det(R + \langle K_{S'} \rangle)|$ is a nonzero square number.*

We denote this nonzero square number as

$$D := |\det(R + \langle K_{S'} \rangle)|.$$

The following is well known.

LEMMA 3.5. *Assume that p is a cyclic singularity such that $f^{-1}(p)$ has l components $A_1, \dots, A_l$, with $A_i^2 = -n_i$ forming a string of smooth rational curves $\overset{-n_1}{\circ} - \overset{-n_2}{\circ} - \cdots - \overset{-n_l}{\circ}$. Then* $\text{disc}(R_p)$ *is a cyclic group generated by*

$$e_p := A_l^* = -\frac{1}{q} \sum_{i=1}^{l} u_i A_i,$$

where $u_i = |[n_1, n_2, \dots, n_{i-1}]|$ as in Notation 2.3. This cyclic group has the properties that

$$e_p A_l = 1, \quad e_p A_j = 0 \;\; (1 \le j \le l-1), \quad \textit{and} \quad e_p^2 = -\frac{u_l}{q} = -\frac{q_l}{q}.$$

Proof. We know that $\text{disc}(R_p) := \text{Hom}(R_p, \mathbb{Z})/R_p$ is a cyclic group of order $q = |[n_1, n_2, \dots, n_l]|$. Let $A_l^* \in \text{Hom}(R_p, \mathbb{Z})$ be the dual element of A_l, and write

$$A_l^* = \sum a_i A_i$$

for some rational numbers a_i. Then the equalities

$$A_l^* A_l = 1, \qquad A_l^* A_j = 0 \quad (1 \le j \le l-1)$$

give a system of linear equations for the a_i. Now, by Cramer's rule, we have

$$a_i = -\frac{u_i}{q}.$$

Since $u_1 = 1$, it follows that A_l^* has order q in $\text{disc}(R_p)$. □

The next lemma will also prove to be useful.

LEMMA 3.6 [HK2, Lemma 3]. *Let S be a $\mathbb{Q}$-homology projective plane with cyclic singularities such that $H_1(S^0, \mathbb{Z}) = 0$. Let $f : S' \to S$ be a minimal resolution. Then*:

(1) *$H^2(S', \mathbb{Z})$ is torsion free* (*i.e., $H^2(S', \mathbb{Z}) = H^2(S', \mathbb{Z})_{\text{free}}$*);
(2) *R is a primitive sublattice of the unimodular lattice $H^2(S', \mathbb{Z})$*;
(3) $\operatorname{disc}(R)$ *is a cyclic group—in particular, the orders $|G_p| = |\det(R_p)|$ are pairwise relatively prime*;
(4) *K_S is not numerically trivial* (*i.e., K_S is either ample or anti-ample*);
(5) *$D = |\det(R)| K_S^2$ and is a nonzero square number*; *and*
(6) *the Picard group* $\operatorname{Pic}(S')$ *is generated over $\mathbb{Z}$ by the exceptional curves and a $\mathbb{Q}$-divisor M of the form*

$$M = \frac{1}{\sqrt{D}} f^* K_S + \sum_{p \in \operatorname{Sing}(S)} b_p e_p$$

for some integers b_p, where e_p is the generator of $\operatorname{disc}(R_p)$ *as in Lemma 3.5.*

Finally we generalize Lemma 3.6 to the case without the condition $H_1(S^0, \mathbb{Z}) = 0$. We will encounter this general situation later in our proof (see Sections 5 and 6).

Let S be a $\mathbb{Q}$-homology projective plane with cyclic singularities, and let $f : S' \to S$ be a minimal resolution. Denote by $\operatorname{Pic}(S')_{\text{free}}$ the group of numerical equivalence classes of divisors; thus,

$$\operatorname{Pic}(S')_{\text{free}} := \operatorname{Pic}(S')/(\text{torsion}).$$

With the intersection pairing, $\operatorname{Pic}(S')_{\text{free}}$ becomes a unimodular lattice isometric to $H^2(S', \mathbb{Z})_{\text{free}}$. Denote by

$$\bar{R} \subset \operatorname{Pic}(S')_{\text{free}}$$

the primitive closure of $R \subset \operatorname{Pic}(S')_{\text{free}}$, the sublattice spanned by the numerical equivalence classes of exceptional curves of f.

LEMMA 3.7. *Let S be a $\mathbb{Q}$-homology projective plane with cyclic singularities, and let $f : S' \to S$ be a minimal resolution. Assume that K_S is not numerically trivial. Then we have the following five claims.*

(1) *$D = |\det(R)| K_S^2$ and is a nonzero square number.*

(2) $\operatorname{disc}(\bar{R})$ *is a cyclic group of order $|\det(\bar{R})| = |\det(R)|/c^2$, where c is the order of $\bar{R}/R$.*

(3) *Define*

$$D' := |\det(\bar{R})| K_S^2 = \frac{D}{c^2}.$$

Then $\operatorname{Pic}(S')_{\text{free}}$ *is generated over $\mathbb{Z}$ by the numerical equivalence classes of exceptional curves, an element $T \in \operatorname{Pic}(S')_{\text{free}}$ giving a generator of $\bar{R}/R$, and a $\mathbb{Q}$-divisor of the form*

$$M = \frac{1}{\sqrt{D'}} f^* K_S + z;$$

here z is a generator of $\operatorname{disc}(\bar{R})$ *and hence of the form $z = \sum_{p \in \operatorname{Sing}(S)} b_p e_p$ for some integers b_p, where e_p is the generator of* $\operatorname{disc}(R_p)$ *as in Lemma 3.5.*

(4) *For each singular point p, denote by $A_{1,p}, A_{2,p}, \dots, A_{l_p,p}$ the exceptional curves of f at p and by q_p the order of the local fundamental group at p. Then every element $E \in \mathrm{Pic}(S')_{\mathrm{free}}$ can be written uniquely as*

$$E = mM + \sum_{p \in \mathrm{Sing}(S)} \sum_{i=1}^{l_p} a_{i,p} A_{i,p} \tag{3.1}$$

for some integer m and some $a_{i,p} \in (1/c)\mathbb{Z}$ for all i, p.

(5) *E is supported on $f^{-1}(\mathrm{Sing}(S))$ if and only if $m = 0$. Moreover, if E is effective (modulo a torsion) and not supported on $f^{-1}(\mathrm{Sing}(S))$, then $m > 0$ when K_S is ample and $m < 0$ when $-K_S$ is ample.*

Proof. Part (1) follows from Lemma 3.4, and part (2) is well known.

For part (3), we slightly modify the proof of [HK2, Lemma 3]. Here $R^\perp$ is generated by

$$v := \frac{\sqrt{D'}}{K_S^2} f^*K_S = \frac{|\det(\bar{R})|}{\sqrt{D'}} f^*K_S,$$

$\mathrm{disc}(R^\perp)$ is generated by

$$\frac{1}{\sqrt{D'}} f^*K_S,$$

and

$$\frac{\mathrm{Pic}(S')_{\mathrm{free}}}{R^\perp \oplus \bar{R}} \subset \mathrm{disc}(R^\perp \oplus \bar{R})$$

is an isotropic subgroup of order $|\det(\bar{R})|$ of $\mathrm{disc}(R^\perp \oplus \bar{R})$ and hence is generated by an element

$$M \in \mathrm{disc}(R^\perp \oplus \bar{R})$$

of order $|\det(\bar{R})|$. Moreover, M is the sum of a generator of $\mathrm{disc}(R^\perp)$ and a generator of $\mathrm{disc}(\bar{R})$, since $\mathrm{Pic}(S')_{\mathrm{free}}$ is unimodular. Replacing M by kM for a suitable choice of an integer k, we obtain M of the desired form. We have shown that $\mathrm{Pic}(S')_{\mathrm{free}}$ is generated over $\mathbb{Z}$ by v, $\bar{R}$, and M. Note that

$$|\det(\bar{R})| M \equiv v \text{ modulo } \bar{R};$$

that is, v is generated by M and $\bar{R}$. Finally, $\bar{R}$ is generated over $\mathbb{Z}$ by R and T.

(4) By part (3), E is a $\mathbb{Z}$-linear combination of M, T, and $A_{i,p}$. Since $cT \in R$, the result follows.

(5) The first assertion is obvious. For the second, observe that

$$E(f^*K_S) = mM(f^*K_S) = \frac{m}{\sqrt{D'}} K_S^2. \qquad \square$$

4. Curves on the Minimal Resolution

Throughout this section, we denote by S a $\mathbb{Q}$-homology projective plane with cyclic singularities and by $f : S' \to S$ its minimal resolution; in addition, we assume that K_S is not numerically trivial. But we do not assume that $H_1(S^0, \mathbb{Z}) = 0$, so the orders of singularities may not be pairwise relatively prime.

Let E be a divisor on S'. Then, by Lemma 3.7(4), the numerical equivalence class of E can be written in the form (3.1). The coefficients of E in (3.1) and the intersection numbers $EA_{j,p}$ are related as follows, where u_j and v_j are as in Notation 2.3.

LEMMA 4.1. *Fix $p \in \mathrm{Sing}(S)$. Then, for $j = 1, \dots, l_p$,*

$$\frac{u_{j,p}}{q_p} m b_p - a_{j,p} = \sum_{k=1}^{j} \frac{v_{j,p} u_{k,p}}{q_p} (EA_{k,p}) + \sum_{k=j+1}^{l_p} \frac{v_{k,p} u_{j,p}}{q_p} (EA_{k,p}).$$

Proof. Note that, by Lemma 3.5, for each $p \in \mathrm{Sing}(S)$ we have

$$MA_{j,p} = 0 \quad \text{for} \quad j = 1, \dots, l_p - 1, \qquad MA_{l_p,p} = b_p.$$

We fix p and, for simplicity, omit the subscript p. Thus we obtain the following system of equalities:

$$\begin{aligned} EA_1 &= -n_1 a_1 + a_2, \\ EA_2 &= a_1 - n_2 a_2 + a_3, \\ EA_3 &= a_2 - n_3 a_3 + a_4, \\ &\vdots \\ EA_{l-1} &= a_{l-2} - n_{l-1} a_{l-1} + a_l, \\ EA_l &= a_{l-1} - n_l a_l + mb. \end{aligned}$$

This system implies that

$$\begin{aligned} a_1 &= \frac{1}{n_1} a_2 - \frac{1}{n_1} EA_1 = \frac{u_1}{u_2} a_2 - \frac{1}{u_2} EA_1, \\ a_2 &= \frac{u_2}{u_3} a_3 - \frac{1}{u_3} EA_1 - \frac{u_2}{u_3} EA_2, \\ &\vdots \\ a_j &= \frac{u_j}{u_{j+1}} a_{j+1} - \frac{1}{u_{j+1}} EA_1 - \cdots - \frac{u_k}{u_{j+1}} EA_k - \cdots - \frac{u_j}{u_{j+1}} EA_j, \\ &\vdots \\ a_{l-1} &= \frac{u_{l-1}}{u_l} a_l - \frac{1}{u_l} EA_1 - \cdots - \frac{u_k}{u_l} EA_k - \cdots - \frac{u_{l-1}}{u_l} EA_{l-1}, \\ a_l &= \frac{u_l}{q} mb - \frac{1}{q} EA_1 - \cdots - \frac{u_l}{q} EA_l = \frac{u_l}{q} mb - \sum_{k=1}^{l} \frac{v_l u_k}{q} EA_k. \end{aligned}$$

Plugging the last equation into the previous equation for a_{l-1}, we obtain

$$\begin{aligned} a_{l-1} &= \frac{u_{l-1}}{u_l} \left(\frac{u_l}{q} mb - \frac{1}{q} EA_1 - \cdots - \frac{u_l}{q} EA_l \right) - \frac{1}{u_l} EA_1 - \cdots - \frac{u_{l-1}}{u_l} EA_{l-1} \\ &= \frac{u_{l-1}}{q} mb - \sum_{k=1}^{l-1} \frac{(u_{l-1} + q) u_k}{q u_l} EA_k - \frac{u_{l-1}}{q} EA_l. \end{aligned}$$

By Lemma 2.2(2),

$$u_{l-1} + q = v_l u_{l-1} + q = v_{l-1} u_l;$$

hence the required equation for a_{l-1} follows.

Next, plugging the required equation for a_{l-1} into the equation for a_{l-2}, we obtain the required equation for that term. The other values can be obtained similarly. □

Now we express the intersection numbers $EK_{S'}$ and E^2 in terms of the intersection numbers $EA_{j,p}$ of E and the exceptional curves $A_{j,p}$.

PROPOSITION 4.2. *Let E be a divisor on S'. Write* (*the numerical equivalence class of*) *E as the form* (3.1). *Then the following statements hold.*

(1) $$EK_{S'} = \frac{m}{\sqrt{D'}} K_S^2 - \sum_p \sum_{j=1}^{l_p} \left(1 - \frac{v_{j,p} + u_{j,p}}{q_p}\right) EA_{j,p}.$$

If $EA_{j,p} \geq 0$ for all p and j, then

$$EK_{S'} \leq \frac{m}{\sqrt{D'}} K_S^2 - \sum_p \sum_{j=1}^{l_p} \left(1 - \frac{2}{n_{j,p}}\right) EA_{j,p}.$$

(2) $$E^2 = \frac{m^2}{D'} K_S^2 - \sum_p \sum_{j=1}^{l_p} \left(\sum_{k=1}^{j} \frac{v_{j,p} u_{k,p}}{q_p} (EA_{k,p}) + \sum_{k=j+1}^{l_p} \frac{v_{k,p} u_{j,p}}{q_p} (EA_{k,p}) \right) EA_{j,p}.$$

If $EA_{j,p} \geq 0$ for all p and j, then

$$E^2 \leq \frac{m^2}{D'} K_S^2 - \sum_p \sum_{j=1}^{l_p} \frac{v_{j,p} u_{j,p}}{q_p} (EA_{j,p})^2.$$

(3) *For each $p \in \mathrm{Sing}(S)$, supppose E has a nonzero intersection number with at most two components of $f^{-1}(p)$* (*i.e., suppose $EA_{j,p} = 0$ for $j \neq s_p, t_p$ with $1 \leq s_p < t_p \leq l_p$*); *then*

$$E^2 = \frac{m^2}{D'} K_S^2 - \sum_p \left(\frac{v_{s_p} u_{s_p}}{q_p} (EA_{s_p})^2 + \frac{v_{t_p} u_{t_p}}{q_p} (EA_{t_p})^2 + \frac{2 v_{t_p} u_{s_p}}{q_p} (EA_{s_p})(EA_{t_p}) \right).$$

Proof. (1) Note that

$$K_{S'} = f^*(K_S) - \sum_{p \in \mathrm{Sing}(S)} \sum_{j=1}^{l_p} \left(1 - \frac{v_{j,p} + u_{j,p}}{q_p}\right) A_{j,p}.$$

Intersecting both sides with E yields

$$EK_{S'} = Ef^*(K_S) - \sum_p \sum_{j=1}^{l_p} \left(1 - \frac{v_{j,p} + u_{j,p}}{q_p}\right) EA_{j,p}.$$

Intersecting both sides of

$$E = mM + \sum_p \sum_{i=1}^{l_p} a_{i,p} A_{i,p}$$

with $f^*(K_S)$, we get

$$Ef^*(K_S) = mMf^*(K_S) = \frac{m}{\sqrt{D'}} f^*(K_S)^2 = \frac{m}{\sqrt{D'}} K_S^2.$$

This proves the equality. The inequality follows from the equality by Lemma 2.4(4).

(2) Intersecting both sides of

$$E = mM + \sum_p \sum_{j=1}^{l_p} a_{j,p} A_{j,p}$$

with E yields

$$E^2 = mEM + \sum_p \sum_{j=1}^{l_p} a_{j,p} EA_{j,p}.$$

Intersecting both sides of

$$M = \frac{1}{\sqrt{D'}} f^* K_S + \sum_p b_p e_p$$

with E, we obtain

$$\begin{aligned} mEM &= \frac{m}{\sqrt{D'}} Ef^*(K_S) + m \sum_p b_p Ee_p \\ &= \frac{m}{\sqrt{D'}} \frac{m}{\sqrt{D'}} K_S^2 + m \sum_p b_p (mMe_p + a_{l,p}) \\ &= \frac{m^2}{D'} K_S^2 + m \sum_p b_p (mb_p e_p^2 + a_{l,p}) \\ &= \frac{m^2}{D'} K_S^2 + m \sum_p b_p \left(-\frac{mb_p u_{l,p}}{q} + a_{l,p}\right) \quad \text{(by Lemma 3.5)} \\ &= \frac{m^2}{D'} K_S^2 - m \sum_p b_p \left(\sum_{k=1}^{l_p} \frac{v_{l,p} u_{k,p}}{q} EA_{k,p}\right) \quad \text{(by Lemma 4.1).} \end{aligned}$$

Therefore,

$$\begin{aligned} E^2 &= \frac{m^2}{D'} K_S^2 - m \sum_p b_p \left(\sum_{j=1}^{l_p} \frac{v_{l,p} u_{j,p}}{q} EA_{j,p}\right) + \sum_p \sum_{j=1}^{l_p} a_{j,p} EA_{j,p} \\ &= \frac{m^2}{D'} K_S^2 - \sum_p \sum_{j=1}^{l_p} \left(\frac{mb_p u_{j,p}}{q} - a_{j,p}\right) EA_{j,p}. \end{aligned}$$

Now the equality follows from Lemma 4.1.

If $EA_{j,p} \geq 0$ for all p and j, then

$$\sum_{k=1}^{j} \frac{v_{j,p}u_{k,p}}{q_p}(EA_{k,p}) + \sum_{k=j+1}^{l_p} \frac{v_{k,p}u_{j,p}}{q_p}(EA_{k,p}) \geq \frac{v_{j,p}u_{j,p}}{q_p}(EA_{j,p}),$$

so the inequality follows.

(3) If $EA_{j,p} = 0$ for $j \neq s_p, t_p$ with $1 \leq s_p < t_p \leq l_p$, then

$$\begin{aligned}
&\sum_{j=1}^{l_p}\left(\sum_{k=1}^{j} \frac{v_{j,p}u_{k,p}}{q_p} EA_{k,p} + \sum_{k=j+1}^{l_p} \frac{v_{k,p}u_{j,p}}{q_p} EA_{k,p}\right) EA_{j,p} \\
&\quad = \left(\frac{v_{s_p}u_{s_p}}{q_p} EA_{s_p} + \frac{v_{t_p}u_{s_p}}{q_p} EA_{t_p}\right)(EA_{s_p}) \\
&\qquad + \left(\frac{v_{t_p}u_{s_p}}{q_p} EA_{s_p} + \frac{v_{t_p}u_{t_p}}{q_p} EA_{t_p}\right)(EA_{t_p}).
\end{aligned}$$

In this case, the equality follows from (2). □

Let

$$L = L_S := \operatorname{rank}(R)$$

be the number of the irreducible exceptional curves of $f\colon S' \to S$. We have

$$b_2(S') = 1 + L.$$

Note that $H^1(S', \mathcal{O}_{S'}) = H^2(S', \mathcal{O}_{S'}) = 0$. Hence, by the Noether formula,

$$K_{S'}^2 = 12 - e(S') = 10 - b_2(S') = 9 - L.$$

We close this section with the following two general results for the case where S is not rational.

PROPOSITION 4.3. *Let S be a $\mathbb{Q}$-homology projective plane with quotient singular points. If S is not rational, then the following statements hold.*

(1) *K_S is ample or numerically trivial.*
(2) *K_S is numerically trivial iff $K_{S'}$ is numerically trivial iff S' is an Enriques surface.*
(3) *If $L_S \geq 10$, then K_S is ample and S' contains a (-1)-curve.*
(4) *If one of the singularities of S is not a rational double point, then K_S is ample.*

Proof. (1) If $-K_S$ is ample, then S is rational.

(2) Note that $p_g(S') = q(S') = 0$. Thus the second equivalence follows from the classification theory of algebraic surfaces.

If K_S is numerically trivial, then the adjunction formula gives

$$K_{S'} \underset{\text{num}}{\equiv} f^*K_S - \sum_{p \in \operatorname{Sing}(S)} \mathcal{D}_p \underset{\text{num}}{\equiv} - \sum_{p \in \operatorname{Sing}(S)} \mathcal{D}_p.$$

Since S' is not rational, $\mathcal{D}_p = 0$ for every singular point $p \in S$. Therefore, $K_{S'}$ is numerically trivial.

If $K_{S'}$ is numerically trivial, then S' is an Enriques surface and every smooth rational curve on S' is a (-2)-curve; hence S has only rational double points. Then, by the adjunction formula, $K_{S'} = f^*K_S$ and so K_S is numerically trivial.

(3) Since $L_S \geq 10$, it follows that $K_{S'}^2 = 9 - L_S < 0$; hence S' is not minimal. If K_S is numerically trivial then S' is an Enriques surface by (2) and so $L_S = 9$, a contradiction.

(4) Note that $\mathcal{D}_p = 0$ for a singular point p if and only if p is a rational double point. Now the statement follows from the adjunction formula. $\square$

REMARK 4.4. The converse of Proposition 4.3(4) does not hold. There is a minimal surface of general type with $p_g = 0$ and $K^2 = 1$ that has eight (-2)-curves of Dynkin type $4A_2$ [K1]. By contracting the eight curves, we get a $\mathbb{Q}$-homology projective plane S with K_S ample but having rational double points only.

LEMMA 4.5. *Let S be a $\mathbb{Q}$-homology projective plane with cyclic singularities. Assume that S is not rational. If $L \geq 10$, then there is a (-1)-curve E on S' of the form* (3.1) *with* $0 < m \leq \sqrt{D'}/(L-9)$.

Proof. Since S is not rational and since $L \geq 10$, it follows from Proposition 4.3 that K_S is ample. Thus $m > 0$ for any (-1)-curve E by Lemma 3.7(5).

Since $K_{S'}^2 = 9 - L < 0$, we know that S' is not a minimal surface. Let

$$g\colon S' = S_k \to S_{k-1} \to S_{k-2} \to \cdots \to S_1 \to S_0 = S_{\min}$$

be a morphism of S' to its minimal model. A consequence of $K_{S_{\min}}^2 \geq 0$ is that

$$k \geq L - 9.$$

One can write

$$K_{S'} = g^*K_{S_{\min}} + \sum_{i=1}^{k} E_i,$$

where E_i is the total transform of the exceptional curve of the blowup $S_i \to S_{i-1}$. Note that $E_1, \dots, E_k$ are effective but not necessarily irreducible divisors that satisfy $E_i^2 = -1$ and $E_iE_j = 0$ for $i \neq j$.

Let m_0 be the leading coefficient of $g^*K_{S_{\min}}$ written in the form (3.1). Since S is not rational, $K_{S_{\min}}$ is a nef $\mathbb{Q}$-divisor on $S_{\min}$ and so $g^*K_{S_{\min}}$ is a nef $\mathbb{Q}$-divisor on S'. Since K_S is ample, it follows that

$$m_0 \geq 0.$$

Let m_i be the leading coefficient of E_i written in the form (3.1), and note that $\sqrt{D'}$ is the leading coefficient of $K_{S'}$ written in the form (3.1). Therefore,

$$\sqrt{D'} = m_0 + \sum_{i=1}^{k} m_i.$$

If E_s is a (-1)-curve and is a component of E_t for some $t \neq s$, then one can write $E_t = aE_s + F$ for $a \geq 1$ an integer and F an effective divisor. It follows that $m_t \geq am_s \geq m_s$. Let

$$m := \min\{m_1, m_2, \dots, m_k\}.$$

Then there is an irreducible member E among $E_1, \dots, E_k$ whose leading coefficient is m. This member is a (-1)-curve, and

$$\sqrt{D'} = m_0 + \sum_{i=1}^{k} m_i \geq \sum_{i=1}^{k} m_i \geq km \geq (L-9)m. \qquad \square$$

5. First Reduction Steps for Cases with $|\mathrm{Sing}(S)| \geq 4$

Let S be a $\mathbb{Q}$-homology projective plane with cyclic quotient singularities such that $H_1(S^0, \mathbb{Z}) = 0$. By Lemma 3.6(3), the orders of singularities are pairwise relatively prime. Since $e_{\mathrm{orb}}(S) \geq 0$ (Theorems 3.2 and 3.3), one sees immediately that S can have at most four singular points (see [HK1, Kol2]).

Assume that $|\mathrm{Sing}(S)| = 4$. Then we enumerate all possible 4-tuples of orders of local fundamental groups as follows:

(1) $(2, 3, 5, q)$, $q \geq 7$, $\gcd(q, 30) = 1$;
(2) $(2, 3, 7, q)$, $11 \leq q \leq 41$, $\gcd(q, 42) = 1$;
(3) $(2, 3, 11, 13)$.

For (2) and (3), there are exactly 1092 different possible types for R, the sublattice of $H^2(S', \mathbb{Z})_{\mathrm{free}}$ generated by all exceptional curves of the minimal resolution $f : S' \to S$. There are two types ([3] and [2, 2]) of order 3; four types ([7], [4, 2], [3, 2, 2], and A_6) of order 7; and $\phi(q)/2 + 1$ types of order q. Hence the total number of types of R for the case $(2, 3, 7, q)$ is

$$2 \times 4 \times \left(\frac{\phi(q)}{2} + 1\right) = 4(\phi(q) + 2),$$

where ϕ is the Euler function. Here we identify $\frac{1}{q}(1, q_1)$ with $\frac{1}{q}(1, q_l)$. By Lemma 3.6(5), the number

$$D = |\det(R)| K_S^2$$

must be a nonzero square number. Among the 1092 cases, a computer calculation of the number D shows that only 24 cases satisfy this property. Table 1 describes these 24 cases.

The number D can be computed as follows. First note that

$$|\det(R)| = \text{the product of orders.}$$

To compute K_S^2, we use the equality

$$K_S^2 = K_{S'}^2 + \sum_p \mathcal{D}_p K_{S'} = K_{S'}^2 - \sum_p \mathcal{D}_p^2$$

from Section 3.1. By the Noether formula we have

$$K_{S'}^2 = 9 - L,$$

Table 1

No.	Type of R	Orders	K_S^2		$3e_{\mathrm{orb}}(S)$
1	$[2]+A_2+[7]+[13]$	$(2,3,7,13)$	$\frac{1536}{91}$	$>$	$\frac{29}{182}$
2	$[2]+A_2+[7]+[3,2,2,2,2,2,2,2,2]$	$(2,3,7,19)$	$\frac{6}{133}$	$<$	$\frac{23}{266}$
3	$[2]+A_2+[7]+[5,4]$	$(2,3,7,19)$	$\frac{1350}{133}$	$>$	$\frac{23}{266}$
4	$[2]+A_2+[7]+[3,4,2]$	$(2,3,7,19)$	$\frac{1014}{133}$	$>$	$\frac{23}{266}$
5	$[2]+A_2+[4,2]+[2,2,4,2,2,2]$	$(2,3,7,31)$	$\frac{150}{217}$	$>$	$\frac{11}{434}$
6	$[2]+A_2+[4,2]+[6,2,2,2,2,2]$	$(2,3,7,31)$	$\frac{486}{217}$	$>$	$\frac{11}{434}$
7	$[2]+[3]+[3,2,2]+[4,2,2,2,3]$	$(2,3,7,29)$	$\frac{968}{609}$	$>$	$\frac{13}{406}$
8	$[2]+A_2+[3,2,2]+[7,2,2,2]$	$(2,3,7,25)$	$\frac{24}{7}$	$>$	$\frac{17}{350}$
9	$[2]+A_2+[7]+[2,2,3,2,2,2,2,2,2]$	$(2,3,7,31)$	$\frac{54}{217}$	$>$	$\frac{11}{434}$
10	$[2]+[3]+[4,2]+[3,3,2,2,3]$	$(2,3,7,41)$	$\frac{2888}{861}$	$>$	$\frac{1}{574}$
11	$[2]+A_2+[3,2,2]+[7,2,2,2,2,2]$	$(2,3,7,37)$	$\frac{384}{259}$	$>$	$\frac{5}{518}$
12	$[2]+A_2+[4,2]+[11,2,2]$	$(2,3,7,31)$	$\frac{2166}{217}$	$>$	$\frac{11}{434}$
13	$[2]+[3]+A_6+[2,6,2,2]$	$(2,3,7,29)$	$\frac{56}{87}$	$>$	$\frac{13}{406}$
14	$[2]+[3]+[3,2,2]+[4,3]$	$(2,3,7,11)$	$\frac{1058}{231}$	$>$	$\frac{31}{154}$
15	$[2]+[3]+[3,2,2]+[3,2,2,2,2]$	$(2,3,7,11)$	$\frac{50}{231}$	$>$	$\frac{31}{154}$
16	$[2]+[3]+[3,2,2]+[4,2,2,3]$	$(2,3,7,23)$	$\frac{1250}{483}$	$>$	$\frac{19}{322}$
17	$[2]+[3]+[3,2,2]+[6,5]$	$(2,3,7,29)$	$\frac{5000}{609}$	$>$	$\frac{13}{406}$
18	$[2]+A_2+[3,2,2]+[3,5,2]$	$(2,3,7,25)$	$\frac{24}{7}$	$>$	$\frac{17}{350}$
19	$[2]+A_2+[3,2,2]+[13,2]$	$(2,3,7,25)$	$\frac{1944}{175}$	$>$	$\frac{17}{350}$
20	$[2]+A_2+[4,2]+[4,2,2,2]$	$(2,3,7,13)$	$\frac{216}{91}$	$>$	$\frac{29}{182}$
21	$[2]+A_2+[4,2]+[5,2,2]$	$(2,3,7,13)$	$\frac{384}{91}$	$>$	$\frac{29}{182}$
22	$[2]+A_2+[4,2]+[4,2,2,2,2,2]$	$(2,3,7,19)$	$\frac{54}{133}$	$>$	$\frac{23}{266}$
23	$[2]+[3]+[3,2,2,2,2]+[4,2,2,2]$	$(2,3,11,13)$	$\frac{8}{429}$	$>$	$\frac{1}{286}$
24	$[2]+[3]+[3,2,2,2,2]+[5,2,2]$	$(2,3,11,13)$	$\frac{800}{429}$	$>$	$\frac{1}{286}$

where $L := \mathrm{rank}(R)$ is the number of the exceptional curves of f. Finally, the self-intersection number $\mathcal{D}_p^2$ is given in Lemma 3.1.

REMARK 5.1. None of the 24 cases listed in Table 1 can be ruled out by any further lattice-theoretic argument. In fact, in each case the lattice R can be embedded into a unimodular lattice $I_{1,L}$(odd) or $II_{1,L}$(even) of signature $(1, L)$. This can be checked by the local–global principle and the computation of ε-invariants (see e.g. [HK1, Sec. 6]).

Table 2

	[2]	[2,2]		[7]	[3,2,2,2,2,2,2,2,2]								
j	1	1	2	1	1	2	3	4	5	6	7	8	9
$1-\frac{v_j+u_j}{q}$	0	0	0	$\frac{5}{7}$	$\frac{9}{19}$	$\frac{8}{19}$	$\frac{7}{19}$	$\frac{6}{19}$	$\frac{5}{19}$	$\frac{4}{19}$	$\frac{3}{19}$	$\frac{2}{19}$	$\frac{1}{19}$

LEMMA 5.2. *In all cases (except the second) of Table 1, $-K_S$ is ample. In the second case, S is rational.*

Proof. The 23 cases do not satisfy the inequality $K_S^2 \le 3e_{\mathrm{orb}}(S)$ in Theorem 3.2. From this, the first assertion follows.

Consider the second case, $A_1 + A_2 + [7] + [3,2,2,2,2,2,2,2,2]$. In this case we have

$$K_S^2 = \frac{6}{133}, \quad D = |\det(R)|K_S^2 = 36, \quad L = 13.$$

Suppose that S is not rational. By Lemma 4.5, S' contains a (-1)-curve E with $0 < m \le \sqrt{D}/(L-9) = 6/4$; that is, $m = 1$. By Proposition 4.2(1), we obtain

$$\sum_p \sum_j \left(1 - \frac{v_{j,p} + u_{j,p}}{q_p}\right)(EA_{j,p}) = -EK_{S'} + \frac{m}{\sqrt{D}}K_S^2 = 1 + \frac{1}{6}\cdot\frac{6}{133} = \frac{134}{133}.$$

Looking at Table 2, we see that there are nonnegative integers x, y such that

$$\frac{5x}{7} + \frac{y}{19} = \frac{134}{133}.$$

But it is easy to check that this equation has no solution. □

Next we consider the cases $(2,3,5,q)$ for $q \ge 7$ and $\gcd(q,30) = 1$.

LEMMA 5.3. *In the cases $(2,3,5,q)$, where $q \ge 7$ and $\gcd(q,30) = 1$, the order-3 singularity must be of type $\frac{1}{3}(1,1)$.*

Proof. Suppose this order-3 singularity is of type A_2. We divide the proof into three cases according to the type of the third singularity.

Case 1: $A_1 + A_2 + A_4 + \frac{1}{q}(1,q_1)$. In this case,

$$K_S^2 = \sum_{j=1}^{l} n_j - 3l + \frac{q_1 + q_l + 2}{q}$$

and

$$D = 30\left\{q_1 + q_l + \left(\sum_{j=1}^{l} n_j - 3l\right)q + 2\right\}.$$

Since D is a square number, 3 divides $q_1+q_l+(\mathrm{tr}-3l)q+2 \equiv q_1+q_l+(\mathrm{tr})q+2$. Then, by Proposition 2.2, q is a multiple of 3—a contradiction.

Case 2: $A_1 + A_2 + \frac{1}{5}(1,2) + \frac{1}{q}(1,q_1)$. In this case,

$$K_S^2 = \sum_{j=1}^{l} n_j - 3l + \frac{12}{5} + \frac{q_1+q_l+2}{q}$$

and

$$D = 6\left[5(q_1+q_l) + \left\{5\left(\sum_{j=1}^{l} n_j - 3l\right) + 12\right\}q + 10\right].$$

Thus 3 divides $5(q_1+q_l) + \{5(\mathrm{tr}-3l)+12\}q + 10 \equiv -(q_1+q_l) - (\mathrm{tr})q + 1$. Then, by Proposition 2.2, q is a multiple of 3—a contradiction.

Case 3: $A_1 + A_2 + \frac{1}{5}(1,1) + \frac{1}{q}(1,q_1)$. In this case,

$$K_S^2 = \sum_{j=1}^{l} n_j - 3l + \frac{24}{5} + \frac{q_1+q_l+2}{q}$$

and

$$D = 6\left[5(q_1+q_l) + \left\{5\left(\sum_{j=1}^{l} n_j - 3l\right) + 24\right\}q + 10\right].$$

Thus 3 divides $5(q_1+q_l) + \{5(\mathrm{tr}-3l)+24\}q + 10$. Then, by Proposition 2.2, q is a multiple of 3—a contradiction. □

In the following two lemmas, we do not assume that $H_1(S^0,\mathbb{Z}) = 0$. As a result, the orders may not be pairwise relatively prime.

LEMMA 5.4. *Let S be a $\mathbb{Q}$-homology projective plane with exactly four cyclic singular points p_1, p_2, p_3, p_4 of orders $(2,3,5,q)$, $q \geq 7$. (We do not assume that* $\gcd(q,30) = 1$.) *Regard $\mathcal{F} := f^{-1}(\mathrm{Sing}(S))$ as a reduced integral divisor on S', and assume that S' contains a (-1)-curve E. Then*

$$E.\mathcal{F} \geq 2.$$

Equality holds iff $E.f^{-1}(p_i) = 0$ for $i = 1,2,3$ and $E.f^{-1}(p_4) = 2$.

Proof. Assume that $E.\mathcal{F} = 1$. Blowing up the intersection point and then contracting the proper transform of E as well as the proper transforms of all irreducible components of $\mathcal{F}$, we obtain a $\mathbb{Q}$-homology projective plane $\bar{S}$ with five quotient singular points. Then, by [HK1], the minimal resolution of $\bar{S}$ is an Enriques surface and hence has no (-1)-curve, which is a contradiction. This proves that $E.\mathcal{F} \geq 2$.

Now assume that $E.\mathcal{F} = 2$. We will prove first that E does not meet any end component of $f^{-1}(p_i)$ for $1 \leq i \leq 3$. So suppose that E does meet such an end component. To derive a contradiction, we divide the proof into three cases.

Case 1: $EF = 1$. Then $EF' = 1$ for some other component F' of $f^{-1}(p_j)$, where $j = 1, 2, 3, 4$ and may be equal to i. Assume that $E \cap F \cap F' = \emptyset$. Blowing up the intersection point of E and F' sufficiently many times before contracting the proper transform of E with a string of (-2)-curves and the proper transforms of all irreducible components of $\mathcal{F}$, we obtain a $\mathbb{Q}$-homology projective plane $\bar{S}$ with four quotient singular points such that $e_{\text{orb}} < 0$ (see Lemma 2.4(5)); this violates the orbifold Bogomolov–Miyaoka–Yau inequality. Next assume that $E \cap F \cap F' \neq \emptyset$. Blowing up the intersection point once and then contracting the proper transform of E and the proper transforms of all irreducible components of $\mathcal{F}$, we obtain a $\mathbb{Q}$-homology projective plane $\bar{S}$ with six quotient singular points—in contradiction to [HK1].

Case 2: E intersects F at two distinct points. In this case we get a similar contradiction. Blowing up one of the two intersection points of E and F sufficiently many times before contracting the proper transform of E with the adjacent string of (-2)-curves and the proper transforms of all irreducible components of $\mathcal{F}$, we obtain a $\mathbb{Q}$-homology projective plane $\bar{S}$ with four quotient singular points such that $e_{\text{orb}} < 0$. Here we also use Lemma 2.4(5).

Case 3: E intersects F at one point with multiplicity 2. Blowing up the intersection point twice and then contracting the proper transform of E, a (-2)-curve, and the proper transforms of all irreducible components of $\mathcal{F}$, we obtain a $\mathbb{Q}$-homology projective plane $\bar{S}$ with six quotient singular points; this contradicts [HK1].

We have proved that E does not meet any end component of $f^{-1}(p_i)$ for $1 \leq i \leq 3$. This implies that $E.f^{-1}(p_1) = E.f^{-1}(p_2) = 0$ and $E.f^{-1}(p_3) = 0$ if $f^{-1}(p_3)$ has at most two components. We will show that $E.f^{-1}(p_3) = 0$ even if $f^{-1}(p_3)$ has more than two components (i.e., even if p_3 is of type $A_4 = [2, 2, 2, 2]$). Suppose that p_3 is of type A_4 and let F_1, F_2, F_3, F_4 be its four components whose dual graph is $F_1 - F_2 - F_3 - F_4$. We split the proof into four cases.

Case A: E meets F_2 at two distinct points. Blowing up one of the two intersection points of E and F_2 once and then contracting the proper transform of E and the proper transforms of all irreducible components of $\mathcal{F}$, we obtain a $\mathbb{Q}$-homology projective plane $\bar{S}$ with one noncyclic quotient singularity of type

$$\langle 3; 2, 1; 2, 1; 3, 2 \rangle := \begin{array}{ccccccc} \overset{-2}{\circ} & - & \overset{-3}{\circ} & - & \overset{-2}{\circ} & - & \overset{-2}{\circ} \\ & & | & & & & \\ & & \underset{-2}{\circ} & & & & \end{array};$$

the order of this singularity is 48, and it has three cyclic singular points of order $2, 3, q$ (see [B] or [HK1, Table 1] for the notation of dual graphs of noncyclic singularities). For this surface,

$$e_{\text{orb}} = -1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{q} + \frac{1}{48} < 0,$$

which violates the orbifold Bogomolov–Miyaoka–Yau inequality.

Case B: $EF_2 = EF_3 = 1$ *and* $E \cap F_2 \cap F_3 = \emptyset$. Blowing up the intersection point of E and F_3 once and then contracting the proper transform of E and the proper transforms of all irreducible components of $\mathcal{F}$, we obtain a $\mathbb{Q}$-homology projective plane $\bar{S}$ with one noncyclic quotient singularity of type

$$\langle 2; 2, 1; 2, 1; 5, 2\rangle := \begin{array}{ccccccc} \overset{-2}{\circ} & - & \overset{-2}{\circ} & - & \overset{-3}{\circ} & - & \overset{-2}{\circ} \\ & & | & & & & \\ & & \underset{-2}{\circ} & & & & \end{array};$$

the order of this singularity is 60, and it has three cyclic singular points of order $2, 3, q$. For this surface,

$$e_{\text{orb}} = -1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{q} + \frac{1}{60} < 0,$$

which also violates the orbifold Bogomolov–Miyaoka–Yau inequality.

Case C: $EF_2 = EF_3 = 1$ *and* $E \cap F_2 \cap F_3 \neq \emptyset$. Blowing up the intersection point once before contracting the proper transform of E and the proper transforms of all irreducible components of $\mathcal{F}$, we obtain a $\mathbb{Q}$-homology projective plane $\bar{S}$ with six quotient singular points—in contradiction to [HK1].

Case D: $EF_2 = 1$ *and* $EF = 1$ *for some component* F *of* $f^{-1}(p_i)$ *for some* $i \neq 3$. Blowing up the intersection point of E and F three times and then contracting all curves except the (-1)-curve coming from the last blowup, we obtain a $\mathbb{Q}$-homology projective plane $\bar{S}$ with one noncyclic quotient singularity of type

$$\langle 2; 2, 1; 3, 2; 4, 3\rangle := \begin{array}{ccccccc} \overset{-2}{\circ} & - & \overset{-2}{\circ} & - & \overset{-2}{\circ} & - & \overset{-2}{\circ} \\ & & | & & & & \\ & & \underset{-2}{\circ} & - & \underset{-2}{\circ} & - & \underset{-2}{\circ} \end{array};$$

the order of this singularity is 48, and it has three cyclic singular points of respective order ≥ 2, ≥ 3, and $\geq q$. For this surface,

$$e_{\text{orb}} \leq -1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{q} + \frac{1}{48} < 0,$$

which violates the orbifold Bogomolov–Miyaoka–Yau inequality.

This completes the proof of $E.f^{-1}(p_3) = 0$, from which it follows that $E.f^{-1}(p_4) = 2$. $\square$

In our next lemma it is not assumed that $H_1(S^0, \mathbb{Z}) = 0$.

Lemma 5.5. *Let* S *be a* $\mathbb{Q}$*-homology projective plane with exactly four cyclic singular points* p_1, p_2, p_3, p_4 *of orders* $(2, 3, 5, q)$. (*We do not assume that* $\gcd(q, 30) = 1$.) *Assume that* K_S *is ample and that the order-3 singularity is of type* $\frac{1}{3}(1, 1)$. *Then*:

(1) $L \geq 12$ *except possibly four cases* (*1–4 in Table 3*) *in which* S *is rational and* $L = 11$; *and*

(2) $q \geq 20$ *except possibly one case* (*1 in Table 3*).

Proof. (1) We must consider the following types:

- $A_1 + \frac{1}{3}(1,1) + A_4 + \frac{1}{q}(1,q_1)$,
- $A_1 + \frac{1}{3}(1,1) + \frac{1}{5}(1,2) + \frac{1}{q}(1,q_1)$,
- $A_1 + \frac{1}{3}(1,1) + \frac{1}{5}(1,1) + \frac{1}{q}(1,q_1)$.

Let $[n_1,\dots,n_l]$ be the Hirzebruch–Jung continued fraction corresponding to the singularity p_4. Since K_S is ample, Theorem 3.2 implies that

$$0 < K_{S'}^2 - \mathcal{D}_{p_2}^2 - \mathcal{D}_{p_3}^2 - \mathcal{D}_{p_4}^2 = K_S^2 \le 3e_{\mathrm{orb}}(S) = \frac{1}{10} + \frac{3}{q}.$$

Since $K_{S'}^2 = 9 - L$ and $\mathcal{D}_{p_2}^2 = -\frac{1}{3}$, Lemma 3.1 implies that

$$\begin{aligned} L - 7 + 2l - \frac{1}{3} + \mathcal{D}_{p_3}^2 - \frac{q_1+q_l+2}{q} \\ < \sum n_j \le L - 7 + 2l - \frac{1}{3} + \mathcal{D}_{p_3}^2 - \frac{q_1+q_l-1}{q} + \frac{1}{10}. \end{aligned}$$

In particular, if L is bounded then so is the number of possible cases for $[n_1,\dots,n_l]$.

Assume that $L \le 11$. If p_3 is of type A_4 then $L = l+6$, $\mathcal{D}_{p_3}^2 = 0$, and the preceding inequality shows that $\sum n_j = 3l-2$ or $3l-3$. Therefore, up to permutation of $n_1,\dots,n_l$, we have

$$\begin{aligned} [n_1,\dots,n_l] = {} & [5,2,2,2,2],[4,3,2,2,2],[3,3,3,2,2]; \\ & [4,2,2,2,2],[3,3,2,2,2]; \\ & [4,2,2,2],[3,3,2,2]; \\ & [3,2,2,2]; \\ & [3,2,2]; \\ & [2,2,2]; \\ & [2,2]. \end{aligned}$$

Hence there are 42 possible cases for $[n_1,\dots,n_l]$. Here we identify $[n_1,\dots,n_l]$ with its reverse, $[n_l,\dots,n_1]$.

If p_3 is of type $\frac{1}{5}(1,2)$ then $L = l+4$, $\mathcal{D}_{p_3}^2 = -\frac{2}{5}$, and $\sum n_j = 3l-4$ or $3l-5$; hence, up to permutation of $n_1,\dots,n_l$,

$$\begin{aligned} [n_1,\dots,n_l] = {} & [5,2,2,2,2,2,2],[4,3,2,2,2,2,2],[3,3,3,2,2,2,2]; \\ & [4,2,2,2,2,2,2],[3,3,2,2,2,2,2]; \\ & [4,2,2,2,2,2],[3,3,2,2,2,2]; \\ & [3,2,2,2,2,2]; \\ & [3,2,2,2,2]; \\ & [2,2,2,2,2]; \\ & [2,2,2,2]. \end{aligned}$$

There are consequently 80 possible cases for $[n_1,\dots,n_l]$ if $l \le 7$.

If p_3 is of type $\frac{1}{5}(1,1)$ then $L = l+3$, $\mathcal{D}_{p_3}^2 = -\frac{9}{5}$, and $\sum n_j = 3l-7$ or $3l-8$; hence, up to permutation of $n_1,\dots,n_l$, we have

Table 3

No.	Type of R	q	K_S^2		$3e_{\mathrm{orb}}$
1	$A_1+\frac{1}{3}(1,1)+\frac{1}{5}(1,1)+[2,2,2,2,2,2,2,2]$	9	$\frac{2}{15}$	$<$	$\frac{13}{30}$
2	$A_1+\frac{1}{3}(1,1)+\frac{1}{5}(1,2)+[4,2,2,2,2,2,2]$	22	$\frac{1}{165}$	$<$	$\frac{13}{55}$
3	$A_1+\frac{1}{3}(1,1)+\frac{1}{5}(1,2)+[3,3,2,2,2,2,2]$	33	$\frac{2}{55}$	$<$	$\frac{21}{110}$
4	$A_1+\frac{1}{3}(1,1)+\frac{1}{5}(1,2)+[3,2,2,3,2,2,2]$	43	$\frac{8}{645}$	$<$	$\frac{73}{430}$
5	$A_1+\frac{1}{3}(1,1)+\frac{1}{5}(1,2)+[2,2,2,4,2,2,2]$	40	$\frac{1}{3}$	$>$	$\frac{7}{40}$
6	$A_1+\frac{1}{3}(1,1)+\frac{1}{5}(1,2)+[3,3,3,2,2,2,2]$	73	$\frac{1058}{1095}$	$>$	$\frac{103}{730}$
7	$A_1+\frac{1}{3}(1,1)+\frac{1}{5}(1,2)+[2,3,4,2,2,2,2]$	70	$\frac{25}{21}$	$>$	$\frac{1}{7}$
8	$A_1+\frac{1}{3}(1,1)+\frac{1}{5}(1,2)+[2,3,3,3,2,2,2]$	97	$\frac{1682}{1455}$	$>$	$\frac{127}{970}$
9	$A_1+\frac{1}{3}(1,1)+\frac{1}{5}(1,2)+[2,2,4,3,2,2,2]$	78	$\frac{81}{65}$	$>$	$\frac{9}{65}$
10	$A_1+\frac{1}{3}(1,1)+\frac{1}{5}(1,2)+[3,3,2,2,3,2,2]$	87	$\frac{128}{145}$	$>$	$\frac{39}{290}$
11	$A_1+\frac{1}{3}(1,1)+\frac{1}{5}(1,2)+[2,3,3,2,2,3,2]$	103	$\frac{1568}{1545}$	$>$	$\frac{133}{1030}$

$$[n_1,\dots,n_l]=[3,2,2,2,2,2,2,2],[2,2,2,2,2,2,2,2];\\ [2,2,2,2,2,2,2].$$

Thus there are six possible cases for $[n_1,\dots,n_l]$ if $l\le 8$.

Among these $42+80+6=128$ cases, a direct calculation of $D=|\det(R)|K_S^2$ shows that only 11 cases satisfy the condition that D be a positive square number (see Lemma 3.6(5)). Table 3 describes these 11 cases, among which only the first four satisfy the orbifold Bogomolov–Miyaoka–Yau inequality $K_S^2\le 3e_{\mathrm{orb}}$.

One can check that none of these four cases can be ruled out by any further lattice-theoretic argument; that is, in each case the lattice R can be embedded into an odd unimodular lattice of signature $(1,L)$. This can be checked by the local–global principle and the computation of ε-invariants (see e.g. [HK1, Sec. 6]).

To prove the rationality in each of the first four cases of Table 3, we will use the formulas from Proposition 4.2. First note that $L=11$ in each of these four cases. We assume throughout the proof that S is not rational.

Case 1. Note that $D=36$. Since $\mathrm{disc}(\bar{R})$ is a cyclic group (Lemma 3.7), we see that $\det(\bar{R})=\det(R)/3^2$ and so $D'=D/3^2=4$. By Lemma 4.5, S' contains a (-1)-curve E with $0<m\le\sqrt{D'}/(L-9)=1$ (i.e., $m=1$). By Proposition 4.2(1), we obtain

$$\sum_p\sum_j\left(1-\frac{v_{j,p}+u_{j,p}}{q_p}\right)(EA_{j,p})=1+\frac{m}{\sqrt{D'}}K_S^2=\frac{16}{15}.$$

Looking at Table 4, we see that there are nonnegative integers x, y such that

Table 4

	[2]	[3]	[5]	[2,2,2,2,2,2,2,2]							
j	1	1	1	1	2	3	4	5	6	7	8
$1-\frac{v_j+u_j}{q}$	0	$\frac{1}{3}$	$\frac{3}{5}$	0	0	0	0	0	0	0	0

Table 5

	[2]	[3]	[2,3]		[3,3,2,2,2,2,2]						
j	1	1	1	2	1	2	3	4	5	6	7
$1-\frac{v_j+u_j}{q}$	0	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{19}{33}$	$\frac{24}{33}$	$\frac{20}{33}$	$\frac{16}{33}$	$\frac{12}{33}$	$\frac{8}{33}$	$\frac{4}{33}$
$\frac{v_j u_j}{q}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{3}{5}$	$\frac{2}{5}$	$\frac{13}{33}$	$\frac{18}{33}$	$\frac{40}{33}$	$\frac{52}{33}$	$\frac{54}{33}$	$\frac{46}{33}$	$\frac{28}{33}$

$$\frac{x}{3}+\frac{3y}{5}=\frac{16}{15}.$$

It is easy to check that the equation has no solution.

Case 2. Note that $D=4$. Since $\operatorname{disc}(\bar{R})$ is a cyclic group (Lemma 3.7), we see that $D'=D/2^2=1$. By Lemma 4.5, S' contains a (-1)-curve E with $0<m\leq\sqrt{D'}/(L-9)=1/2$, a contradiction.

Case 3. Note that $D=36$. Since $\operatorname{disc}(\bar{R})$ is a cyclic group (Lemma 3.7), we see that $D'=D/3^2=4$. By Lemma 4.5, S' contains a (-1)-curve E with $0<m\leq\sqrt{D'}/(L-9)=1$ (i.e., $m=1$). By Proposition 4.2(1), we obtain

$$\sum_p\sum_j\left(1-\frac{v_{j,p}+u_{j,p}}{q_p}\right)(EA_{j,p})=1+\frac{m}{\sqrt{D'}}K_S^2=\frac{56}{55}.$$

Looking at Table 5, we see that there are nonnegative integers x,y,z such that

$$\frac{x}{3}+\frac{y}{5}+\frac{z}{33}=\frac{56}{55}.$$

This equation has three solutions $(x,y,z)=(0,1,27),(1,1,16),(2,1,5)$. Again by Table 5, we can rule out the third solution. By Proposition 4.2(2), we obtain

$$\sum_p\sum_j\frac{v_j u_j}{q}(EA_j)^2\leq 1+\frac{m^2}{D'}K_S^2=\frac{111}{110},$$

which rules out the first two solutions.

Case 4. Note that $D=4^2$. Since the orders are pairwise relatively prime, $D'=D$. By Lemma 4.5, S' contains a (-1)-curve E with $0<m\leq\sqrt{D}/(L-9)=2$; that is, $m=1$ or 2. By Proposition 4.2, we obtain

Table 6

	[2]	[3]	[2,3]		[3,2,2,3,2,2,2]						
j	1	1	1	2	1	2	3	4	5	6	7
$1-\frac{v_j+u_j}{q}$	0	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{23}{43}$	$\frac{26}{43}$	$\frac{29}{43}$	$\frac{32}{43}$	$\frac{24}{43}$	$\frac{16}{43}$	$\frac{8}{43}$

Table 7

q	Singularity types with $l \geq 6$
7	A_6
8	A_7
9	A_8
10	A_9
11	A_{10}
12	A_{11}
13	$[3,2,2,2,2,2]$, A_{12}
14	A_{13}
15	$[3,2,2,2,2,2,2]$, A_{14}
16	A_{15}
17	$[2,3,2,2,2,2]$, $[3,2,2,2,2,2,2,2]$, A_{16}
18	A_{17}
19	$[2,2,3,2,2,2]$, $[4,2,2,2,2,2]$, $[3,2,2,2,2,2,2,2,2]$, A_{18}

$$\sum_p \sum_j \left(1-\frac{v_{j,p}+u_{j,p}}{q_p}\right)(EA_{j,p}) = 1+\frac{m}{\sqrt{D}}K_S^2 = \frac{647}{645} \text{ or } \frac{649}{645}.$$

Looking at Table 6, we see that there are nonnegative integers x, y, z such that

$$\frac{x}{3}+\frac{y}{5}+\frac{z}{43} = \frac{647}{645} \text{ or } \frac{649}{645}.$$

But it is easy to check that both equations have no solution.

To prove part (2) of the lemma, first suppose that $q \leq 19$. By (1) we may assume that $L \geq 11$, and $L = 11$ if and only if one of the first four cases in Table 3 occurs. If $L = 11$, then only the first case in Table 3 satisfies the assumption $q \leq 19$.

Now we assume that $L \geq 12$. In this case $l \geq 6$, where l is the length of the singularity type of p_4. Table 7 lists all the possibilities.

If p_4 is of type $[2,3,2,2,2,2]$, then the third singularity p_3 is of type A_4 and

$$K^2 = K_{S'}^2 - \sum_p D_p^2 = (9-12)+\frac{1}{3}+\frac{10}{17} < 0,$$

a contradiction. The cases $[2,2,3,2,2,2]$ and $[4,2,2,2,2,2]$ can be similarly removed.

If p_4 is of type A_{q-1} then, since $-D_{p_3}^2 \le \frac{9}{5}$ and $L \ge 12$,

$$K^2 = K_{S'}^2 - \sum_p D_p^2 \le (9-L) + \frac{1}{3} + \frac{9}{5} < 0,$$

a contradiction.

If p_4 is of type $[3,2,2,\dots,2]$, then

$$D_{p_4}^2 = 2l - \mathrm{tr} + 2 - \frac{q_1 + q_l + 2}{q} = 1 - \frac{l + 2l - 1 + 2}{2l+1} = -\frac{l}{2l+1}$$

and so

$$\begin{aligned} K^2 = K_{S'}^2 - \sum_p D_p^2 &\le (9-L) + \frac{1}{3} + \frac{9}{5} + \frac{l}{2l+1} \\ &< (9-L) + \frac{1}{3} + \frac{9}{5} + \frac{1}{2} < 0, \end{aligned}$$

a contradiction. □

LEMMA 5.6. *Let S be a $\mathbb{Q}$-homology projective plane with exactly four cyclic singular points p_1, p_2, p_3, p_4 of orders $(2,3,7,q)$, $11 \le q \le 41$, or $(2,3,11,13)$. Regard $\mathcal{F} := f^{-1}(\mathrm{Sing}(S))$ as a reduced integral divisor on S' and assume that S' contains a (-1)-curve E. Then*

$$E.\mathcal{F} \ge 2.$$

Moreover, if $E.\mathcal{F} = 2$ then E does not meet an end component of $f^{-1}(p_i)$ for any $i = 1,2,3,4$.

Proof. The proof of the first assertion is the same as that of Lemma 5.4. To prove the second assertion, assume that $E.\mathcal{F} = 2$. Suppose that E meets an end component F of $f^{-1}(p_i)$ for some $1 \le i \le 4$.

If $EF = 1$, then $EF' = 1$ for some other component F' of $f^{-1}(p_j)$, where j may or may not be i. Assume that $E \cap F \cap F' = \emptyset$. Blowing up the intersection point of E and F' sufficiently many times and then contracting the proper transform of E with a string of (-2)-curves and the proper transforms of all irreducible components of $\mathcal{F}$, we obtain a $\mathbb{Q}$-homology projective plane $\bar{S}$ with four quotient singular points such that $e_{\mathrm{orb}} < 0$ (see Lemma 2.4(6)); this violates the orbifold Bogomolov–Miyaoka–Yau inequality. Assume that $E \cap F \cap F' \ne \emptyset$. Blowing up the intersection point once before contracting the proper transform of E and the proper transforms of all irreducible components of $\mathcal{F}$, we obtain a $\mathbb{Q}$-homology projective plane $\bar{S}$ with six quotient singular points—in contradiction to [HK1].

If E intersects F at two distinct points then we derive a similar contradiction. Blowing up one of the two intersection points of E and F sufficiently many times and then contracting the proper transform of E with the adjacent string of (-2)-curves and the proper transforms of all irreducible components of $\mathcal{F}$, we obtain a $\mathbb{Q}$-homology projective plane $\bar{S}$ with four quotient singular points such that $e_{\mathrm{orb}} < 0$.

If E intersects F at one point with multiplicity 2, then blowing up the intersection point twice before contracting the proper transform of E with a (-2)-curve and the proper transforms of all irreducible components of $\mathcal{F}$ yields a $\mathbb{Q}$-homology projective plane $\bar{S}$ with six quotient singular points, contradicting [HK1].

In all cases, we get a contradiction. This proves the second assertion. □

6. Proof of Theorem 1.2

Let S be a $\mathbb{Q}$-homology projective plane with cyclic quotient singularities such that

- $H_1(S^0, \mathbb{Z}) = 0$ and
- S is not rational.

Assume that $|\mathrm{Sing}(S)| = 4$. In Section 5 we enumerated all possible 4-tuples of orders of local fundamental groups:

(1) $(2,3,5,q)$, $q \geq 7$, $\gcd(q,30) = 1$;
(2) $(2,3,7,q)$, $11 \leq q \leq 41$, $\gcd(q,42) = 1$;
(3) $(2,3,11,13)$.

For (2) and (3), we listed in Table 1 the 24 different possible types for R, the sublattice of $H^2(S', \mathbb{Z})_{\mathrm{free}}$ generated by all exceptional curves of the minimal resolution $f: S' \to S$. Lemma 5.2 rules out all these 24 cases, since we assume that S is not rational.

For (1), the order-3 singularity is of type $\frac{1}{3}(1,1)$ (Lemma 5.3); it therefore remains to consider the following cases:

- $A_1 + \frac{1}{3}(1,1) + A_4 + \frac{1}{q}(1,q_1)$, $q \geq 7$, $\gcd(q,30) = 1$;
- $A_1 + \frac{1}{3}(1,1) + \frac{1}{5}(1,2) + \frac{1}{q}(1,q_1)$, $q \geq 7$, $\gcd(q,30) = 1$;
- $A_1 + \frac{1}{3}(1,1) + \frac{1}{5}(1,1) + \frac{1}{q}(1,q_1)$, $q \geq 7$, $\gcd(q,30) = 1$.

Since S is not rational, K_S is ample by Lemma 3.6(4). By Lemma 5.5, we may also assume that $q \geq 20$ and $L \geq 12$.

We will show that none of the cases just listed occurs. In the proof we do not assume that $\gcd(q,30) = 1$ (and so do not assume that $H_1(S^0, \mathbb{Z}) = 0$). That is, we consider the cases

- $A_1 + \frac{1}{3}(1,1) + A_4 + \frac{1}{q}(1,q_1)$, $q \geq 20$, $L \geq 12$;
- $A_1 + \frac{1}{3}(1,1) + \frac{1}{5}(1,2) + \frac{1}{q}(1,q_1)$, $q \geq 20$, $L \geq 12$;
- $A_1 + \frac{1}{3}(1,1) + \frac{1}{5}(1,1) + \frac{1}{q}(1,q_1)$, $q \geq 20$, $L \geq 12$.

As before, we assume that S is not rational.

Note first that, since $L \geq 12$, it follows from Proposition 4.3 that K_S is ample. We will show that none of the listed cases occurs. We refrain from assuming $\gcd(q,30) = 1$ because part of the proof uses induction on $L = \mathrm{rank}(R)$. After blowing down a suitable (-1)-curve E on S',

$$S' \to S'_1,$$

we contract Hirzebruch–Jung chains of rational curves,

$$S_1' \to S_1,$$

to get a new $\mathbb{Q}$-homology projective plane S_1 with $L_{S_1} = L - 1$; here the plane has cyclic quotient singularities whose orders may not be pairwise relatively prime.

By Lemma 4.5, there is a (-1)-curve E on S' of the form (3.1) with

$$0 < \frac{m}{\sqrt{D'}} \le \frac{1}{L-9} \le \frac{1}{3}.$$

We will show that the existence of such a curve E leads to a contradiction.

STEP 1. *We have the following inequalities*:

(1) $K_S^2 \le \dfrac{1}{4}$;

(2) $\dfrac{m}{\sqrt{D'}} K_S^2 \le \dfrac{1}{12}$;

(3) $\dfrac{m^2}{D'} K_S^2 \le \dfrac{1}{36}$.

Proof. Since $q \ge 20$, we have

$$3e_{\mathrm{orb}}(S) = \frac{1}{10} + \frac{3}{q} \le \frac{1}{10} + \frac{3}{20} = \frac{1}{4}.$$

Since K_S is ample, (1) follows from the orbifold Bogomolov–Miyaoka–Yau inequality. Both (2) and (3) follow from (1) and the inequality $m/\sqrt{D'} \le 1/3$. $\square$

Let p_1, p_2, p_3, p_4 be the four singular points. Assume that the singularity p_4 is of type $[n_1, \ldots, n_l]$. Since $L \ge 12$, we see that $l \ge 6$.

STEP 2. $E.f^{-1}(p_4) = 2$ *and* $E.f^{-1}(p_i) = 0$ *for* $i = 1, 2, 3$.

Proof. By Proposition 4.2(1),

$$\sum_p \sum_{j=1}^{l_p} \left(1 - \frac{v_{j,p} + u_{j,p}}{q_p}\right)(EA_{j,p}) = 1 + \frac{m}{\sqrt{D'}} K_S^2.$$

By Lemma 2.4 we see that $1 - \frac{v_{j,p}+u_{j,p}}{q_p} \ge 0$ for all j, p and so, looking at only the terms with $p = p_4$, we obtain

$$\begin{aligned} E.f^{-1}(p_4) - \sum_{j=1}^{l} \left(\frac{v_j + u_j}{q}\right)(EA_j) &= \sum_{j=1}^{l} \left(1 - \frac{v_j + u_j}{q}\right)(EA_j) \\ &\le 1 + \frac{m}{\sqrt{D'}} K_S^2, \end{aligned}$$

where $A_j := A_{j,p_4}$, $v_j := v_{j,p_4}$, and $u_j := u_{j,p_4}$. By Proposition 4.2(2),

$$\sum_{j=1}^{l} \frac{v_j u_j}{q} (EA_j)^2 \le 1 + \frac{m^2}{D'} K_S^2.$$

Adding these two inequalities side by side yields

$$E.f^{-1}(p_4) - \sum_{j=1}^{l}\left(\frac{v_j+u_j}{q}\right)(EA_j) + \sum_{j=1}^{l}\frac{v_ju_j}{q}(EA_j)^2 \\ \le 2 + \frac{m}{\sqrt{D'}}K_S^2 + \frac{m^2}{D'}K_S^2.$$

By Lemma 2.5,

$$\sum_{j=1}^{l}\left(\frac{v_j+u_j}{q}\right)(EA_j) \le \sum_{j=1}^{l}\frac{v_ju_j}{q}(EA_j)^2 + \frac{2}{q}.$$

Thus

$$E.f^{-1}(p_4) \le 2 + \frac{m}{\sqrt{D'}}K_S^2 + \frac{m^2}{D'}K_S^2 + \frac{2}{q} < 3,$$

which proves that $E.f^{-1}(p_4) \le 2$.

Now assume that $E.f^{-1}(p_4) = 2$. By parts (1) and (2) of Proposition 4.2,

$$\sum_{p\neq p_4}\sum_{j=1}^{l_p}\left(1-\frac{v_{j,p}+u_{j,p}}{q_p}\right)(EA_{j,p}) \\ = 1 + \frac{m}{\sqrt{D'}}K_S^2 - E.f^{-1}(p_4) + \sum_{j=1}^{l}\left(\frac{v_j+u_j}{q}\right)(EA_j),$$

$$\sum_{p\neq p_4}\sum_{j=1}^{l_p}\frac{v_{j,p}u_{j,p}}{q_p}(EA_{j,p})^2 \le 1 + \frac{m^2}{D'}K_S^2 - \sum_{j=1}^{l}\frac{v_ju_j}{q}(EA_j)^2.$$

Adding these two side by side and then using Lemma 2.5, we have

$$\begin{aligned} &\sum_{p\neq p_4}\sum_{j=1}^{l_p}\left(\left(1-\frac{v_{j,p}+u_{j,p}}{q_p}\right)(EA_{j,p}) + \frac{v_{j,p}u_{j,p}}{q_p}(EA_{j,p})^2\right) \\ &\le \frac{m}{\sqrt{D'}}K_S^2 + \frac{m^2}{D'}K_S^2 + \sum_{j=1}^{l}\left(\frac{v_j+u_j}{q}\right)(EA_j) - \sum_{j=1}^{l}\frac{v_ju_j}{q}(EA_j)^2 \\ &\le \frac{m}{\sqrt{D'}}K_S^2 + \frac{m^2}{D'}K_S^2 + \frac{2}{q} \\ &\le \frac{1}{12} + \frac{1}{36} + \frac{2}{20} < \frac{1}{3}. \end{aligned}$$

From Table 8 it is easy to see that $E.f^{-1}(p_i) = 0$ for $i = 1, 2, 3$.

Assume that $E.f^{-1}(p_4) = 1$; that is, $EA_s = 1$ for some s and $EA_j = 0$ for all $j \neq s$. Lemma 2.5 then gives

$$\sum_{j=1}^{l}\left(\frac{v_j+u_j}{q}\right)(EA_j) \le \sum_{j=1}^{l}\frac{v_ju_j}{q}(EA_j)^2 + \frac{1}{q}.$$

Hence

Table 8

	[2]	[3]	[5]	[3, 2]		[2, 2, 2, 2]			
j	1	1	1	1	2	1	2	3	4
$1-\frac{v_j+u_j}{q}$	0	$\frac{1}{3}$	$\frac{3}{5}$	$\frac{2}{5}$	$\frac{1}{5}$	0	0	0	0
$\frac{v_ju_j}{q}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{6}{5}$	$\frac{6}{5}$	$\frac{4}{5}$

$$\begin{aligned}\sum_{p\neq p_4}\sum_{j=1}^{l_p}&\left(\left(1-\frac{v_{j,p}+u_{j,p}}{q_p}\right)(EA_{j,p})+\frac{v_{j,p}u_{j,p}}{q_p}(EA_{j,p})^2\right)\\ &\leq 1+\frac{m}{\sqrt{D'}}K_S^2+\frac{m^2}{D'}K_S^2+\frac{1}{q}\\ &\leq 1+\frac{1}{12}+\frac{1}{36}+\frac{1}{20}<\frac{7}{6}.\end{aligned}$$

On the other hand, if $E.(f^{-1}(p_1)+f^{-1}(p_2)+f^{-1}(p_3))\geq 2$ then Table 8 gives

$$\sum_{p\neq p_4}\sum_{j=1}^{l_p}\left(\left(1-\frac{v_{j,p}+u_{j,p}}{q_p}\right)(EA_{j,p})+\frac{v_{j,p}u_{j,p}}{q_p}(EA_{j,p})^2\right)\geq\frac{7}{6},$$

where the equality holds if and only if $E.f^{-1}(p_1)=E.f^{-1}(p_2)=1$, $E.f^{-1}(p_3)=0$. It follows that

$$E.(f^{-1}(p_1)+f^{-1}(p_2)+f^{-1}(p_3))\leq 1,$$

which contradicts Lemma 5.4.

Now we assume that $E.f^{-1}(p_4)=0$. In this case,

$$\sum_{p\neq p_4}\sum_{j=1}^{l_p}\left(1-\frac{v_{j,p}+u_{j,p}}{q_p}\right)(EA_{j,p})=1+\frac{m}{\sqrt{D'}}K_S^2.$$

Since $0<\left(m/\sqrt{D'}\right)K_S^2\leq 1/12$, we have

$$1<\sum_{p\neq p_4}\sum_{j=1}^{l_p}\left(1-\frac{v_{j,p}+u_{j,p}}{q_p}\right)(EA_{j,p})\leq 1+\frac{1}{12}.$$

It is easy to see that Table 8 contains no solution to this inequality. □

These considerations leave us with the following four cases:

(1) $E.f^{-1}(p_i)=0$ for $i=1,2,3$, and E meets one component of $f^{-1}(p_4)$ with multiplicity 2;

(2) $E.f^{-1}(p_i)=0$ for $i=1,2,3$, and E meets two non-end components of $f^{-1}(p_4)$;

(3) $E.f^{-1}(p_i) = 0$ for $i = 1, 2, 3$, and E meets both end components of $f^{-1}(p_4)$;
(4) $E.f^{-1}(p_i) = 0$ for $i = 1, 2, 3$, and E meets an end component and a non-end component of $f^{-1}(p_4)$.

STEP 3. *Case (1) cannot occur.*

Proof. Suppose to the contrary that case (1) occurs; that is, $EA_s = 2$ for some $1 \le s \le l$ and $EA_j = 0$ for $j \neq s$.

If $1 < s < l$, then parts (1) and (3) of Proposition 4.2 give

$$1 - \frac{m}{\sqrt{D'}} K_S^2 = 2\left(\frac{v_s + u_s}{q}\right)$$

and

$$1 + \frac{m^2}{D'} K_S^2 = 4\frac{v_s u_s}{q}.$$

Subtracting the first equality multiplied by 2 from the second yields

$$\frac{m^2}{D'} K_S^2 + 2\frac{m}{\sqrt{D'}} K_S^2 - 1 = 4\frac{v_s u_s}{q} - 4\left(\frac{v_s + u_s}{q}\right) \ge 0,$$

where the inequality follows from $vu - (v + u) = (v - 1)(u - 1) - 1 \ge 0$ for $v \ge 2$, $u \ge 2$, and $v + u \ge 4$. (Note that $l \ge 6$ implies $v_j + u_j \ge 7$ for every j.) Yet by Step 1,

$$\frac{m^2}{D'} K_S^2 + 2\frac{m}{\sqrt{D'}} K_S^2 - 1 \le \frac{1}{36} + \frac{2}{12} - 1 < 0,$$

a contradiction.

If $s = 1$, then parts (1) and (3) of Proposition 4.2 give

$$1 - \frac{m}{\sqrt{D'}} K_S^2 = 2\left(\frac{v_1 + 1}{q}\right)$$

and

$$1 + \frac{m^2}{D'} K_S^2 = 4\frac{v_1}{q}.$$

Eliminating v_1/q yields

$$1 = \frac{m^2}{D'} K_S^2 + 2\frac{m}{\sqrt{D'}} K_S^2 + \frac{4}{q} \le \frac{1}{36} + \frac{2}{12} + \frac{4}{20} < 1,$$

a contradiction. □

STEP 4. *Case (2) cannot occur.*

Proof. Suppose that case (2) does occur; that is, $EA_s = EA_t = 1$ for some $1 < s < t < l$ and $EA_j = 0$ for $j \neq s, t$. Then parts (1) and (2) of Proposition 4.2 give

$$1 - \frac{m}{\sqrt{D'}} K_S^2 = \frac{v_s + u_s}{q} + \frac{v_t + u_t}{q}$$

and

$$1 + \frac{m^2}{D'} K_S^2 = \frac{v_s u_s}{q} + \frac{v_t u_t}{q} + 2\frac{v_s u_t}{q} \ge \frac{v_s u_s}{q} + \frac{v_t u_t}{q}.$$

Subtracting the equality multiplied by $\frac{4}{3}$ from the inequality yields

$$1+\frac{m^2}{D'}K_S^2-\frac{4}{3}+\frac{4m}{3\sqrt{D'}}K_S^2 \geq \frac{v_su_s}{q}+\frac{v_tu_t}{q}-\frac{4}{3}\left(\frac{v_s+u_s}{q}+\frac{v_t+u_t}{q}\right) \geq 0,$$

where the last inequality follows from

$$vu-\frac{4}{3}(v+u)=\left(v-\frac{4}{3}\right)\left(u-\frac{4}{3}\right)-\frac{16}{9} \geq 0$$

for $v \geq 2$, $u \geq 2$, and $v+u \geq 6$ (once again, $l \geq 6$ implies $v_j+u_j \geq 7$ for every j). Because

$$\frac{m^2}{D'}K_S^2+\frac{4m}{3\sqrt{D'}}K_S^2 < \frac{1}{3},$$

we have a contradiction. □

STEP 5. *Case (3) cannot occur.*

Proof. Suppose by way of contradiction that case (3) occurs; that is, $EA_1 = EA_l = 1$ and $EA_j = 0$ for $j \neq 1, l$. Then, by Proposition 4.2(1),

$$\frac{q_1+q_l+2}{q}=1-\frac{m}{\sqrt{D'}}K_S^2.$$

Also, by Proposition 4.2(3) we obtain

$$\frac{q_1+q_l+2}{q}=1+\frac{m^2}{D'}K_S^2.$$

From these two equations it follows that $m = -\sqrt{D'}$ and so, by Lemma 3.7(5), $-K_S$ is ample. □

STEP 6. *Case (4) cannot occur.*

Proof. Suppose that case (4) does occur; that is, $EA_1 = EA_t = 1$ for some $1 < t < l$ and $EA_j = 0$ for $j \neq 1, t$. Then parts (1) and (3) of Proposition 4.2 give

$$1-\frac{m}{\sqrt{D'}}K_S^2=\frac{q_1+1}{q}+\frac{v_t+u_t}{q}=\frac{q_1-1}{q}+\frac{v_t+(u_t+2)}{q}$$

and

$$1+\frac{m^2}{D'}K_S^2=\frac{q_1}{q}+\frac{v_tu_t}{q}+2\frac{v_t}{q}=\frac{q_1}{q}+\frac{v_t(u_t+2)}{q}.$$

Subtracting the first equality multiplied by $\frac{3}{2}$ from the second yields

$$\begin{aligned}1+\frac{m^2}{D'}K_S^2-\frac{3}{2}+\frac{3m}{2\sqrt{D'}}K_S^2 &\\ =\frac{q_1}{q}-\frac{3(q_1-1)}{2q}+\frac{v_t(u_t+2)}{q}-\frac{3}{2}\left(\frac{v_t+(u_t+2)}{q}\right)&\\ \geq \frac{q_1}{q}-\frac{3(q_1-1)}{2q}=-\frac{q_1-3}{2q},&\end{aligned}$$

where the inequality follows from

$$vu' - \frac{3}{2}(v + u') = \left(v - \frac{3}{2}\right)\left(u' - \frac{3}{2}\right) - \frac{9}{4} \geq 0$$

for $v \geq 2, u' \geq 4$, and $v + u' \geq 8$. (Here $l \geq 6$ implies $v + u' = v + (u+2) \geq 9$.) Thus

$$\frac{q_1}{2q} > \frac{q_1 - 3}{2q} \geq \frac{1}{2} - \frac{m^2}{D'}K_S^2 - \frac{3m}{2\sqrt{D'}}K_S^2 \geq \frac{1}{2} - \frac{1}{36} - \frac{3}{2} \cdot \frac{1}{12} = \frac{25}{72};$$

hence

$$\frac{q_1}{q} > \frac{25}{36} > \frac{1}{2}$$

and, in particular,

$$n_1 = 2.$$

We claim that $n_t = 2$. Suppose instead that $n_t > 2$. Let

$$\sigma : S' \to S_1'$$

be the blowdown of the (-1)-curve E, and let

$$g : S_1' \to S_1$$

be the contraction to another $\mathbb{Q}$-homology projective plane S_1 with

$$L_{S_1} := b_2(S_1') - 1 = L - 1.$$

The map g contracts the images under σ of all exceptional curves of f except the image of $A_1 = A_{1,p_4}$ that is a (-1)-curve. Observe that S_1 has three singularities $\bar{p}_1, \bar{p}_2, \bar{p}_3$ of order $2, 3, 5$ of the same type as S as well as a singularity $\bar{p}_4$ of order q' with $q' < q$. The latter claim follows from Lemma 2.4(5).

Since $L_{S_1} = L - 1 \geq 11$, it follows from Proposition 4.3 that K_{S_1} is ample. If S_1 has $L_{S_1} < 12$ or $q' < 20$, then we are done by Lemma 5.5. Otherwise, we can find a (-1)-curve E' on S_1' of the form (3.1) with

$$0 < \frac{m}{\sqrt{D'}} \leq \frac{1}{L_{S_1} - 9} \leq \frac{1}{3}.$$

We restart with E' on S_1' from Step 1. Then, by Steps 1–5, we may assume that E' satisfies the case (4); in other words, we may assume that E' meets an end component and a middle (non-end) component of $g^{-1}(\bar{p}_4)$. By the same argument as before we see that the end component is a (-2)-curve. If the middle component has self-intersection ≤ -3 then we repeat the process. Since each process decreases L by 1, we may assume that both the end component and the middle component are (-2)-curves at certain stage. Now, by Lemma 2.4(3),

$$\frac{u_t v_t}{q} \geq \frac{1}{n_t} = \frac{1}{2}.$$

Hence

$$\frac{37}{36} \geq 1 + \frac{m^2}{D'}K_S^2 = \frac{q_1}{q} + \frac{u_t v_t + 2v_t}{q} > \frac{q_1}{q} + \frac{u_t v_t}{q} > \frac{25}{36} + \frac{1}{2} = \frac{43}{36},$$

a contradiction. □

This completes the proof of Theorem 1.2.

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