Michigan Math. J. 61 (2012), 867–874

Filling Loops at Infinity in the Mapping Class Group

Aaron Abrams, Noel Brady, Pallavi Dani, Moon Duchin, & Robert Young

Dehn functions quantify simple connectivity. That is, in a simply connected space, every closed curve is the boundary of some disk; the Dehn function measures the area required to fill the curves of a given length. The growth of the Dehn function is invariant under quasi-isometry, so one can define the Dehn function not just for spaces but also for groups. The Dehn function is not the only group invariant based on a filling problem; for example, one can also define the *Dehn function at infinity,* which is a quasi-isometry invariant that measures the difficulty of filling closed curves with disks that avoid a large ball. The Dehn function at infinity is a special case ($k = 1$) of the *higher divergence functions* Div^k that were defined for groups in [1] and serve to quantify the connectivity at infinity. In that paper we surveyed some results using the growth rates of Div^k to detect geometric features of groups and spaces.

The mapping class group of a surface has quadratic Dehn function because it is automatic, and an automatic structure provides a combing that can be used to shrink a curve to a point using no more area than is needed in a Euclidean space (see [3; 6]). In this paper we study the Dehn function at infinity: If we impose the additional condition that the filling of a loop avoid a large ball, must its area be much worse than quadratic? In [1, Thm. 6.1] we addressed this question and its higher-dimensional analogues in the case of right-angled Artin groups (RAAGs), and we showed that loops can be filled at infinity using area at most polynomial of degree 4.

Here we show that the same result holds in mapping class groups of surfaces of genus $g \geq 5$, contributing to the growing literature comparing mapping class groups to RAAGs. We use two key features of these mapping class groups: first, they have presentations with short relators (a result due to Gervais [5]); second, all abelian subgroups are undistorted [4].

Received September 27, 2011. Revision received May 14, 2012.

This work was supported by a SQuaRE grant from the American Institute of Mathematics. The second author and the fourth author are partially supported by NSF Grant nos. DMS-0906962 and DMS-0906086, respectively. The fifth author would like to thank New York University for its hospitality during the preparation of this paper. We thank the referee for helpful comments.

1. Notation

In any space X we denote by $B_r(x)$ the ball of radius r centered at x. We will use x_0 to denote a basepoint, and we often write B_r for $B_r(x_0)$. Any object in X that is disjoint from B_r is called *r-avoidant* (or often simply *avoidant*).

As usual, for two functions $f, g: \mathbb{R} \to \mathbb{R}$ we write $f \preceq g$ if there is a constant $A > 0$ such that, for all $t \geq 0$,

$$f(t) \leq Ag(At + A) + At + A.$$

Remark 1.1. Note that in Euclidean space $\mathbb{R}^d$, any two points on the sphere of radius r can be joined by an r-avoidant path of length at most πr. Also, it is an exercise that there exists a constant $c > 0$ such that any r-avoidant loop of length l in any $\mathbb{R}^d$ $(d \geq 3)$ can be filled with an r-avoidant disk of area at most cl^2.

2. The Gervais Presentation

For a topological surface $S = S_{g,b}$ (where g is the genus and b is the number of punctures/boundary components), we write $\mathrm{Mod}(S)$ for its group of orientation-preserving diffeomorphisms up to isotopy, or *mapping class group*.

Our strategy for bounding the area of an efficient r-avoidant filling in $\mathrm{Mod}(S)$ is based on the ideas developed in [1] for RAAGs. We begin with an efficient but presumably non-avoidant filling and alter it, "pushing" each original 2-cell to an avoidant 2-cell (i.e., replacing the former with the latter). These new cells are then patched together (still avoidantly) using commuting relations to form the new filling. Careful control of the pushing process allows us to bound the number of 2-cells in the new filling in terms of the number of 2-cells in the original filling.

In order to do this, we present $\mathrm{Mod}(S)$ as a quotient of a RAAG whose generators are Dehn twists (which commute if the corresponding curves are disjoint). In a 2-complex for such a presentation, all cells are either squares coming from the RAAG or among finitely many types of other cells coming from the additional relators of the mapping class group.

Squares coming from commutation relations can be replaced by avoidant squares in a straightforward manner: we push them out radially by post-multiplying with a high power of one of the two commuting letters. The effect of this is to translate along a standard ray in the 2-complex. For the other types of 2-cells, we will be able to carry out a similar pushing operation if we can find a common commuter for all of the letters in the corresponding relator. That is, if σ is a 2-cell with boundary labeled by the word w and if h is a generator that commutes with all letters in w, then post-multiplying by h^R results in a translated copy of σ that is far from x_0. To employ this strategy, we would like a presentation in which every generator is a Dehn twist and every relator w has "small support" in the following sense: the curves corresponding to the letters appearing in w are collectively supported on a subsurface F such that some other generator has support disjoint from F. This generator therefore commutes with every letter in w, and it can be used to push the corresponding cell as described previously.

The Gervais presentation of the mapping class group (see [5]) fits the bill: this is a finite presentation in which every relator is supported on a small subsurface (at most a three-times-punctured torus). Compared to better-known presentations (such as those using Humphries or Lickorish generators), this will have the advantages provided by common commuters to offset the disadvantage of having a far larger number of generators.

All the Gervais generators are Dehn twists supported on the collection of curves shown in Figure 1. We call these the *Gervais curves.* The relations are of three kinds: commuting, braid, and so-called *star* relations. Commuting relations arise from twists around disjoint curves. The braid relations arise whenever two curves intersect once; they have the form $ABA = BAB$. The star relations arise when a collection of seven curves is in a particular topological configuration (see the right side of Figure 1); these have the form $(ABCD)^3 = XYZ$. If two of the twisting curves for A, B, C, D are isotopic (say A and B), then there is a corresponding degenerate star relation (in this case, $(A^2CD)^3 = XY$).

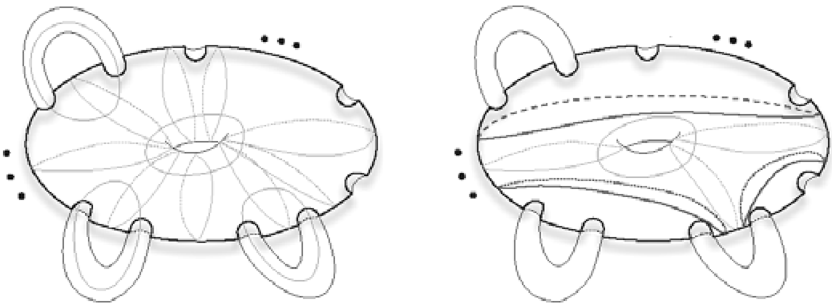

Figure 1 A diagram of the Gervais curves on $S_{g,b}$. On the left are $\alpha_1, \ldots, \alpha_{2g+b-2}$ (appearing as meridians of the central torus) and $\beta_1, \ldots, \beta_g$ (with β_1 as the longitude of the central torus). For every pair α_i, α_j, there is a corresponding curve γ_{ij}; the figure on the right shows three of the γ_{ij}. The star relations are formed by twists around seven curves in the configuration depicted on the right; the central curve β_1 is always used. Note that each such relation is supported on a three-times-punctured torus.

For this presentation of $\mathrm{Mod}(S)$, let X be the universal cover of the presentation 2-complex so that its 1-skeleton is the Cayley graph. Then all of the 2-cells are squares, hexagons, 14-gons, and 15-gons corresponding to the relators just described.

The possible directions to push 2-cells, as well as the possible commuting relations used to patch avoidant 2-cells together, are determined by a particular abelian subgroup of $\mathrm{Mod}(S)$ generated by Dehn twists around the set of curves described in the following lemma.

Lemma 2.1 (Common commuters in the Gervais presentation). *Let S be an orientable surface of genus $g \geq 5$ and with any number $b \geq 0$ of punctures. Then* *there is a set $\mathcal{H}$ of g mutually disjoint Gervais curves, whose associated Dehn twists generate a subgroup $H \le \mathrm{Mod}(S)$, with the following properties.*

(1) *For every relation in the Gervais presentation, there exists an element of $\mathcal{H}$ whose Dehn twist commutes with every letter appearing in the relation.*
(2) *Any Gervais curve intersects at most two curves from $\mathcal{H}$.*

Proof. Gervais gives his presentation in terms of three types of curves: α-curves (separating out the topology), β-curves (mutually disjoint curves, one around each handle, dual to some of the α-curves), and γ-curves (derived from pairs of α-curves); see Figure 1. Each star relation is supported on a three-times-punctured torus, and the relation involves three α-curves, one β-curve, and three γ-curves. One easily verifies that the maximum number of other β-curves intersecting any of these support curves is three, if dual to the α-curves. Thus, if there are at least five β-curves in total, then one of them must be disjoint from the support curves. In this case, the support of any commuting or braid relation clearly also misses some β-curve. The total number of β-curves is g, the genus.

Each α-curve and each γ-curve intersects at most two β-curves, by inspection. Now let $\mathcal{H}$ be the set of all β-curves. □

3. Filling Loops at Infinity

Let $h_1, \dots, h_g$ denote the Dehn twists about the β-curves, and let H denote the subgroup $\langle h_1, \dots, h_g \rangle$. The group H is abelian, since the β curves are disjoint. Abelian subgroups of $\mathrm{Mod}(S)$ are undistorted [4]. That is, if d is the word metric on $\mathrm{Mod}(S)$ with respect to the Gervais presentation and if d_H is the word metric on H, then there is a $C > 1$ such that

$$d(x,y) \le d_H(x,y) \le Cd(x,y) + C. \tag{UND}$$

This means that the intersection of B_r and any coset of H is small, as shown by the following lemma.

LEMMA 3.1. *Let $H' \subset H$ be generated by a subset of the β-curves. Let $v \in \mathrm{Mod}(S)$, and let $y \in v \cdot H'$ be a point in $v \cdot H'$ such that $d(x_0, y) = d(x_0, v \cdot H')$. If $B_r(y)$ denotes the r-ball in $v \cdot H'$ centered at y and if $r \ge 1$, then*

$$B_{r-d(x_0, v\cdot H')}(y) \subset B_r \cap v \cdot H' \subset B_{3Cr}(y).$$

Proof. The first inclusion follows from the triangle inequality and the fact that $d \le d_H$. For the second, suppose that $z \in B_r \cap v \cdot H'$. Then by the definition of y we have $d(x_0, y) \le r$, so $d(z,y) \le 2r$ and

$$d_{v\cdot H'}(z,y) \le 2Cr + C \le 3Cr. \qquad \square$$

We can therefore construct avoidant curves and disks in X from avoidant curves and disks in H' as follows.

LEMMA 3.2 (Avoidance in cosets). *Suppose that $H' \subset H$ is generated by a subset of the β-curves and that H' has rank at least 3. Then there is a constant $D > 0$ depending only on S and such that, for any $v \in \mathrm{Mod}(S)$ and any $r \geq D$, the following statements hold.*

(1) *Let $x_1, x_2 \in v \cdot H'$ be r-avoidant. There is an $\frac{r}{D}$-avoidant path in $v \cdot H'$ from x_1 to x_2 that has length at most $D \cdot d_{H'}(x_1, x_2)$.*
(2) *Let γ be an r-avoidant curve in $v \cdot H'$ of length l. There is an $\frac{r}{D}$-avoidant disk $f \colon D^2 \to v \cdot H'$ that fills γ and has area at most Dl^2.*

Proof. Let $D = 5C$. If $d(x_0, v \cdot H') > \frac{r}{D}$ then the statements are trivial, since the $\frac{r}{D}$-ball does not intersect $v \cdot H'$.

Otherwise, by Lemma 3.1, there is a $y \in v \cdot H'$ such that

$$B_{4r/5}(y) \subset B_r \cap v \cdot H'$$

and

$$B_{r/D} \cap v \cdot H' \subset B_{3r/5}(y).$$

Because x_1, x_2, and γ are r-avoidant, they are outside $B_{4r/5}(y)$. Since H' has rank at least 3, it follows from Remark 1.1 that there is a curve from x_1 to x_2 as well as a disk filling γ that both avoid $B_{3r/5}(y)$; consequently, this curve and disk are $\frac{r}{D}$-avoidant in $\mathrm{Mod}(S)$. $\square$

Thus we can construct avoidant fillings of loops that live in flat cosets; we will use these to build avoidant fillings of arbitrary loops.

THEOREM 3.3 (Filling loops at infinity in the mapping class group). *Suppose S has genus at least 5 and any number of punctures, and let X be the Cayley 2-complex of $\mathrm{Mod}(S)$. There is a constant $c > 0$ such that, for any r, any r-avoidant loop of length l has an $\frac{r}{c}$-avoidant filling of area $\leq cr^2l^2$.*

Proof. We start with an r-avoidant loop of length l in X. Since the Dehn function is quadratic, there exists a (not necessarily avoidant) filling Δ with area $\preceq l^2$. We use Δ as a combinatorial model for an avoidant filling of the same loop. The new filling is obtained by making the following replacements, which are depicted in Figures 2 and 3 and are described more precisely next.

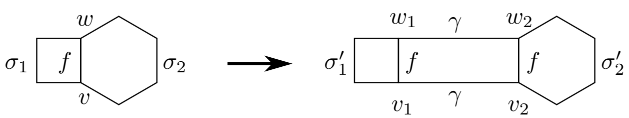

Figure 2 The cells σ'_i are obtained by pushing the σ_i out of the ball of radius r. The path γ is $\frac{r}{D}$-avoidant, and the letters in the corresponding word all commute with f.

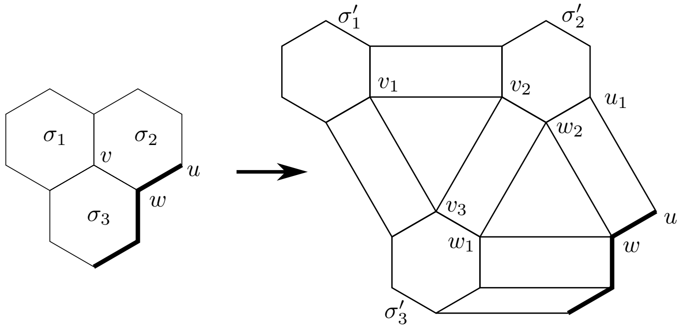

Figure 3 The edges in bold in this figure are part of the boundary loop of Δ. Each strip is $\left(\frac{r}{D}-1\right)$-avoidant and has length $\preceq r$.

Step 1 Each 2-cell of Δ is replaced by ("pushed to") an avoidant copy of itself.

Step 2 Each edge of Δ is replaced by a (possibly degenerate) avoidant strip of squares of length $\preceq r$. An edge belonging to two 2-cells is replaced by a strip connecting the two pushed copies of the cells. An edge belonging to a single 2-cell is necessarily part of the boundary loop and is extended to a strip connecting the edge to the pushed copy of the cell.

Step 3 The result of the previous steps is topologically a punctured disk, with one boundary component equal to the original loop and an additional boundary component corresponding to each vertex in Δ. Each boundary component of the latter kind is filled by an avoidant disk in an appropriate flat.

Step 1: Pushing 2-cells. We replace each 2-cell σ with an avoidant cell σ'. If σ is already r-avoidant, we let $\sigma' = \sigma$. Otherwise, σ is partially contained in the ball of radius r. It corresponds to a relation in the Gervais presentation, and we choose a common commuter h_σ for the generators in this relation, whose existence is guaranteed by Lemma 2.1.

Let $R = (2r+30)C + C$. If the vertices of σ are $v_1, \dots, v_k$ then $v_1 h_\sigma^R, \dots, v_k h_\sigma^R$ are the vertices of a copy of σ (i.e., an isometric 2-cell), since h_σ commutes with the edge labels of σ. Denote this copy by σ'. Since σ is partially contained in the ball of radius r and since relators have length at most 15, it follows that σ' is entirely outside the ball of radius $r+15$. Thus it is r-avoidant by undistortedness (UND).

Step 2: Connecting pushed cells with strips. Consider an edge in Δ with vertices v and w, labeled by a Gervais generator f. Let $H_f \subset H$ be the subgroup of H generated by the generators of H that commute with f; this has rank at least $g-2$, where g is the genus of S, by Lemma 2.1(2).

First, we find vertices v_1 and v_2 corresponding to v in the pushed filling. If the edge is shared by two 2-cells σ_1 and σ_2, then let v_1, v_2 be the vertices of σ'_1 and σ'_2 that correspond to v. Otherwise, the edge belongs to the boundary of Δ. If it is adjacent to a 2-cell σ, let $v_1 = v$ and let v_2 be the vertex in σ' corresponding to v. Otherwise, the edge is used twice in the boundary of Δ and we can let $v_1 = v_2 = v$. In any case, v_1 and v_2 are r-avoidant, both are contained in $v \cdot H_f$, and $d_{H_f}(v_1, v_2) \preceq r$. By Lemma 3.2, there is a $\frac{r}{D}$-avoidant path γ in $v \cdot H_f$ that connects v_1 to v_2; we can interpret this path as a word representing $v_1^{-1}v_2$ whose letters all commute with f. Then there is a strip built out of squares (i.e., commuting relations) whose boundary label is the commutator $[f, \gamma]$; this strip is $\left(\frac{r}{D} - 1\right)$-avoidant and has length $\preceq r$.

Step 3: Filling in the holes. The partial filling constructed in the preceding steps has one boundary component for each vertex of Δ that is sufficiently close to (i.e., is within distance $r + 15$ of) the basepoint. Each boundary component is a polygonal loop whose sides are paths γ belonging to strips from Step 2 (these appear as triangles in Figure 3). The number of sides of the polygon associated to v is the number of edges incident to v in Δ. Each vertex is r-avoidant, and each side is an $\left(\frac{r}{D} - 1\right)$-avoidant curve of length $\preceq r$. Indeed, each vertex is distance at most R away from v, so any two vertices are distance $\leq 2R$ apart. The entire polygon is contained in the coset $v \cdot H$.

To fill these polygonal loops, we first subdivide each one into triangular loops by adding additional $\frac{r}{D}$-avoidant curves in $v \cdot H$ between the vertices; these exist by Lemma 3.2. The resulting triangular loops are $\left(\frac{r}{D} - 1\right)$-avoidant and have length $\preceq r$. Again by Lemma 3.2, they can be filled by $\left(\frac{r}{D^2} - 1\right)$-avoidant disks in $v \cdot H$ of area $\preceq r^2$. Let $\rho = \frac{1}{2D^2}$ so that, when r is sufficiently large, $\frac{r}{D^2} - 1 \geq \rho r$.

The union of the pushed cells, strips, and filled triangles described in these steps is a ρr-avoidant filling of the boundary loop; it remains to estimate the area of this filling. Since the area (the number of 2-cells) of Δ is $\preceq l^2$, the number of vertices and edges in Δ is $\preceq l^2$ as well. Each strip introduced in this construction has length (and area) $\preceq r$. Because the triangular loops lie in cosets, there is a constant M such that the area of any of the triangle fillings is at most Mr^2. Moreover, the number of triangles is certainly bounded above by twice the number of edges in Δ. So the total area of the new filling is $\preceq l^2 + l^2(2r) + l^2(Mr^2) \preceq r^2 l^2$, as desired. $\square$

In the language of [1], which primarily considers the case $l \sim r$, this theorem implies that $\mathrm{Div}^1(\mathrm{Mod}(S)) \preceq r^4$. By considering the lower bound given by the Euclidean filling area, we have $r^2 \preceq \mathrm{Div}^1(\mathrm{Mod}(S))$. In fact, there is some evidence that the true answer is $\mathrm{Div}^1(\mathrm{Mod}(S)) \sim r^3$; good candidates for hard-to-fill loops are found in the 2-flats generated by one Dehn twist and one other mapping class that is pseudo-Anosov on the complementary subsurface. In general, to approach higher divergence in mapping class groups it is natural to focus on flats generated by many partial pseudo-Anosovs. If a boundary in such a flat has an avoidant filling then we can project that filling to the flat. This projection must be a filling of the original boundary; however, since the projection from the mapping class group to the axis of a pseudo-Anosov is strongly contracting [2, Thm. 4.2], the original avoidant filling must be much larger than its projection.

References

[1] A. Abrams, N. Brady, P. Dani, M. Duchin, and R. Young, *Pushing fillings in right-angled Artin groups,* J. London Math. Soc. (2) (to appear).
[2] M. Duchin and K. Rafi, *Divergence of geodesics in Teichmüller space and the mapping class group,* Geom. Funct. Anal. 19 (2009), 722–742.
[3] D. B. A. Epstein, J. W. Cannon, D. F. Holt, S. V. F. Levy, M. S. Paterson, and W. P. Thurston, *Word processing in groups,* Jones & Bartlett, Boston, 1992.
[4] B. Farb, A. Lubotzky, and Y. Minsky, *Rank-1 phenomena for mapping class groups,* Duke Math. J. 106 (2001), 581–597.
[5] S. Gervais, *A finite presentation of the mapping class group of a punctured surface,* Topology 40 (2001), 703–725.
[6] L. Mosher, *Mapping class groups are automatic,* Math. Res. Lett. 1 (1994), 249–255.

A. Abrams
Mathematics Department
Washington and Lee University
Lexington, VA 24450

abrams.aaron@gmail.com

N. Brady
Department of Mathematics
University of Oklahoma
Norman, OK 73019

nbrady@math.ou.edu

P. Dani
Department of Mathematics
Louisiana State University
Baton Rouge, LA 70803

pdani@math.lsu.edu

M. Duchin
Department of Mathematics
Tufts University
Medford, MA 02155

moon.duchin@tufts.edu

R. Young
Department of Mathematics
University of Toronto
Toronto, ON M5S 2E4
Canada

ryoung@math.toronto.edu