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Fano Surfaces with 12 or 30 Elliptic Curves

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Introduction

A Fano surface is a surface of general type that parameterizes the lines of a smooth cubic threefold. First studied by Fano and then by many others—including Bombieri and Swinnerton-Dyer [4], Gherardelli [7], Tyurin [14; 15], Clemens and Griffiths [5], and Collino [6]—these surfaces carry many remarkable properties. In our previous paper [12], we classified Fano surfaces according to the configurations of their elliptic curves. The aim of the present paper is to give various applications of this study when the Fano surface contains 12 or 30 elliptic curves.

The main result of the first part of this paper is as follows.

Proposition 1.

(i) *The Picard number* ρ_S *of a Fano surface* S *satisfies* $1 \leq \rho_S \leq 25$ *and is* 1 *for* S *generic.*
(ii) *A Fano surface that contains* 12 *elliptic curves is a triple ramified cover of the blow-up of* 9 *points of an abelian surface.*
(iii) *The Néron–Severi group of such a surface has rank* 12, 13 *or* $25 = h^{1,1}(S)$.
(iv) *For* S *generic among Fano surfaces with* 12 *elliptic curves, the Néron–Severi group has rank* 12 *and is rationally generated by its* 12 *elliptic curves.*
(v) *An infinite number of Fano surfaces with* 12 *elliptic curves have maximal Picard number* $25 = h^{1,1}(S)$.

Recall that among the K3 surfaces, the Kummer surfaces are recognized as those K3 having 16 disjoint (-2)-curves. They are the double cover of the blow-up over the 2-torsion points of an abelian surface (see [11]). Our theorem is the analogue for Fano surfaces that contains 12 elliptic curves among Fano surfaces.

In the second part, we study the Fano surface S of the Fermat cubic threefold $F \hookrightarrow \mathbb{P}^4$:

$$F = \{x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 = 0\}.$$

Let μ_3 be the group of third roots of unity and let $\alpha \in \mu_3$ be a primitive root. For s a point of S, we denote by L_s the line on F corresponding to the point s and we denote by C_s the incidence divisor that parameterizes the lines in F that cut the line L_s.

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THEOREM 2. *The surface S is the unique Fano surface that contains* 30 *smooth curves of genus* 1. *These curves are numbered*

$$E_{ij}^{\beta},\quad 1 \le i < j \le 5,\ \beta \in \mu_3,$$

in such a way that, for two such curves E_{ij}^{γ} and E_{st}^{β},

$$E_{ij}^{\beta} E_{st}^{\gamma} = \begin{cases} 1 & \textit{if } \{i,j\} \cap \{s,t\} = \emptyset, \\ -3 & \textit{if } E_{ij}^{\beta} = E_{st}^{\gamma}, \\ 0 & \textit{else}. \end{cases}$$

The Néron–Severi group $\mathrm{NS}(S)$ *of S has rank* $25 = \dim H^1(S, \Omega_S)$ *and discriminant* 3^{18}. *These* 30 *elliptic curves generate an index-*3 *sublattice of* $\mathrm{NS}(S)$ *and, with the class of an incidence divisor C_s $(s \in S)$, they generate the Néron–Severi group.*

Given a smooth curve of low genus and with a sufficiently large automorphism group, it is sometimes possible to calculate the period matrix of its Jacobian [3]. In this paper, we calculate also the period lattice of the Albanese variety of the 2-dimensional variety S. This computation is used to determine the Néron–Severi group of S. We determine also the fibrations of S onto an elliptic curve as well as the intersection numbers between the fibers of these fibrations, and we discuss some of the more interesting fibrations.

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1. Preliminaries on Fano Surfaces

1.1. Tangent Bundle Theorem

Let $F \hookrightarrow \mathbb{P}^4$ be a smooth cubic threefold and let S be its Fano surface of lines. We consider the following diagram:

$$\begin{array}{ccccc} \mathcal{U} & \xrightarrow{\psi} & F & \hookrightarrow & \mathbb{P}^4 \\ {\scriptstyle\pi}\downarrow & & & & \\ S & & & & \end{array}$$

where $\mathcal{U}$ is the universal family of lines and π, ψ are the projections.

THEOREM 3 (Tangent Bundle Theorem [5]). *There is an isomorphism*

$$\pi_* \psi^* \mathcal{O}(1) \simeq \Omega_S,$$

where Ω_S is the cotangent sheaf. By this isomorphism, we can identify the spaces $H^0(F, \mathcal{O}(1))$ and $H^0(S, \Omega_S)$, the varieties $\mathbb{P}^4$ and $\mathbb{P}(H^0(S, \Omega_S)^*)$, and the varieties $\mathcal{U}$ and $\mathbb{P}(T_S)$ for $T_S = \Omega_S^*$.

We always work with the identifications of this theorem. In particular, the line $L_s \hookrightarrow F$ corresponding to a point s in S is the projectivized tangent space to S at s.

1.2. Properties of Fano Surfaces with an Elliptic Curve

Let us denote by $\mathcal{C}$ a cone on the cubic F. The following lemmas come from [12].

LEMMA 4. *The cone $\mathcal{C}$ is a hyperplane section of F. The curve E parameterizing the lines on $\mathcal{C}$ is naturally embedded into S and is an elliptic curve. Conversely, if $E \hookrightarrow S$ is an elliptic curve on S, the surface $\psi(\pi^{-1}(E))$ is a cone.*

Let $E \hookrightarrow S$ be an elliptic curve.

LEMMA 5. *We can canonically associate to $E \hookrightarrow S$ an involution $\sigma_E : S \to S$ and a fibration $\gamma_E : S \to E$.*

Let us recall the construction of σ_E and γ_E. For a generic point s of S, the line L_s cuts the hyperplane section $\psi(\pi^{-1}(E))$ into a point p_s. The line between p_s and the vertex of the cone $\psi(\pi^{-1}(E))$ lies inside this cone and is represented by a point $\gamma_E s$ in E. The lines L_s and $L_{\gamma_E s}$ lie on a plane; that plane cuts the cubic into a third residual line, denoted by $L_{\sigma_E s}$.

Let s be a point of E and let C_s be the incident divisor parameterizing the lines in F that cut L_s.

LEMMA 6. *The fiber $\gamma_E^* s$ satisfies $C_s = \gamma_E^* s + E$. We have $C_s^2 = 5$, $C_s E = 1$, and $E^2 = -3$.*

Let A be the Albanese variety of the Fano surface S; its tangent space is $H^0(S, \Omega_S)^*$. We denote by $\vartheta : S \to A$ the Albanese map. Since ϑ is an embedding [5], we consider S as a subvariety of A. To the morphisms σ_E and γ_E correspond an involution $\Sigma_E : A \to A$ of A and a morphism $\Gamma_E : A \to E$ such that $\vartheta \circ \sigma_E = \Sigma_E \circ \vartheta$ and $\Gamma_E \circ \vartheta = \vartheta \circ \gamma_E$.

LEMMA 7 [12, Lemma 29]. *The differentials $d\Sigma_E$ and $d\Gamma_E$ of Σ_E and Γ_E are endomorphisms of $H^0(S, \Omega_S)^*$. They satisfy*

$$I + d\Sigma_E + d\Gamma_E = 0,$$

where I is the identity. The eigenspace of the eigenvalue 1 of the involution $d\Sigma_E$ is the tangent space T_E of the curve $E \hookrightarrow A$ (translated in 0).

Let us denote by f the projectivization of $d\Sigma_E \in \mathrm{GL}(H^0(S, \Omega_S)^*)$: it is an automorphism of $\mathbb{P}^4 = \mathbb{P}(H^0(S, \Omega_S)^*)$. Let p_E be the point of $\mathbb{P}^4$ corresponding to the 1-dimensional space $T_E \subset H^0(S, \Omega_S)^*$.

LEMMA 8. *The involution f preserves the cubic threefold $F \hookrightarrow \mathbb{P}^4$. The point p_E is the vertex of the cone $\psi(\pi^{-1}(E))$. The hyperplane $\mathbb{P}(\mathrm{Ker}(d\Gamma_E))$ and p_E constitute the closed set of fixed points of f.*

Conversely, let f be an involution of $\mathbb{P}^4$ acting on F and fixing an isolated point and a hyperplane. The isolated fixed point is the vertex of a cone on F.

We will use Lemma 7 as in the next example.

EXAMPLE 9. Let $x_1, \dots, x_5$ be homogenous coordinates of $\mathbb{P}^4$. Then the point $(1 : 0 : \cdots : 0)$ is the vertex of a cone on the cubic threefold

$$F = \{x_1^2 x_2 + G(x_2, \dots, x_5) = 0\}$$

(where G is a cubic form such that F is smooth). Let $E \hookrightarrow S$ be the elliptic curve parameterizing the lines of that cone. By Lemma 7, we see that the involution $d\Sigma_E$ satisfies

$$d\Sigma_E \colon (x_1, x_2, \dots, x_5) \to (x_1, -x_2, \dots, -x_5),$$

and we deduce that $d\Gamma_E$ is defined by

$$d\Gamma_E \colon (x_1, x_2, \dots, x_5) \to (-2x_1, 0, \dots, 0).$$

1.3. Theta Polarization

Let S be a Fano surface, let A be its Albanese variety, and let $\vartheta \colon S \hookrightarrow A$ be the Albanese map. By [5, Thm. 13.4], the image Θ of $S \times S$ under the morphism $(s_1, s_2) \to \vartheta(s_1) - \vartheta(s_2)$ is a principal polarization of A. Let τ be an automorphism of S and let τ' be the automorphism of A such that $\vartheta \circ \tau = \tau' \circ \vartheta$. Let (s_1, s_2) be a point of $S \times S$; then $\tau'(\vartheta(s_1) - \vartheta(s_2)) = \vartheta(\tau(s_1)) - \vartheta(\tau(s_2))$. This leads to the following statement.

LEMMA 10. *The automorphism τ' preserves the polarization $\tau'^*\Theta = \Theta$.*

For a variety X, we denote by $H^2(X, \mathbb{Z})_f$ the group $H^2(X, \mathbb{Z})$ modulo torsion. We denote by $\mathrm{NS}(X) = H^{1,1}(X) \cap H^2(X, \mathbb{Z})_f$ its Néron–Severi group and by ρ_X its Picard number. For a divisor D in X, we denote its Chern class by $c_1(D)$.

THEOREM 11.

(a) *If D and D' are two divisors of A, then*

$$\vartheta^*(D)\vartheta^*(D') = \int_A \frac{1}{3!} \bigwedge^3 c_1(\Theta) \wedge c_1(D) \wedge c_1(D').$$

(b) *The following sequence is exact*:

$$0 \to \mathrm{NS}(A) \xrightarrow{\vartheta^*} \mathrm{NS}(S) \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

(c) *The Néron–Severi group of S is generated by $\vartheta^*\mathrm{NS}(A)$ and by the class of an incidence divisor C_s $(s \in S)$. The class of $\vartheta^*(\Theta)$ is equal to $2C_s$.*

(d) *We have $\rho_A = \rho_S \le 25 = \dim H^1(S, \Omega_S)$ and $\rho_S = 1$ for S generic.*

Proof. The morphism ϑ is an embedding, and the homological class of $\vartheta(S)$ is equal to $\frac{1}{3!}\Theta^3$ [2, Prop. 7]. This proves part (a).

Since Θ is a polarization, the bilinear symmetric form

$$Q_\Theta \colon H^2(A, \mathbb{C}) \times H^2(A, \mathbb{C}) \to \mathbb{C}$$

defined by

$$Q_\Theta(\eta_1, \eta_2) = \int_A \frac{1}{3!} \bigwedge^3 c_1(\Theta) \wedge \eta_1 \wedge \eta_2$$

is nondegenerate (Hodge–Riemann bilinear relations; [8, Chap. 0, Sec. 7]). This implies that the morphism

$$\vartheta^*\colon H^2(A, \mathbb{C}) \to H^2(S, \mathbb{C})$$

is injective, and since S and A have the same second Betti number (see [7, (2)]), it follows that the homomorphism

$$\vartheta_*\colon H_2(S, \mathbb{Z})_f \to H_2(A, \mathbb{Z})$$

is injective. By [6, 2.3.5.1] we have the exact sequence

$$H_2(S, \mathbb{Z})_f \xrightarrow{\vartheta_*} H_2(A, \mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z} \to 0;$$

thus

$$0 \to H_2(S, \mathbb{Z})_f \xrightarrow{\vartheta_*} H_2(A, \mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z} \to 0$$

is exact. By duality, this yields:

$$0 \to H^2(A, \mathbb{Z}) \xrightarrow{\vartheta_*} H^2(S, \mathbb{Z})_f \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

Because the spaces $H^{1,1}(A)$ and $H^{1,1}(S)$ have dimension 25 (see [5]), we have $\vartheta^*(H^{1,1}(A)) = H^{1,1}(S)$. This implies that the sequence

$$0 \to \mathrm{NS}(A) \xrightarrow{\vartheta_*} \mathrm{NS}(S) \to \mathbb{Z}/2\mathbb{Z}$$

is exact. This sequence is exact on the right also because $\vartheta^*\Theta = 2C_s$ by [5, Lemma 11.27] (Θ is a principal polarization and is not divisible by 2, so the class of C_s and $\vartheta^*\mathrm{NS}(A)$ generate $\mathrm{NS}(S)$). This proves part (b).

By [6], any Jacobian of an hyperelliptic curve of genus 5 is a limit of the Albanese varieties of Fano surfaces endowed with their principal polarization. By [9], the endomorphism ring of a Jacobian of a generic hyperelliptic curve is isomorphic to $\mathbb{Z}$. If the generic Albanese variety of Fano surface were not simple, then also its limit would be nonsimple. This is a contradiction, so $\rho_A = 1$ for a generic Fano surface. □

2. Fano Surfaces with 12 Elliptic Curves

Let $\lambda \in \mathbb{C}$ and $\lambda^3 \neq 1$; then the cubic threefold

$$F_\lambda = \{x_1^3 + x_2^3 + x_3^3 - 3\lambda x_1 x_2 x_3 + x_4^3 + x_5^3 = 0\} \hookrightarrow \mathbb{P}^4$$

is smooth. Let $e_1, \dots, e_5$ be the dual basis of $x_1, \dots, x_5$. The 12 points

$$\mathbb{C}(e_4 - \beta e_5), \mathbb{C}(e_i - \beta e_j) \in \mathbb{P}^4, \quad 1 \le i < j \le 3, \ \beta^3 = 1$$

are vertices of cones in F_λ. Let S_λ be the Fano surface of F_λ. We denote by $E_{ij}^\beta \hookrightarrow S_\lambda$ the elliptic curve that parameterizes the lines of the cone of vertex

$\mathbb{C}(e_i - \beta e_j)$ (see Lemma 4). Let $(y_1 : y_2 : y_3)$ be projective coordinates of the plane. Let $E_\lambda \hookrightarrow \mathbb{P}^2$ be the elliptic curve

$$E_\lambda = \{y_1^3 + y_2^3 + y_3^3 - 3\lambda y_1 y_2 y_3 = 0\}$$

with neutral element $(1 : -1 : 0)$. By [12], the 9 curves E_{ij}^β $(1 \le i < j \le 3, \beta^3 = 1)$ are disjoint and isomorphic to E_0; the 3 curves E_{45}^β $(\beta^3 = 1)$ are disjoint and isomorphic to E_λ, and

$$E_{45}^\beta E_{ij}^\gamma = 1 \quad \text{for all } 1 \le i < j \le 3,\ \beta^3 = \gamma^3 = 1.$$

Conversely, let S be a Fano surfaces with 12 elliptic curves. Then we have the following result.

LEMMA 12 [12, 3.3]. *There is a set of* 3 *disjoint elliptic curves on S isomorphic to E_μ (for some $\mu \in \mathbb{C}$) such that the* 9 *remaining elliptic curves are disjoint and such that the surface S is isomorphic to S_μ.*

Let Y be the surface $Y = E_\lambda \times E_\lambda$. Let T_1 and T_2 be the elliptic curves

$$T_1 = \{x + 2y = 0/(x, y) \in E_\lambda \times E_\lambda\},$$
$$T_2 = \{2x + y = 0/(x, y) \in E_\lambda \times E_\lambda\}$$

on Y and let $\Delta \hookrightarrow Y$ be the diagonal. Any 2 of the 3 curves T_1, T_2, Δ meet transversally at the 9 points of 3-torsion of Δ. We denote by Z the blow-up of Y at these 9 points.

PROPOSITION 13. *The Fano surface S_λ is a triple cyclic cover of Z branched along the proper transform of $\Delta + T_1 + T_2$ in Z.*

Proof. Let $\alpha \in \mu_3$ be a primitive root. The order-3 automorphism

$$f : x \to (\alpha x_1 : \alpha x_2 : \alpha x_3 : x_4 : x_5)$$

acts on F_λ. The automorphism f acts on the Fano surface of lines of F_λ by an automorphism denoted by τ. Since we know the action of f, we can check immediately that the fixed locus of τ is the smooth divisor $E_{45}^1 + E_{45}^\alpha + E_{45}^{\alpha^2}$. The quotient of S_λ by τ is a smooth surface Z' with Chern numbers $c_1^2 = -9$ and $c_2 = 9$, and the degree-3 quotient map $\eta \colon S_\lambda \to Z'$ is ramified over $E_{45}^1 + E_{45}^\alpha + E_{45}^{\alpha^2}$.

For an elliptic curve $E \hookrightarrow S$, we denote by $\gamma_E \colon S \to E$ the associated fibration (Lemma 5). By Lemma 6, the morphism

$$g = (\gamma_{E_{45}^\alpha}, \gamma_{E_{45}^{\alpha^2}}) : S_\lambda \to Y$$

has degree $3 = (C_s - E_{45}^\alpha)(C_s - E_{45}^{\alpha^2})$. Let be $E = E_{45}^\beta$ (for $\beta^3 = 1$).

Let s be a generic point of S. By definition (see Lemma 4), the line L_s cuts the line $L_{\gamma_E s}$. Because $L_{\gamma_E s}$ is stable by f, the line $f(L_s) = L_{\tau s}$ cuts also the line $L_{\gamma_E s}$; thus, by definition of γ_E, we have $\gamma_E \tau s = \gamma_E s$. This proves that $\gamma_E \circ \tau = \gamma_E$ and $g \circ \tau = g$. Hence, by the property of the quotient map, there is a birational morphism

$$h\colon Z' \to Y$$

such that $g = h \circ \eta$.

Let t be the intersection point of E_{12}^{1} and E_{45}^{1} and let $\vartheta : S_\lambda \to A_\lambda$ be the Albanese map such that $\vartheta(t) = 0$. It is an embedding and we consider S_λ as a subvariety of A_λ. The tangent space to the curve $E_{45}^{\beta} \hookrightarrow A_\lambda$ (translated to 0) is $V_\beta = \mathbb{C}(\beta e_4 - \beta^2 e_5)$. The tangent space of $E_{45}^{\alpha} \times E_{45}^{\alpha^2}$ is $V_\alpha \oplus V_{\alpha^2}$. With the help of Lemma 7 and Example 9, it is easily checked that the images under g of the curves E_{45}^{1}, E_{45}^{α}, and $E_{45}^{\alpha^2}$ are respectively Δ, T_1, and T_2.

Moreover, the morphism g has degree 1 on these 3 elliptic curves and contracts the 9 elliptic curves E_{ij}^{β} ($1 \le i < j \le 3$, $\beta^3 = 1$). This implies that the image under g of $E_{45}^{1} + E_{45}^{\alpha} + E_{45}^{\alpha^2}$ is $T_1 + T_2 + \Delta$ and that Z' is isomorphic to Z. □

Let D be the proper transform of $\Delta + T_2 + T_2$ in Z. By Proposition 13 and [1, Chap. I, Para. 17 & 18], the divisor D is divisible by 3 in $\mathrm{NS}(Z)$.

Since Y is an Abelian surface, there exist 3^4 invertible sheaves $\mathcal{L}$ on Z such that $\mathcal{L}^{\otimes 3} = \mathcal{O}_Z(D)$. Let $S(\mathcal{L}) \to Z$ be the degree-3 cyclic cover of Z that is branched over D and associated to such $\mathcal{L}$.

COROLLARY 14. *The surface $S(\mathcal{L})$ contains* 12 *elliptic curves,*

$$E_{45}^{\beta},\ E_{ij}^{\gamma}, \quad 1 \le i < j \le 3, \quad \beta^3 = \gamma^3 = 1,$$

that have the same configuration as for S_λ; moreover, the divisor

$$K = \sum_{\beta^3=1} 2E_{45}^{\beta} + E_{12}^{\beta} + E_{13}^{\beta} + E_{23}^{\beta}$$

is a canonical divisor of $S(\mathcal{L})$.

Among these 81 *invertible sheaves, if there exists a unique $\mathcal{L}$ such that $S(\mathcal{L})$ is a Fano surface, then that surface is isomorphic to S_λ.*

Proof. See [1, Chap. I, Para. 17 & 18]. For the uniqueness of the invertible sheaf $\mathcal{L}$, suppose that $S(\mathcal{L})$ and $S(\mathcal{L}')$ are Fano surfaces. By construction, they contain 12 elliptic curves, 3 of which are isomorphic to E_λ and cut the 9 others. Thus, by Lemma 12, $S(\mathcal{L})$ and $S(\mathcal{L}')$ are isomorphic to S_λ and therefore $\mathcal{L} = \mathcal{L}'$. □

REMARK 15. The remaining 80 surfaces $S(\mathcal{L})$ are thus “fake” Fano surfaces and are on different components of the moduli space of surfaces with $c_1^2 = 45$ and $c_2 = 27$.

Let α be a third primitive root of unity. Let us now study the Néron–Severi group of S.

PROPOSITION 16. (1) *Suppose that E_λ has no complex multiplication. The Néron–Severi group of S_λ has rank* 12. *The sublattice generated by the elliptic curves and the class of an incidence divisor C_s has rank* 12 *and discriminant* 6^{10}.

(2) *If E_λ has complex multiplication by a field different from $\mathbb{Q}(\alpha)$, then the Néron–Severi group of S_λ has rank* 13.

(3) *If* E_λ *has complex multiplication by* $\mathbb{Q}(\alpha)$, *then the Néron–Severi group of* S_λ *has rank* 25.

Proof. We can easily compute the Picard number of the Abelian variety $E_0^3 \times E_\lambda^2$. By [12], the Albanese variety A of S_λ is isogenous to $E_0^3 \times E_\lambda^2$. Hence their Néron–Severi groups have the same rank and, according to the cases (1), (2), and (3), this rank is 12, 13, or 25. Now Theorem 11 implies that the Picard number of S_λ is 12, 13, or 25, respectively. □

3. The Fano Surface of the Fermat Cubic

3.1. Elliptic Curve Configuration of the Fano Surface of the Fermat Cubic

Let S be the Fano surface of the Fermat cubic $F \hookrightarrow \mathbb{P}^4 = \mathbb{P}(H^0(S, \Omega_S)^*)$:

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 = 0.$$

Let $e_1, \dots, e_5 \in H^0(S, \Omega_S)^*$ be the dual basis of basis of $x_1, \dots, x_5$. Let μ_3 be the group of third roots of unity, let $1 \le i < j \le 5$, and let $\beta \in \mu_3$. The point

$$p_{ij}^\beta = \mathbb{C}(e_i - \beta e_j) \in \mathbb{P}^4$$

is the vertex of a cone on the cubic F. We denote by $E_{ij}^\beta \hookrightarrow S$ the elliptic curve that parameterizes the lines on that cone. The complex reflection group $G(3,3,5)$ (in the basis $e_1, \dots, e_5$ of $H^0(\Omega_S)^*$) is the group generated by the permutation matrices and the matrix with diagonal elements $\alpha, \alpha, \alpha, \alpha, \alpha^2$ (where $\alpha \in \mu_3$ is a primitive root).

We recall the following.

PROPOSITION 17. *The Fano surface of the Fermat cubic possesses* 30 *smooth curves of genus* 1 *numbered*

$$E_{ij}^\beta, \quad 1 \le i < j \le 5, \ \beta \in \mu_3.$$

(i) *Each smooth genus-*1 *curve of the Fano surface is isomorphic to the Fermat plane cubic* $\mathbb{E} := \{x^3 + y^3 + z^3 = 0\}$.

(ii) *Let* E_{ij}^γ *and* E_{st}^β *be two smooth curves of genus* 1. *Then*

$$E_{ij}^\beta E_{st}^\gamma = \begin{cases} 1 & \textit{if } \{i,j\} \cap \{s,t\} = \emptyset, \\ -3 & \textit{if } E_{ij}^\beta = E_{st}^\gamma, \\ 0 & \textit{else}. \end{cases}$$

(iii) *Let* $E \hookrightarrow S$ *be a smooth curve of genus* 1. *The fibration* γ_E *has* 20 *sections and contracts* 9 *elliptic curves.*

(iv) *The automorphism group of* S *is isomorphic to the complex reflection* $G(3,3,5)$.

Proof. See [12]. For part (iv) we use the fact that an automorphism of F must preserve the configuration of the 30 vertices of cones and the 100 lines that contain 3 such vertices. □

3.2. The Albanese Variety of the Fano Surface of the Fermat Cubic

Let S be the Fano surface of the Fermat cubic F. Our main aim is to compute the full Néron–Severi group of S, which will be done in Section 3.3. We first need to study the Albanese variety A of S.

3.2.1. *Construction of Fibrations*

In order to know the period lattice of the Albanese variety of S, we construct morphisms of the Fano surface onto an elliptic curve and then study their properties.

Let $\vartheta : S \to A$ be a fixed Albanese map. It is an embedding, and we consider S as a subvariety of A. Recall that if τ is an automorphism of S, we denote by $\tau' \in \mathrm{Aut}(A)$ the unique automorphism such that $\tau' \circ \vartheta = \vartheta \circ \tau$.

By [12], the reflection group $G(3,3,5)$ is the analytic representation of the automorphisms $\tau', \tau \in \mathrm{Aut}(S)$. The ring $\mathbb{Z}[G(3,3,5)] \subset \mathrm{End}(H^0(\Omega_S)^*)$ is then the analytic representation of a subring of endomorphisms of the Abelian variety A.

Let us denote by Λ_A^* the rank-5 sub-$\mathbb{Z}[\alpha]$-module of $H^0(\Omega_S)$ generated by the forms

$$x_i - \beta x_j \quad (i < j,\ \beta \in \mu_3).$$

Let ℓ be an element of Λ_A^*. The endomorphism of $H^0(\Omega_S)^* = H^0(S,\Omega_S)^*$ defined by $x \to \ell(x)(e_1 - e_2)$ is an element of $\mathbb{Z}[G(3,3,5)]$. Let us denote by $\Gamma_\ell : A \to \mathbb{E}$ the corresponding morphism of Abelian varieties, where $\mathbb{E} \hookrightarrow A$ is the elliptic curve with tangent space $\mathbb{C}(e_1 - e_2)$. We denote by $\gamma_\ell : S \to \mathbb{E}$ the morphism $\Gamma_\ell \circ \vartheta$.

For $1 \le i < j \le 5$ and $\beta \in \mu_3$, the space

$$\mathbb{C}(e_i - \beta e_j) \subset H^0(\Omega_S)^*$$

is the tangent space to the elliptic curve $E_{ij}^{\beta} \hookrightarrow A$ translated to 0 (Lemma 7).

Let $H_1(A,\mathbb{Z}) \subset H^0(\Omega_S)^*$ be the period lattice of A. The elliptic curve $\mathbb{E}$ has complex multiplication by the principal ideal domain $\mathbb{Z}[\alpha]$. There exists a $c \in \mathbb{C}^*$ such that

$$H_1(A,\mathbb{Z}) \cap \mathbb{C}(e_1 - e_2) = \mathbb{Z}[\alpha]c(e_1 - e_2).$$

Up to the basis change of $e_1, \dots, e_5$ by $ce_1, \dots, ce_5$, we may suppose that $c = 1$. Since $G(3,3,5)$ acts transitively on the 30 spaces $\mathbb{C}(e_i - \beta e_j)$, we have

$$H_1(A,\mathbb{Z}) \cap \mathbb{C}(e_i - \beta e_j) = \mathbb{Z}[\alpha](e_i - \beta e_j).$$

We define the Hermitian product of two forms $\ell, \ell' \in \Lambda_A^*$ by

$$\langle \ell, \ell' \rangle := \sum_{k=1}^{k=5} \ell(e_k)\overline{\ell'(e_k)}$$

and the norm of ℓ by $\|\ell\| = \sqrt{\langle \ell, \ell \rangle}$. Let C_s be an incidence divisor.

THEOREM 18. *Let ℓ be a nonzero element of Λ_A^* and let F_ℓ be a fiber of γ_ℓ.*

(a) *The intersection number of F_ℓ and $E_{ij}^\beta \hookrightarrow S$ is equal to*

$$E_{ij}^\beta F_\ell = |\ell(e_i - \beta e_j)|^2.$$

(b) *We have $F_\ell C_s = 2\|\ell\|^2$, and the fiber F_ℓ has genus*

$$g(F_\ell) = 1 + 3\|\ell\|^2.$$

(c) *Let ℓ and ℓ' be two linearly independent elements of $\Lambda_A^* \subset H^0(\Omega_S)$. The morphism $\tau_{\ell,\ell'} = (\gamma_\ell, \gamma_{\ell'}): S \to \mathbb{E} \times \mathbb{E}$ has degree equal to $F_\ell F_{\ell'}$, and*

$$F_\ell F_{\ell'} = \|\ell\|^2\|\ell'\|^2 - \langle \ell, \ell' \rangle \langle \ell', \ell \rangle.$$

REMARK 19. The known intersection numbers $F_\ell E_{ij}^\beta$ and $F_\ell C_s$ enable us to write the numerical equivalence class of the fiber F_ℓ in the $\mathbb{Z}$-basis given in Theorem 29 (to follow).

Proof of Theorem 18. Proving part (a) involves a trick. For $1 \leq i < j \leq 5$ and $\beta \in \mu_3$, we can interpret the intersection number $E_{ij}^\beta F_\ell$ geometrically as the degree of the restriction of γ_ℓ to $E_{ij}^\beta \hookrightarrow S$. It is also the degree of the restriction of Γ_ℓ to $E_{ij}^\beta \hookrightarrow A$. Since this restriction is the multiplication map by $\ell(e_i - \beta e_j)$, the degree of the morphism Γ_ℓ on E_{ij}^β is equal to $|\ell(e_i - \beta e_j)|^2$. Thus $F_\ell E_{ij}^\beta = |\ell(e_i - \beta e_j)|^2$.

Let us now study the genus of F_ℓ.

LEMMA 20. *The fiber of F_ℓ has genus $1 + 3\|\ell\|^2$ and $C_s F_\ell = 2\|\ell\|^2$ (where s is a point of S and C_s is the incidence divisor).*

Proof. Let Σ be the sum of the 30 elliptic curves on S. We have

$$\Sigma F_\ell = \sum_{i,j,\beta} F_\ell E_{ij}^\beta = \sum_{i,j,\beta} |\ell(e_i - \beta e_j)|^2 = 12\|\ell\|^2.$$

Because F_ℓ is a fiber, we have $F_\ell^2 = 0$. Since Σ is twice a canonical divisor [5], we deduce that F_ℓ has genus $1 + \frac{1}{2}(0 + \frac{1}{2}\Sigma F_\ell) = 1 + 3\|\ell\|^2$.

The divisor $3C_s$ is numerically equivalent to a canonical divisor [5]. Therefore $C_s F_\ell = 2\|\ell\|^2$. □

We identify the Chern class of a divisor of the Abelian variety A with an alternating form on the tangent space $H^0(\Omega_S)^*$ of A (cf. [3, Thm. 2.12]). Let Θ be the principal polarization defined in Section 1.3.

LEMMA 21. *The Chern class of Θ is equal to*

$$a \frac{i}{\sqrt{3}} \sum_{j=1}^{5} dx_j \wedge d\bar{x}_j,$$

where a is a scalar and $i^2 = -1$.

Proof. Let H be the matrix (in the basis $e_1, \dots, e_5$) of the Hermitian form associated to $c_1(\Theta)$ (see [3, Lemma 2.17]). The automorphism τ' induced by $\tau \in \mathrm{Aut}(S)$ preserves the polarization Θ (Lemma 10). This implies that, for all $M = (m_{jk})_{1\le j,k\le 5} \in G(3,3,5)$, we have

$$^{t}MH\bar{M} = H,$$

where $\bar{M}$ is the matrix $\bar{M} = (\bar{m}_{jk})_{1\le j,k\le 5}$. This proves that

$$H = \frac{2}{\sqrt{3}} a I_5,$$

where I_5 is the identity matrix and $a \in \mathbb{C}$. Therefore, $c_1(\Theta) = a\frac{i}{\sqrt{3}} \sum_{j=1}^{5} dx_j \wedge d\bar{x}_j$. □

Because $H_1(A,\mathbb{Z}) \cap \mathbb{C}(e_1 - e_2) = \mathbb{Z}[\alpha](e_1 - e_2)$, the Néron–Severi group of the elliptic curve $\mathbb{E}$ is the $\mathbb{Z}$-module generated by

$$\eta = \frac{i}{\sqrt{3}} dz \wedge d\bar{z},$$

where z is the coordinate on the space $\mathbb{C}(e_1 - e_2)$.

Let $\ell = a_1x_1 + \cdots + a_5x_5$ be an element of Λ_A^*. The pull-back of the form η by the morphism $\Gamma_\ell \colon A \to \mathbb{E}$ is

$$\Gamma_\ell^*\eta = \frac{i}{\sqrt{3}} d\ell \wedge d\bar{\ell}.$$

The form $\Gamma_\ell^*\eta$ is the Chern class of the divisor Γ_ℓ^*0 and $\gamma_\ell^*\eta = \vartheta^*\Gamma_\ell^*\eta$ is the Chern class of the divisor F_ℓ.

LEMMA 22. *Let ℓ and ℓ' be two elements of Λ_A^*. Then*

$$F_\ell F_{\ell'} = \|\ell\|^2\|\ell'\|^2 - \langle \ell,\ell'\rangle\langle \ell',\ell\rangle$$

and $c_1(\Theta) = \frac{i}{\sqrt{3}} \sum_{i=1}^{i=5} dx_i \wedge d\bar{x}_i$.

Proof. By Theorem 11, $\vartheta^*c_1(\Theta)$ is the Chern class of the divisor $2C_s$ $(s \in S)$ and

$$2C_sF_\ell = \vartheta^*c_1(\Theta)\vartheta^*\Gamma_\ell^*\eta = \int_A \frac{1}{3!} \bigwedge^4 c_1(\Theta) \wedge \Gamma_\ell^*\eta;$$

as a result, we have

$$2C_sF_\ell = \left(\frac{i}{\sqrt{3}}\right)^5 \int_A \Big(\sum a_j dx_j\Big) \wedge \Big(\sum \bar{a}_j d\bar{x}_j\Big) \wedge 4a^4 \sum_{1\le k\le 5} \Big(\bigwedge_{j\neq k} (dx_j \wedge d\bar{x}_j)\Big)$$

and

$$2C_sF_\ell = \left(\frac{4}{a} \sum_{k=1}^{k=5} a_k\bar{a}_k\right) \frac{1}{5!} \int_A \bigwedge^5 c_1(\Theta).$$

Since Θ is a principal polarization, we have $\frac{1}{5!}\int_A \bigwedge^5 c_1(\Theta) = 1$; hence $2C_sF_\ell = \frac{4}{a}\|\ell\|^2$. We have seen in Lemma 20 that $C_sF_\ell = 2\|\ell\|^2$. Thus we deduce that $a = 1$.

By Theorem 11, for $\ell = a_1x_1 + \cdots + a_5x_5$ and $\ell' = b_1x_1 + \cdots + b_5x_5 \in \Lambda_A^*$,

$$F_\ell F_{\ell'} = \int_A \frac{1}{3!} \bigwedge^3 c_1(\Theta) \wedge \Gamma_\ell^* \eta \wedge \Gamma_{\ell'}^* \eta.$$

Since

$$\frac{1}{3!}\left(\frac{i}{\sqrt{3}}\right)^2 d\ell \wedge d\bar{\ell} \wedge d\ell' \wedge d\bar{\ell}' \wedge \big(\bigwedge^3 c_1(\Theta)\big) = \left(\sum_{k\neq j} a_k\bar{a}_k b_j\bar{b}_j - a_k\bar{a}_j b_j\bar{b}_k\right)\frac{1}{5!}\bigwedge^5 c_1(\Theta),$$

the result follows. □

Let ℓ and ℓ' be two linearly independent elements of Λ_A^*. The degree of the morphism $\tau_{\ell,\ell'} = (\gamma_\ell, \gamma_{\ell'})$ is equal to $F_\ell F_{\ell'}$ because $\tau_{\ell,\ell'}^*(\mathbb{E} \times \{0\}) = F_{\ell'} \in \mathrm{NS}(S)$, $\tau_{\ell,\ell'}^*(\{0\} \times \mathbb{E}) = F_\ell \in \mathrm{NS}(S)$, and the intersection number of the divisors $\{0\} \times \mathbb{E}$ and $\mathbb{E} \times \{0\}$ is equal to 1.

This completes the proof of Theorem 18. □

3.2.2. *Period Lattice of A.*
We compute here the period lattice of the Albanese variety A in the basis $e_1, \ldots, e_5$.

THEOREM 23.

(1) *The lattice $H_1(A, \mathbb{Z})$ is equal to*:

$$\mathbb{Z}[\alpha](e_1 - e_5) + \mathbb{Z}[\alpha](e_2 - e_5) + \mathbb{Z}[\alpha](e_3 - e_5) + \mathbb{Z}[\alpha](e_4 - e_5) + \frac{1+\alpha}{1-\alpha}\mathbb{Z}[3\alpha](\alpha^2 e_1 + \alpha^2 e_2 + \alpha e_3 + \alpha e_4 + e_5).$$

(2) *The variety A is isomorphic to $\mathbb{E}^4 \times \mathbb{E}'$, where $\mathbb{E} = \mathbb{C}/\mathbb{Z}[\alpha]$ and $\mathbb{E}' = \mathbb{C}/\mathbb{Z}[3\alpha]$.*

(3) *The image of the morphism $\vartheta^*: \mathrm{NS}(A) \to \mathrm{NS}(S)$ is the sublattice of rank* 25 *and discriminant $2^2 3^{18}$ generated by the divisors*

$$F_{x_i - \beta^2 x_j} = C_s - E_{ij}^\beta, \quad 1 \le i < j \le 5, \ \beta \in \mu_3, \ \sum_{i<j} E_{ij}^1.$$

Proof. The group $G(3,3,5)$ acts on $H_1(A, \mathbb{Z})$, and

$$H_1(A, \mathbb{Z}) \cap \mathbb{C}(e_i - \beta e_j) = \mathbb{Z}[\alpha](e_i - \beta e_j);$$

hence $H_1(A, \mathbb{Z})$ contains the lattice

$$\Lambda_0 = \sum_{i \le j, \beta \in \mu_3} \mathbb{Z}[\alpha](e_i - \beta e_j).$$

For $1 \le i < j \le 5$ and $\beta \in \mu_3$, the differential of $\Gamma_{x_i - \beta x_j}$ is the morphism $x \to (x_i - \beta x_j)(e_1 - e_2)$. Thus,

$$\lambda_i - \beta\lambda_j \in \mathbb{Z}[\alpha] \quad \forall \lambda = (\lambda_1, \ldots, \lambda_5) \in H_1(A, \mathbb{Z}).$$

Let us define

$$\Lambda = \{x = (x_1, \ldots, x) \in \mathbb{C}^5 / x_i - \beta x_j \in \mathbb{Z}[\alpha],\ 1 \le i < j \le 5,\ \beta \in \mu_3\}.$$

This lattice Λ contains $H_1(A, \mathbb{Z})$ and is equal to

$$\mathbb{Z}[\alpha]e_1 \oplus \cdots \oplus \mathbb{Z}[\alpha]e_4 \oplus \frac{1}{\alpha - 1}\mathbb{Z}[\alpha]w,$$

where $w = e_1 + \cdots + e_5$. Let $\phi \colon \Lambda \to \Lambda/\Lambda_0$ be the quotient map. The group Λ/Λ_0 is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^2$ and contains 6 subgroups. The reciprocal images of these groups are the lattices

$$\Lambda_0 = \phi^{-1}(0), \qquad \Lambda_{\alpha^2} = \Lambda_0 + \frac{\alpha^2}{\alpha - 1}\mathbb{Z}w,$$

$$\Lambda_1 = \Lambda_0 + \frac{1}{\alpha - 1}\mathbb{Z}w, \qquad \Lambda_{\alpha-1} = \Lambda_0 + \mathbb{Z}w,$$

$$\Lambda_\alpha = \Lambda_0 + \frac{\alpha}{\alpha - 1}\mathbb{Z}w, \qquad \Lambda = \Lambda_0 + \frac{1}{\alpha - 1}\mathbb{Z}[\alpha]w.$$

These are the 6 lattices Λ' that verify $\Lambda_0 \subset \Lambda' \subset \Lambda$, so the lattice $H_1(A, \mathbb{Z})$ must be equal to one of them.

Let ω be the alternating form $\omega = \frac{i}{\sqrt{3}} \sum_{k=1}^{k=5} dx_k \wedge d\bar{x}_k$ (see Lemma 22). We have

$$\frac{1}{\alpha - 1}w,\ \frac{\alpha}{\alpha - 1}w \in \Lambda.$$

However,

$$\omega\left(\frac{1}{\alpha - 1}w,\ \frac{\alpha}{\alpha - 1}w\right) = -\frac{5}{3}$$

is not an integer and so Λ is different from $H_1(A, \mathbb{Z})$.

The Pfaffian of $c_1(\Theta)$ relative to $H_1(A, \mathbb{Z})$ is equal to 1 because Θ is a principal polarization. The Pfaffian of ω relative to the lattice Λ_0 is equal to 9; hence Λ_0 is different from $H_1(A, \mathbb{Z})$.

We have $\Lambda_{1-\alpha} = \bigoplus \mathbb{Z}[\alpha]e_i$, and the principally polarized Abelian variety $(\mathbb{C}^5/\Lambda_{1-\alpha}, \omega)$ is isomorphic to a product of Jacobians. Since $(A, c_1(\Theta))$ cannot be isomorphic to a product of Jacobians ([5], 0.12), it follows that $H_1(A, \mathbb{Z}) \neq \Lambda_{1-\alpha}$.

The lattice Λ_{α^i} is equal to

$$\mathbb{Z}[\alpha](e_1 - e_5) + \mathbb{Z}[\alpha](e_2 - e_5) + \mathbb{Z}[\alpha](e_3 - e_5) + \mathbb{Z}[\alpha](e_4 - e_5)$$
$$+ \frac{\alpha^i}{1 - \alpha}\mathbb{Z}[3\alpha](\alpha^2 e_1 + \alpha^2 e_2 + \alpha e_3 + \alpha e_4 + e_5).$$

The lattices Λ_1 and Λ_α depend on a choice of α such that $\alpha^2 + \alpha + 1 = 0$, hence the lattice $H_1(A, \mathbb{Z})$ is equal to Λ_{α^2}.

Let

$$u_1 = e_1 - e_2, \quad u_2 = e_2 - e_3, \quad u_3 = e_3 - e_4, \quad u_4 = e_4 - e_5,$$
$$u_5 = \frac{\alpha^2}{1 - \alpha}(\alpha^2 e_1 + \alpha^2 e_2 + \alpha e_3 + \alpha e_4 + e_5).$$

The Hermitian form $H' = \frac{2}{\sqrt{3}} I_5$ in the basis $u_1, \ldots, u_5$ defines a principal polarization of A. Let $\mathrm{End}^s(A)$ be the group of symmetrical morphisms for the Rosati involution associated to H'. An endomorphism of A can be represented by a size-5 matrix M in the basis $u_1, \dots, u_5$. The symmetrical endomorphisms satisfy ${}^tMH' = H'\bar{M}$; that is, ${}^tA = \bar{A}$. Since we already know $H_1(A, \mathbb{Z})$, we can easily compute a basis $\mathcal{B}$ of $\mathrm{End}^s(A)$.

By [3, Prop. 5.2.1] and [3, Rem. 5.2.2], the map

$$\begin{aligned}\phi_{H'}\colon \mathrm{End}^s(A) &\to \mathrm{NS}(A)\\ M &\mapsto \Im m(\cdot {}^tMH'\bar{\cdot})\end{aligned}$$

is an isomorphism of groups. We obtain the base of the Néron–Severi group of A by taking the image by $\phi_{H'}$ of the base $\mathcal{B}$. Then we get the image of the morphism $\vartheta^*\colon \mathrm{NS}(A) \to \mathrm{NS}(S)$. □

REMARK 24. By Theorem 23, the variety A is biregular to $E^4 \times E'$. Yet by [5, (0.12)], this cannot be an isomorphism of principally polarized Abelian varieties.

3.2.3. *Study of Some Fibrations, Remarks*

Let X be a smooth surface, C a smooth curve, and $\gamma\colon X \to C$ a fibration with connected fibers. A point of X is called a *critical point of* γ if it is a zero of the differential

$$d\gamma\colon T_X \to \gamma^* T_C.$$

A fiber of γ is singular at a point if and only if this point is a critical point [1, Chap. III, Sec. 8].

Let us suppose that C is an elliptic curve. The critical points of γ are then the zeros of the form $\gamma^*\omega \in H^0(X, \Omega_X)$, where ω is a generator of the trivial sheaf Ω_C.

Let us assume that the cotangent sheaf of X is generated by global sections. There are morphisms

$$\begin{array}{ccc}\mathbb{P}(T_X) & \xrightarrow{\psi} & \mathbb{P}(H^0(X, \Omega_X)^*)\\ {\scriptstyle\pi}\downarrow & & \\ X & & \end{array}$$

where π is the natural projection and the morphism ψ, called the *cotangent map* [13], is defined by $\pi_*\psi^*\mathcal{O}(1) = \Omega_X$.

A point x of X is a critical point of γ if and only if the line $L_x = \psi(\pi^{-1}(x))$ lies in the hyperplane

$$\{\gamma^*\omega = 0\} \hookrightarrow \mathbb{P}(H^0(X, \Omega_X)^*).$$

The initial motivation for studying Fano surfaces is that, for those surfaces, the cotangent map is known: it is the projection map of the universal family of lines. The example of Fano surfaces illustrates that the knowledge of the image of the cotangent map is powerful for the study of a surface.

Let S be the Fano surface of the Fermat cubic F.

NOTATION 25. For $1 \le i < j \le 5$, we define $B_{ij} = B_{ji} = \sum_{\beta \in \mu_3} E_{ij}^{\beta}$.

■ Let $1 \leq i \leq 5$ and $j < r < s < t$ be such that $\{i, j, r, s, t\} = \{1, 2, 3, 4, 5\}$. We define $\ell_i = (1-\alpha)x_i \in \Lambda_A^*$.

COROLLARY 26. *The fibration $\gamma_{\ell_i}\colon S \to \mathbb{E}$ is stable and has connected fibers of genus* 10, *and its only singular fibers are*

$$B_{jr} + B_{st}, \quad B_{js} + B_{rt}, \quad B_{jt} + B_{rs}.$$

The 27 *intersection points of the curves E_{jr}^{β} and E_{st}^{τ} $(\beta, \gamma \in \mu_3)$ constitute the set of critical points of this fibration.*

Proof. By Theorem 18(a), a fiber of γ_{ℓ_i} has genus $1 + 3|1-\alpha|^2 = 10$.

Let $\beta \in \mu_3$, $h, k \in \{j, r, s, t\}$, and $h < k$. The form ℓ_i is zero on the space $\mathbb{C}(e_h - \beta e_k)$; hence E_{hk}^{β} is contracted to a point and is a component of the fiber of γ_{ℓ_i}.

The divisor $D_1 = B_{jr} + B_{st}$ is connected, satisfies $(B_{jr} + B_{st})^2 = 0$, and has genus 10. Its irreducible components are contracted by γ_{ℓ_i}. Hence, it is a fiber and γ_{ℓ_i} has connected fibers. Likewise, the divisors $D_2 = B_{js} + B_{rt}$ and $D_3 = B_{jt} + B_{rs}$ are fibers of γ_{ℓ_i}.

The 27 lines inside the intersection of the Fermat cubic and the hyperplane $\{\ell_i = 0\}$ correspond to the 27 intersection points of the curves E_{hk}^{β} and E_{lm}^{γ} such that $h < k$, $l < m$, and $\{i, h, k, l, m\} = \{1, 2, 3, 4, 5\}$. These 27 critical points lie in the fibers D_1, D_2, D_3, which are thus the only singular fibers of γ_{ℓ_i}.

The singularities of D_1, D_2, and D_3 are double ordinary, and the surface possesses no rational curve. Therefore, the fibration is stable. □

■ Let $(a_1, \dots, a_5) \in \mu_3^5$ be such that $a_1 \dots a_5 = 1$, and let

$$\ell = (1-\alpha)(a_1 x_1 + \cdots + a_5 x_5) \in \Lambda_A^*.$$

COROLLARY 27. *The divisor*

$$D = \sum_{1 \leq i < j \leq 5} E_{ij}^{a_i/a_j}$$

is a singular fiber of the Stein factorization of γ_ℓ.

Proof. The connected divisor D satisfies $D^2 = 0$ and has genus 16. Moreover, by Theorem 18, the irreducible components of D are contracted by γ_ℓ.

Let $w \in H^0(\Omega_S)^*$ be $w = e_1 + \cdots + e_5$. Then

$$H^1(A, \mathbb{Z}) \cap \mathbb{C}w = \frac{\alpha^2}{1-\alpha}\mathbb{Z}[3\alpha]w.$$

The morphism $x \to (a_1 x_1 + \cdots + a_5 x_5)w \in \mathrm{End}(H^0(\Omega_S)^*)$ is an element of $\mathbb{Z}[G(3,3,5)]$. It is the differential of a morphism $\Gamma_\ell'\colon A \to \mathbb{E}'$, where $\mathbb{E}' = \big(\mathbb{C}/\frac{\alpha^2}{1-\alpha}\mathbb{Z}[3\alpha]\big)w$.

The morphism Γ_ℓ has a factorization by Γ_ℓ' and a degree-3 isogeny between $\mathbb{E}'$ and $\mathbb{E}$. The divisor D is a connected fiber of the morphism $\vartheta \circ \Gamma_\ell'$, the Stein factorization of γ_ℓ. □

■ The curve $E_{ij}^{\beta^2}$ is the closed set of critical points of the fibration

$$\gamma_{(1-\alpha)(x_i+\beta x_j)}.$$

This fibration has only one singular fiber, and this fiber is not reduced.

■ We can construct an infinite number of fibrations with 9 sections and that contract 9 elliptic curves. Let us take $a \in \mathbb{Z}[\alpha]$ and

$$\ell = x_1 - (1 + (1-\alpha)a)x_2 \in \Lambda_A^*.$$

COROLLARY 28. *The* 9 *curves* E_{13}^{β}, E_{14}^{β}, *and* E_{15}^{β} $(\beta \in \mu_3)$ *are sections of* γ_ℓ. *The* 9 *curves* E_{34}^{β}, E_{35}^{β}, *and* E_{45}^{β} $(\beta \in \mu_3)$ *are contracted.*

Proof. This follows from Theorem 18 and the following equalities for $\beta \in \mu_3$:

$$|\ell(e_1 - \beta e_3)| = |\ell(e_1 - \beta e_4)| = |\ell(e_1 - \beta e_5)| = 1,$$
$$|\ell(e_3 - \beta e_4)| = |\ell(e_3 - \beta e_5)| = |\ell(e_4 - \beta e_5)| = 0.$$

Since $F_\ell E_{12}^1 = 1$, the fibration γ_ℓ has connected fibers. □

3.3. The Néron–Severi Group of the Fano Surface of the Fermat Cubic

Let S be the Fano surface of the Fermat cubic, and let $\mathrm{NS}(S)$ be the Néron–Severi group of S.

THEOREM 29. *The Néron–Severi group of S has rank* $25 = \dim H^1(S, \Omega_S)$. *The* 30 *elliptic curves generate an index-*3 *sublattice of* $\mathrm{NS}(S)$. *The group* $\mathrm{NS}(S)$ *is generated by these* 30 *curves and the class of an incidence divisor* C_s $(s \in S)$; *it has discriminant* 3^{18}. *The relations between the* 30 *elliptic curves in* $\mathrm{NS}(S)$ *are generated by*

$$B_{jr} + B_{st} = B_{js} + B_{rt} = B_{jt} + B_{rs}$$

for $1 \le j < r < s < t \le 5$.

Proof. By Theorem 17, we know the intersection matrix $\mathcal{I}$ of the 30 elliptic curves. As we can verify, the matrix $\mathcal{I}$ has rank 25. The intersection matrix of the 25 elliptic curves *other* than the 5 curves

$$E_{13}^{\alpha},\ E_{15}^{\alpha},\ E_{24}^{\alpha},\ E_{34}^{\alpha},\ E_{45}^{\alpha}$$

has determinant equal to 3^{20}, and these 25 curves form a $\mathbb{Z}$-basis of the lattice generated by the 30 elliptic curves.

By Theorem 23, the image of the morphism

$$\mathrm{NS}(A) \xrightarrow{\vartheta_*} \mathrm{NS}(S)$$

is a lattice of discriminant $2^2 3^{18}$ generated by the class of $C_s - E_{ij}^{\beta}$ and by $\sum_{1 \le i < j \le 5} E_{ij}^1$. Theorem 11 implies that $\mathrm{NS}(S)$ is generated by these classes and the class of an incidence divisor C_s. This lattice is also generated by the classes of the 30 elliptic curves and C_s. □

We remark that Corollary 26 gives a geometric interpretation of the numerical equivalence relations of Theorem 29.

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