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A Strong Comparison Principle for Plurisubharmonic Functions with Finite Pluricomplex Energy

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1. Introduction

Let Ω be a hyperconvex domain in $\mathbb{C}^n$—in other words, Ω is a bounded, open, and connected subset of $\mathbb{C}^n$ and there exists a continuous plurisubharmonic (psh) function ρ in Ω such that $\{z \in \Omega : \rho(z) < -c\} \subset\subset \Omega$ for any constant $c > 0$. Denote by $\mathrm{PSH}(\Omega)$ the set of psh functions in Ω and by $\mathrm{PSH}^-(\Omega)$ the set of nonpositive psh functions in Ω. Write $d = \partial + \bar\partial$ and $d^c = i(\bar\partial - \partial)$. The complex Monge–Ampère operator $(dd^c)^n$ is well-defined on all locally bounded psh functions; see Bedford and Taylor's fundamental paper [BeT]. It plays a central role in pluripotential theory just as the Laplace operator does in classical potential theory. We refer to excellent surveys [Be; Ki2] and books [Kl; K] for references. The monotone convergence theorem and the comparison principle of Bedford and Taylor are both of theoretical interest and extremely useful in pluripotential theory. They are used in almost all papers dealing with the Monge–Ampère operator. We know that the comparison principle not only gives the uniqueness theorem of the Dirichlet problem for the Monge–Ampère operator but also is one of main tools in solving Monge–Ampère equations. In [X1] we obtained the following type of comparison theorem.

Strong Comparison Principle. *Let $u, v \in \mathrm{PSH}(\Omega) \cap L^\infty(\Omega)$ be such that $\liminf_{z\to\partial\Omega}(u(z) - v(z)) \ge 0$. Then for any constant $r \ge 0$ and all $w_j \in \mathrm{PSH}(\Omega)$ with $-1 \le w_j \le 0$, $j = 1, 2, \dots, n$, we have*

$$\frac{1}{(n!)^2}\int_{u<v}(v-u)^n\, dd^c w_1 \wedge \cdots \wedge dd^c w_n + \int_{u<v}(r - w_1)(dd^c v)^n \le \int_{u<v}(r - w_1)(dd^c u)^n.$$

The strong comparison principle has many applications (see [X1; X2; X3]) and, moreover, it implies several important inequalities in pluripotential theory. Let's show some of its direct consequences.

Proposition 1 (First version of Bedford and Taylor's comparison principle; see [BeT]). *If $u, v \in \mathrm{PSH} \cap L^\infty(\Omega)$ satisfy $\liminf_{z\to\partial\Omega}(u(z) - v(z)) \ge 0$, then*

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$$\int_{u<v} (dd^c v)^n \le \int_{u<v} (dd^c u)^n.$$

Proof. Dividing both sides of the inequality of the strong comparison principle by r and letting $r \to \infty$, we get the required inequality. □

PROPOSITION 2 (Second version of Bedford and Taylor's comparison principle; see [BeT]). *If $u, v \in \mathrm{PSH} \cap L^\infty(\Omega)$ satisfy $(dd^c v)^n \ge (dd^c u)^n$ in Ω and if*

$$\liminf_{z\to\partial\Omega} (u(z) - v(z)) \ge 0,$$

then $u \ge v$ in Ω.

Proof. Since $\{u + \varepsilon < v\} \subset\subset \Omega$ for any $\varepsilon > 0$, applying the strong comparison principle for $u + \varepsilon$, v, and $w_1 = w_2 = \cdots = w_n = |z|^2(\sup_\Omega |z|^2)^{-1} - 1$ yields that $u + \varepsilon \ge v$ almost everywhere with respect to the Lebesgue measure, which implies the required result. □

PROPOSITION 3 (Cegrell inequality; [Ce2]). *Let $w \in \mathrm{PSH}^-(\Omega)$. If $u, v \in \mathcal{E}_0(\Omega)$ with $v \ge u$ in Ω, then*

$$\int_\Omega (-w)(dd^c v)^n \le \int_\Omega (-w)(dd^c u)^n.$$

Proof. Applying the strong comparison principle for u and δv with $\delta < 1$, $w_1 = \max(w/s, -1)$, and $r = 0$ and then letting $\delta \nearrow 1$, we get the inequality

$$\int_\Omega -\max(w, -s)(dd^c v)^n \le \int_\Omega -\max(w, -s)(dd^c u)^n.$$

Letting $s \to \infty$, we obtain the required result. □

REMARK. For w in $\mathcal{E}_0(\Omega)$, Proposition 3 yields a basic fact. Based on this fact, Cegrell [Ce2] found the largest subclass $\mathcal{E}(\Omega)$ of $\mathrm{PSH}^-(\Omega)$ on which the Monge–Ampère operator is well-defined in some sense.

PROPOSITION 4 (Blocki Inequality; [B]). *If $u, v, w_1, w_2, \ldots, w_n \in \mathrm{PSH} \cap L^\infty(\Omega)$ are such that $v \ge u$ in Ω, $\lim_{z\to\partial\Omega}(v(z) - u(z)) = 0$, and $0 \ge w_j \ge -1$, $j = 1, 2, \ldots, n$, then*

$$\int_\Omega (v - u)^n \, dd^c w_1 \wedge \cdots \wedge dd^c w_n \le (n!)^2 \int_\Omega (-w_1)(dd^c u)^n.$$

Proof. Use the strong comparison principle for $r = 0$. □

It is known that the nonlinear Monge–Ampère operator cannot be reasonably defined on the whole class $\mathrm{PSH}(\Omega)$; see Kiselman's paper [Ki1]. However, $(dd^c u)^n$ can make sense on certain classes of unbounded psh functions u in such a way that $(dd^c)^n$ is continuous under monotone limits and the comparison principle is valid. In [Ce1], Cegrell introduced the class $\mathcal{F}_p(\Omega)$ of psh functions whose Monge–Ampère measures have finite total masses as well as the larger class $\mathcal{E}_p(\Omega)$ of psh functions whose Monge–Ampère measures may not have finite total masses. He gave a complete description of Monge–Ampère measures $(dd^c\mathcal{F}_p)^n$ and proved comparison theorems of Bedford and Taylor type for functions in $\mathcal{F}_p(\Omega)$. However, comparison theorems like the strong comparison principle or those of Bedford and Taylor type are less useful in $\mathcal{E}_p(\Omega)$ because, for functions in $\mathcal{E}_p(\Omega)$, the integrals given in these types of comparison theorems are supposed to be infinite except in the special case where $\{u < v\}$ locates in some compact subset of Ω. So, a more natural and effective generalization of the strong comparison principle and the comparison principle of Bedford and Taylor to the class $\mathcal{E}_p(\Omega)$ should include only finite integrals. In this paper our principal aim is to prove such a comparison theorem.

THEOREM 2. *Let $u, v \in \mathcal{E}_p(\Omega)$ with $p > 0$. Then for all constants $r \geq 0$ and all $w_1, w_j \in \mathrm{PSH}^-(\Omega) \cap L^\infty(\Omega)$ with $w_j \geq -1$, $j = 2, 3, \dots, n$, the inequality*

$$A_{n,p}\int_{u<v}(v-u)^{n+p}\, dd^c w_1 \wedge \cdots \wedge dd^c w_n + \int_{u<v}(r-w_1)(v-u)^p (dd^c v)^n \leq \int_{u<v}(r-w_1)(v-u)^p (dd^c u)^n$$

holds, where the constant $A_{n,p} = (n!\,(n+p)(n+p-1)\cdots(p+1))^{-1}$.

It is worth pointing out that, in order to retain finiteness of the integrals in Theorem 2, the number p in the exponents cannot be replaced by another because for each p one can always find a function $u \in \mathcal{E}_p(\Omega)$ with $u \notin \mathcal{E}_{p_1}(\Omega)$ for all $p_1 \neq p$. As applications of Theorem 2 we obtain a modification of the Bedford and Taylor comparison principle in $\mathcal{E}_p(\Omega)$ as well as an estimate of the sublevel of functions in $\mathcal{E}_p(\Omega)$ that strengthens a result in [CeKZ].

In this paper we also study convergence theorems for the Monge–Ampère operator in $\mathcal{E}_p(\Omega)$. It is known [Ce1] that $(dd^c u_j)^n \to (dd^c u)^n$ if $u_j \in \mathcal{E}_p(\Omega)$ converges monotonically to $u \in \mathcal{E}_p(\Omega)$. Now our result is as follows.

COROLLARY 3. *Suppose that $u, u_j \in \mathcal{E}_p(\Omega)$, $p > 0$, are such that u_j converges monotonically to u in Ω as $j \to \infty$. Then, for each $0 \leq p_1 < \infty$, the weak convergence $(-\phi)^{p_1}(dd^c u_j)^n \to (-\phi)^{p_1}(dd^c u)^n$ is uniform for all $\phi \in \mathrm{PSH}(\Omega)$ with $-1 \leq \phi \leq 0$. In other words, for any $\psi \in C_0^\infty(\Omega)$ we have that*

$$\int_\Omega \psi(-\phi)^{p_1}(dd^c u_j)^n \longrightarrow \int_\Omega \psi(-\phi)^{p_1}(dd^c u)^n \ \text{ as } j \to \infty$$

uniformly for all $\phi \in \mathrm{PSH}(\Omega)$ with $-1 \leq \phi \leq 0$ in Ω.

2. An Energy Estimate in $\mathcal{E}_p(\Omega)$

In this section we give an energy estimate in $\mathcal{E}_p(\Omega)$, which is useful for the sequel. Denote by $\mathcal{E}_0(\Omega)$ the class of functions u in $\mathrm{PSH}^-(\Omega) \cap L^\infty(\Omega)$ such that $\lim_{z\to\zeta} u(z) = 0$ for all $\zeta \in \partial\Omega$ and $\int_\Omega (dd^c u)^n < \infty$. Following [Ce1], for each $p \geq 0$ we denote by $\mathcal{E}_p(\Omega)$ the class of psh functions u in Ω such that there exists a decreasing sequence $u_j \in \mathcal{E}_0(\Omega)$ with $u_j \searrow u$ as $j \to \infty$ and $\sup_j \int_\Omega (-u_j)^p (dd^c u_j)^n < \infty$. If furthermore the u_j can be chosen so that $\sup_j \int_\Omega (dd^c u_j)^n < \infty$, then we write $u \in \mathcal{F}_p(\Omega)$. It is known that $\mathcal{F}_p(\Omega) \subset \mathcal{F}_{p_1}(\Omega)$ if $p \geq p_1$, but the corresponding property for $\mathcal{E}_p(\Omega)$ is not true. In [Ce1] Cegrell gave one characterization of a positive measure that is the Monge–Ampère measure of some function in $\mathcal{F}_p$. One crucial tool used in his paper is the following energy estimate [Ce1; Pe].

PROPOSITION 5. *Suppose that $u, v \in \mathcal{E}_0(\Omega)$ and $p \geq 1$. Then for each $0 \leq j \leq n$ there exists a $D_{j,p} > 0$ such that*

$$\int_\Omega (-u)^p (dd^c u)^j \wedge (dd^c v)^{n-j} \leq D_{j,p} e_p(u)^{(p+j)/(p+n)} e_p(v)^{(n-j)/(p+n)},$$

where the pluricomplex p-energy of u is defined by $e_p(u) := \int_\Omega (-u)^p (dd^c u)^n$.

Proposition 5 was extended to $1 > p > 0$ in [ACP]. In fact, the energy estimate holds for all functions in $\mathcal{E}_p(\Omega)$.

PROPOSITION 6. *Suppose that $u_0, u_1, \ldots, u_n \in \mathcal{E}_p(\Omega)$ and $p \geq 0$. Then*

$$\int_\Omega (-u_0)^p \, dd^c u_1 \wedge \cdots \wedge dd^c u_n \leq D_{n,p} e_p(u_0)^{p/(p+n)} e_p(u_1)^{1/(p+n)} \cdots e_p(u_n)^{1/(p+n)},$$

where the constants $D_{n,p} > 0$ depend only on n and p and where, moreover, $D_{n,p} \leq 1$ when $0 \leq p \leq 1$.

REMARK. Proposition 6 has been proved in the case of $p = 0$; see [Ce2, Cor. 5.6].

Proof of Proposition 6. Assume $p > 0$. By [CeKZ, Lemma 2.1] there exist sequences $\{u_{kj}\}$ of functions in $\mathcal{E}_0(\Omega)$ such that $u_{kj} \searrow u_k$ in Ω as $j \nearrow \infty$ and $\lim_{j\to\infty} e_p(u_{kj}) = e_p(u_k)$ for each $0 \leq k \leq n$. From [Pe, Thm. 3.4] and [ACP, Thm. 3.3] it turns out that

$$\int_\Omega (-u_{0j})^p \, dd^c u_{1j} \wedge \cdots \wedge dd^c u_{nj} \leq D_{n,p} e_p(u_{0j})^{p/(p+n)} e_p(u_{1j})^{1/(p+n)} \cdots e_p(u_{nj})^{1/(p+n)}.$$

By [Ce2, Thm. 4.2] we know that measures $dd^c u_{1j} \wedge \cdots \wedge dd^c u_{nj}$ converge weakly to $dd^c u_1 \wedge \cdots \wedge dd^c u_n$ in Ω. On the other hand, since the sequence $\{(-u_{0j})^p\}$ of lower semicontinuous functions decreases to $(-u_0)^p$ in Ω, letting $j \to \infty$ yields the required inequality and so the proof is complete. $\square$

COROLLARY 1. *If $u \in \mathrm{PSH}^-(\Omega)$ and $v \in \mathcal{E}_p(\Omega)$, $p \geq 0$, satisfy $u \geq v$ in Ω, then $u \in \mathcal{E}_p(\Omega)$ and $e_p(u) \leq D_{n,p}^{(p+n)/p} e_p(v)$, where $D_{n,p}$ is the same constant as in Proposition 6.*

Proof. By [ACP; Ce1; Ce2] we have $u \in \mathcal{E}_p(\Omega)$. It then follows from Proposition 6 that

$$e_p(u) \le \int_\Omega (-v)^p (dd^c u)^n \le D_{n,p} e_p(v)^{p/(p+n)} e_p(u)^{n/(p+n)},$$

which implies that $e_p(u)^{p/(p+n)} \le D_{n,p} e_p(v)^{p/(p+n)}$; hence we have proved Corollary 1. □

3. A Convergence Theorem in $\mathcal{E}_p(\Omega)$

In this section we prove a convergence theorem for functions in $\mathcal{E}_p(\Omega)$. Let C_n be the inner capacity given by Bedford and Taylor in [BeT] and as defined by $C_n(E) = C_n(E,\Omega) = \sup\bigl\{\int_E (dd^c u)^n : u \in \mathrm{PSH}(\Omega),\ -1 \le u \le 0\bigr\}$ for all Borel subsets E of Ω. Recall that a sequence of functions u_j is said to be convergent to a function u in C_n on a set E if for any $\delta > 0$ we have that $C_n\{z \in E : |u_j(z) - u(z)| > \delta\} \longrightarrow 0$ as $j \to \infty$. It was proved in [Ce1] that $(dd^c u_j)^n \to (dd^c u)^n$ if $u, u_j \in \mathcal{E}_p(\Omega)$ are such that $u_j \nearrow u$ or $u_j \searrow u$ in Ω. Now we prove a stronger result.

THEOREM 1. *Suppose that $u_k, u_{kj} \in \mathcal{E}_p(\Omega)$, $p > 0$, $1 \le k \le n$, and that $\sup_{j,k} e_p(u_{kj}) < \infty$ and $u_{kj} \to u_k$ in C_n on each $E \subset\subset \Omega$ as $j \to \infty$. Then the following statements hold.*

(a) *If a locally uniformly bounded sequence $\{\phi_j\}$ in $\mathrm{PSH}^-(\Omega)$ converges weakly to a psh function ϕ in Ω, then for any constant $0 \le p_1 < \infty$ we have that*

$$(-\phi_j)^{p_1} dd^c u_{1j} \wedge \cdots \wedge dd^c u_{nj} \longrightarrow (-\phi)^{p_1} dd^c u_1 \wedge \cdots \wedge dd^c u_n$$

weakly in Ω.

(b) *Let B be a locally uniformly bounded subset of $\mathrm{PSH}^-(\Omega)$ and let $0 \le p_1 < \infty$. Then for any $\psi \in C_0^\infty(\Omega)$ we have that*

$$\int_\Omega \psi(-\phi)^{p_1}\, dd^c u_{1j} \wedge \cdots \wedge dd^c u_{nj} \longrightarrow \int_\Omega \psi(-\phi)^{p_1}\, dd^c u_1 \wedge \cdots \wedge dd^c u_n$$

uniformly for all ϕ in B.

(c) *For any $0 \le p_1 < p$ we have that*

$$(-u_{0j})^{p_1} dd^c u_{1j} \wedge \cdots \wedge dd^c u_{nj} \longrightarrow (-u_0)^{p_1} dd^c u_1 \wedge \cdots \wedge dd^c u_n$$

weakly in Ω.

(d) *If $\int_\Omega (-u_{0j})^p\, dd^c u_{1j} \wedge \cdots \wedge dd^c u_{nj} \longrightarrow \int_\Omega (-u_0)^p\, dd^c u_1 \wedge \cdots \wedge dd^c u_n$, then*

$$(-u_{0j})^p dd^c u_{1j} \wedge \cdots \wedge dd^c u_{nj} \longrightarrow (-u_0)^p dd^c u_1 \wedge \cdots \wedge dd^c u_n$$

weakly in Ω.

Proof. To prove (a), for any $c > 0$ we write

$$\begin{aligned}
&(-\phi_j)^{p_1} dd^c u_{1j} \wedge \cdots \wedge dd^c u_{nj} - (-\phi)^{p_1} dd^c u_1 \wedge \cdots \wedge dd^c u_n \\
&\quad = (-\phi_j)^{p_1} [dd^c u_{1j} \wedge \cdots \wedge dd^c u_{nj} \\
&\qquad\qquad - dd^c \max(u_{1j}, -c) \wedge \cdots \wedge dd^c \max(u_{nj}, -c)] \\
&\qquad + (-\phi_j)^{p_1} [dd^c \max(u_{1j}, -c) \wedge \cdots \wedge dd^c \max(u_{nj}, -c) \\
&\qquad\qquad - dd^c \max(u_1, -c) \wedge \cdots \wedge dd^c \max(u_n, -c)] \\
&\qquad + (-\phi_j)^{p_1} [dd^c \max(u_1, -c) \wedge \cdots \wedge dd^c \max(u_n, -c) \\
&\qquad\qquad - dd^c u_1 \wedge \cdots \wedge dd^c u_n] \\
&\qquad + ((-\phi_j)^{p_1} - (-\phi)^{p_1}) dd^c u_1 \wedge \cdots \wedge dd^c u_n \\
&\quad := A^1_{c,j} + A^2_{c,j} + A^3_{c,j} + A^4_j.
\end{aligned}$$

From the inequality $dd^c u_1 \wedge \cdots \wedge dd^c u_n \le (dd^c(u_1 + \cdots + u_n))^n$ with $u_1, \ldots, u_n \in \mathcal{E}_p(\Omega)$ and [X4, Cor. 1], it turns out that for each $E \subset\subset \Omega$ we have

$$\int_E |\phi_j - \phi| \, dd^c u_1 \wedge \cdots \wedge dd^c u_n \longrightarrow 0 \text{ as } j \to \infty.$$

If $0 < p_1 < 1$ then, by using Hölder inequality, we get that

$$\begin{aligned}
&\int_E |(-\phi_j)^{p_1} - (-\phi)^{p_1}| \, dd^c u_1 \wedge \cdots \wedge dd^c u_n \\
&\quad \le \int_E |\phi_j - \phi|^{p_1} \, dd^c u_1 \wedge \cdots \wedge dd^c u_n \\
&\quad \le \left(\int_E dd^c u_1 \wedge \cdots \wedge dd^c u_n \right)^{1-p_1} \left(\int_E |\phi_j - \phi| \, dd^c u_1 \wedge \cdots \wedge dd^c u_n \right)^{p_1} \\
&\qquad\qquad \longrightarrow 0 \text{ as } j \to \infty.
\end{aligned}$$

If $p_1 \ge 1$ then, by the inequality $|a^{p_1} - b^{p_1}| \le p_1 a^{p_1 - 1} |a - b|$ for $a \ge b \ge 0$, we also have that

$$\begin{aligned}
&\int_E |(-\phi_j)^{p_1} - (-\phi)^{p_1}| \, dd^c u_1 \wedge \cdots \wedge dd^c u_n \\
&\qquad \le p_1 \sup_{j,\Omega} |\phi_j|^{p_1 - 1} \int_E |\phi_j - \phi| \, dd^c u_1 \wedge \cdots \wedge dd^c u_n \longrightarrow 0 \text{ as } j \to \infty.
\end{aligned}$$

Hence we have obtained that $A^4_j \to 0$ weakly in Ω as $j \to \infty$. Given $\psi \in C_0^\infty(\Omega)$, by [X4, Lemma 2] or [ACP, Prop. 4.1] it follows that

$$\begin{aligned}
\left| \int_\Omega \psi A^1_{c,j} \right| = \Bigg| \sum_{q=1}^n \int_\Omega & \psi (-\phi_j)^{p_1} \, dd^c u_{1j} \wedge \cdots \wedge dd^c u_{(q-1)j} \\
& \wedge [dd^c u_{qj} - dd^c \max(u_{qj}, -c)] \\
& \wedge dd^c \max(u_{(q+1)j}, -c) \wedge \cdots \wedge dd^c \max(u_{nj}, -c) \Bigg|
\end{aligned}$$

$$= \Bigg| \sum_{q=1}^{n} \int_{u_{qj} \le -c} \psi(-\phi_j)^{p_1}\, dd^c u_{1j}$$
$$\wedge \cdots \wedge [dd^c u_{qj} - dd^c \max(u_{qj}, -c)] \wedge \cdots \wedge dd^c \max(u_{nj}, -c) \Bigg|$$
$$\le c^{-p} \max_{j,\Omega}(|\psi|(-\phi_j)^{p_1}) \sum_{q=1}^{n} \int_{u_{qj} \le -c} (-u_{qj})^p\, dd^c u_{1j}$$
$$\wedge \cdots \wedge [dd^c u_{qj} + dd^c \max(u_{qj}, -c)] \wedge \cdots \wedge dd^c \max(u_{nj}, -c),$$

which, by Proposition 6 and then Corollary 1, does not exceed

$$c^{-p} \max_{j,\Omega}(|\psi|(-\phi_j)^{p_1}) \sum_{q=1}^{n} D_{n,p}(D_{n,p}^{(n-q)/p} + D_{n,p}^{(n-q+1)/p})$$
$$\cdot\, e_p(u_{qj})^{p/(p+n)} e_p(u_{1j})^{1/(p+n)} \cdots e_p(u_{nj})^{1/(p+n)}$$
$$\le c^{-p} \max_{j,\Omega}(|\psi|(-\phi_j)^{p_1}) \sup_{j,k} e_p(u_{kj}) \sum_{q=1}^{n} D_{n,p}(D_{n,p}^{(n-q)/p} + D_{n,p}^{(n-q+1)/p})$$
$$\longrightarrow 0 \ \text{ as } c \to \infty.$$

We have obtained that $\left|\int_\Omega \psi A^1_{c,j}\right| \to 0$ uniformly for all j as $c \to \infty$. Similarly, we also have that $\left|\int_\Omega \psi A^3_{c,j}\right| \to 0$ uniformly for all j as $c \to \infty$. We claim now that $A^2_{c,j} \to 0$ weakly in Ω as $j \to \infty$ for each fixed $c > 0$. If the claim is true then the proof of (a) is complete. To prove the claim, write $p_1 = l + s$, where l is an integer and $0 \le s < 1$. Hence $(-\phi_j)^{p_1} = (-1)^{l+1}\phi_j^l(-(-\phi_j)^s)$. Since ϕ_j and $-(-\phi_j)^s$ are locally uniformly bounded psh functions, by subtracting a constant if necessary we can assume that they are positive psh functions. Then, applying the equality $2fg = (f+g)^2 - f^2 - g^2$ step by step, we get that $(-\phi_j)^{p_1}$ can be written as a sum of finite terms of the form $\pm h$, where the h are locally uniformly bounded psh functions in Ω. Hence from [X4, Thm. 1] it turns out that $A^2_{c,j} \to 0$ weakly as $j \to \infty$ for each fixed $c > 0$ and the claim is proved. We have obtained (a). We omit the proof of (b) because it is similar to the proof of (a). Now for each $c > 0$ we write

$$(-u_{0j})^{p_1} dd^c u_{1j} \wedge \cdots \wedge dd^c u_{nj} - (-u_0)^{p_1} dd^c u_1 \wedge \cdots \wedge dd^c u_n$$
$$= [(-u_{0j})^{p_1} - (-\max(u_{0j}, -c))^{p_1}] dd^c u_{1j} \wedge \cdots \wedge dd^c u_{nj}$$
$$+ [(-\max(u_{0j}, -c))^{p_1} dd^c u_{1j} \wedge \cdots \wedge dd^c u_{nj}$$
$$- (-\max(u_0, -c))^{p_1} dd^c u_1 \wedge \cdots \wedge dd^c u_n]$$
$$+ [(-\max(u_0, -c))^{p_1} - (-u_0)^{p_1}] dd^c u_1 \wedge \cdots \wedge dd^c u_n$$
$$:= B^{1,p_1}_{c,j} + B^{2,p_1}_{c,j} + B^{3,p_1}_{c}.$$

To prove (c), by Proposition 6 and Lebesgue's dominated convergence theorem we obtain that $B^{3,p_1}_c \to 0$ weakly in Ω as $c \to \infty$. On the other hand, let χ_E be the characteristic function of a set E. Then we have

$$0 \le B^{1,p_1}_{c,j} \le \chi_{\{u_{0j} < -c\}}[(-u_{0j})^{p_1} - c^{p_1}] dd^c u_{1j} \wedge \cdots \wedge dd^c u_{nj}$$
$$\le c^{p_1 - p}(-u_{0j})^p dd^c u_{1j} \wedge \cdots \wedge dd^c u_{nj}$$

as measures, where by Proposition 6 the total masses of the $(-u_{0j})^p dd^c u_{1j}\wedge\cdots\wedge dd^c u_{nj}$ are uniformly bounded for all j. Therefore, $B^{1,p_1}_{c,j}\to 0$ weakly in Ω as $c\to\infty$ uniformly for all j. Finally, for all fixed c large enough, by (a) we get that $B^{2,p_1}_{c,j}\to 0$ weakly in Ω as $j\to\infty$. Hence we have obtained (c).

To prove (d), it follows from (a) that

$$(-\max(u_{0j},-c))^p dd^c u_{1j}\wedge\cdots\wedge dd^c u_{nj} \longrightarrow (-\max(u_0,-c))^p dd^c u_1\wedge\cdots\wedge dd^c u_n$$

as $j\to\infty$. Thus we have

$$\liminf_{j\to\infty}\int_\Omega(-\max(u_{0j},-c))^p dd^c u_{1j}\wedge\cdots\wedge dd^c u_{nj} \ge \int_\Omega(-\max(u_0,-c))^p dd^c u_1\wedge\cdots\wedge dd^c u_n.$$

Hence, by the limit assumption we obtain that

$$\limsup_{j\to\infty}\int_\Omega B^{1,p}_{c,j}\le -\int_\Omega B^{3,p}_c\le\int_{u_0<-c}(-u_0)^p\, dd^c u_1\wedge\cdots\wedge dd^c u_n\longrightarrow 0$$

as $c\to\infty$. Therefore, for any $\varepsilon>0$ there exist constants c_0 and j_0 large enough such that $\int_\Omega B^{1,p}_{c_0,j}-\int_\Omega B^{3,p}_{c_0}\le\varepsilon$ for all $j\ge j_0$. It then follows from (a) that $B^{2,p}_{c_0,j}\to 0$ weakly as $j\to\infty$. By the arbitrariness of $\varepsilon>0$ we have proved (d), so the proof of Theorem 1 is complete. □

As a consequence of Theorem 1 and Corollary 1 we have the following result.

COROLLARY 2. *Suppose that* $u_k, u_{kj}\in \mathrm{PSH}^-(\Omega)$, $0\le k\le n$, *are such that* $u_{kj}\to u_k$ *in* C_n *on each* $E\subset\subset\Omega$ *as* $j\to\infty$. *If there exist* g_k *in* $\mathcal{E}_p(\Omega)$, $p>0$, $0\le k\le n$, *such that* $u_{kj}\ge g_k$ *for all* j *and* k, *then the following statements hold.*

(a) *For any* $0\le p_1<\infty$ *and* $\psi\in C_0^\infty(\Omega)$ *we have that*

$$\int_\Omega\psi(-\phi)^{p_1}\, dd^c u_{1j}\wedge\cdots\wedge dd^c u_{nj}\longrightarrow\int_\Omega\psi(-\phi)^{p_1}\, dd^c u_1\wedge\cdots\wedge dd^c u_n$$

uniformly for all $\phi\in\mathrm{PSH}(\Omega)$ *with* $-1\le\phi\le 0$ *in* Ω.

(b) *For any* $0\le p_1<p$ *we have that*

$$(-u_{0j})^{p_1}dd^c u_{1j}\wedge\cdots\wedge dd^c u_{nj}\longrightarrow(-u_0)^{p_1}dd^c u_1\wedge\cdots\wedge dd^c u_n$$

weakly in Ω.

(c) *If* $\int_\Omega(-u_{0j})^p\, dd^c u_{1j}\wedge\cdots\wedge dd^c u_{nj}\to\int_\Omega(-u_0)^p\, dd^c u_1\wedge\cdots\wedge dd^c u_n$, *then*

$$(-u_{0j})^p dd^c u_{1j}\wedge\cdots\wedge dd^c u_{nj}\longrightarrow(-u_0)^p dd^c u_1\wedge\cdots\wedge dd^c u_n$$

weakly in Ω.

Hence, by quasicontinuity of psh functions and the Dini theorem, we have our next corollary.

COROLLARY 3. *Suppose that* $u_k, u_{kj} \in \mathcal{E}_p(\Omega)$, $p > 0$, *and that for each* $0 \le k \le n$ *we have that* $u_{kj} \searrow u_k$ *or* $u_{kj} \nearrow u_k$ *in* Ω *as* $j \nearrow \infty$. *Then we have* (a), (b), *and* (c) *as in Corollary 2.*

4. A Strong Comparison Principle in $\mathcal{E}_p(\Omega)$

In this section we prove a strong comparison principle for functions in the class $\mathcal{E}_p(\Omega)$. Since Monge–Ampère measures of functions in $\mathcal{E}_p(\Omega)$ may have infinite total masses in Ω, a natural and more useful generalization of the strong comparison principle to this class is the following type of comparison theorem, in which all the integrals are finite.

THEOREM 2. *Let* $u, v \in \mathcal{E}_p(\Omega)$ *with* $p > 0$. *Then for all constants* $r \ge 0$ *and all* $w_1, w_j \in \mathrm{PSH}^-(\Omega) \cap L^\infty(\Omega)$ *with* $w_j \ge -1$, $j = 2, 3, \dots, n$, *the inequality*

$$A_{n,p} \int_{u<v} (v-u)^{n+p}\, dd^c w_1 \wedge \cdots \wedge dd^c w_n + \int_{u<v} (r - w_1)(v-u)^p (dd^c v)^n \le \int_{u<v} (r - w_1)(v-u)^p (dd^c u)^n$$

holds, where the constant $A_{n,p} = (n!\,(n+p)(n+p-1)\cdots(p+1))^{-1}$.

Recall that a family of positive measures is said to be uniformly absolutely continuous with respect to C_n in a set E if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that, for each Borel subset $E_1 \subset E$ with $C_n(E_1) < \delta$, the inequality $\mu(E_1) < \varepsilon$ holds for all measures μ in the family. First we need to prove a lemma.

LEMMA 1. *Suppose that* $v, u \in \mathcal{E}_p(\Omega)$ *with* $p > 0$, $w \in \mathrm{PSH}^-(\Omega) \cap L^\infty(\Omega)$, *and* ρ *is a continuous exhaustion function for* Ω *vanishing on* $\partial\Omega$. *We let*

$$w_{v,u,s} = \max(w, s\rho, -s(-v)^{p-p_1}, -s(-u)^{p-p_1})$$

for all constants $s > 0$, $p_1 > 0$, *and* $0 < p - p_1 < 1$. *Then the following statements hold.*

(a) *The measures* $(-w_{v,u,s})|v_1 - u_1|^{p_1}(dd^c v_2)^n$ *have uniformly bounded total masses in* Ω *for all* $v_1, v_2, u_1 \in \mathcal{E}_p(\Omega)$ *with* $v_1, v_2 \ge v$ *and* $u_1 \ge u$ *in* Ω.
(b) *The measures* $(-w_{v,u,s})|v_1 - u_1|^{p_1}(dd^c v_2)^n$ *are uniformly absolutely continuous in* Ω *with respect to* C_n *for all* $v_1, v_2, u_1 \in \mathcal{E}_p(\Omega)$ *with* $v_1, v_2 \ge v$ *and* $u_1 \ge u$ *in* Ω.
(c) *For any* $\varepsilon > 0$ *there exists a closed subset* F *in* Ω *such that*

$$\int_{\Omega \setminus F} (-w_{v,u,s})|v_1 - u_1|^{p_1}(dd^c v_2)^n < \varepsilon$$

for all $v_1, v_2, u_1 \in \mathcal{E}_p(\Omega)$ *with* $v_1, v_2 \ge v$ *and* $u_1 \ge u$ *in* Ω.
(d) *If a sequence* $\{v_j\}$ *in* $\mathcal{E}_0(\Omega) \cap C(\bar{\Omega})$ *decreases to* v *in* Ω, *then for any* $\phi \in C_0^\infty(\Omega)$ *we have that*

$$\int_\Omega \phi(-w_{v,u,s})|v_1-u_1|^{p_1}(dd^c v_j)^n \longrightarrow \int_\Omega \phi(-w_{v,u,s})|v_1-u_1|^{p_1}(dd^c v)^n \text{ as } j\to\infty$$

uniformly for all $v_1, u_1 \in \mathcal{E}_p(\Omega)$ *with* $v_1 \geq v$ *and* $u_1 \geq u$ *in* Ω.

Proof. It is no restriction to assume that $e_p(v) \leq 1$ and $e_p(u) \leq 1$. By Corollary 1 we have that $e_p(v_2) \leq D_{n,p}^{(p+n)/p}$. From Proposition 6 it follows that

$$\begin{aligned}\int_\Omega (-w_{v,u,s})|v_1-u_1|^{p_1}(dd^c v_2)^n &\leq s\int_\Omega (-u)^{p-p_1}(-u_1)^{p_1}(dd^c v_2)^n\\ &\leq s\int_\Omega (-u)^p(dd^c v_2)^n \leq sD_{n,p}e_p(u)^{p/(p+n)}e_p(v_2)^{n/(p+n)} \leq sD_{n,p}^{1+n/p} < \infty,\end{aligned}$$

which yields (a).

To prove (b) we observe that

$$\begin{aligned}\int_{u\leq -c} (-w_{v,u,s})|v_1-u_1|^{p_1}(dd^c v_2)^n &\leq \sup_\Omega|w|\int_{u\leq -c}(-u_1)^{p_1}(dd^c v_2)^n\\ &\leq c^{p_1-p}\sup_\Omega|w|\int_{u\leq -c}(-u)^p(dd^c v_2)^n \leq c^{p_1-p}\sup_\Omega|w|D_{n,p}^{1+n/p}\end{aligned}$$

for all constants $c > 0$. Similarly, we get that

$$\int_{v\leq -c}(-w_{v,u,s})|v_1-u_1|^{p_1}(dd^c v_2)^n \leq c^{p_1-p}\sup_\Omega|w|D_{n,p}^{1+n/p}.$$

Hence, by [X4, Lemma 2], for any $E \subset \Omega$ we have

$$\begin{aligned}\int_E (-w_{v,u,s})|v_1-u_1|^{p_1}(dd^c v_2)^n &\leq 2c^{p_1-p}\sup_\Omega|w|D_{n,p}^{1+n/p} + \sup_\Omega|w|\int_{E\cap\{v,u>-c\}}(-u)^{p_1}(dd^c v_2)^n\\ &= 2c^{p_1-p}\sup_\Omega|w|D_{n,p}^{1+n/p}\\ &\quad + \sup_\Omega|w|\int_{E\cap\{v_2,u>-c\}}(-\max(u,-c))^{p_1}(dd^c \max(v_2,-c))^n\\ &\leq 2c^{p_1-p}\sup_\Omega|w|D_{n,p}^{1+n/p} + \sup_\Omega|w|c^{p_1+n}C_n(E).\end{aligned}$$

Given $\varepsilon > 0$, take $c > 0$ such that the first term on the right-hand side in the last inequality is less than ε. Then choose $\delta > 0$ such that the second term on the right-hand side is also less than ε for all $E \subset \Omega$ with $C_n(E) < \delta$. Therefore, we have obtained (b). Assertion (c) follows from the proof of (b) since $w_{v,u,s}$ vanishes on $\partial\Omega$.

To prove (d), by [Ce2, Thm. 2.1] there exist sequences $\{v_{1,t}\}, \{u_{1,t}\}$ in $\mathcal{E}_0(\Omega) \cap C(\bar{\Omega})$ such that $v_{1,t} \searrow v$ and $u_{1,t} \searrow u$ in Ω as $t \nearrow \infty$. By Corollary 1 we have that $v_{1,t}, u_{1,t} \in \mathcal{E}_p(\Omega)$ for all t. Write

$$
\begin{aligned}
&(-w_{v,u,s})|v_1 - u_1|^{p_1}(dd^c v_j)^n - (-w_{v,u,s})|v_1 - u_1|^{p_1}(dd^c v)^n \\
&\quad = (-w_{v,u,s})[|v_1 - u_1|^{p_1}(dd^c v_j)^n \\
&\qquad\qquad - |\max(v_1, -c) - \max(u_1, -c)|^{p_1}(dd^c \max(v_j, -c))^n] \\
&\qquad + (-w_{v,u,s})[|\max(v_1, -c) - \max(u_1, -c)|^{p_1} \\
&\qquad\qquad - |\max(v_{1,t}, -c) - \max(u_{1,t}, -c)|^{p_1}](dd^c \max(v_j, -c))^n \\
&\qquad + (-w_{v,u,s})|\max(v_{1,t}, -c) \\
&\qquad\qquad - \max(u_{1,t}, -c)|^{p_1}[(dd^c \max(v_j, -c))^n - (dd^c \max(v, -c))^n] \\
&\qquad + (-w_{v,u,s})[|\max(v_{1,t}, -c) - \max(u_{1,t}, -c)|^{p_1} \\
&\qquad\qquad - |\max(v_1, -c) - \max(u_1, -c)|^{p_1}](dd^c \max(v, -c))^n \\
&\qquad + (-w_{v,u,s})[|\max(v_1, -c) - \max(u_1, -c)|^{p_1}(dd^c \max(v, -c))^n \\
&\qquad\qquad - |v_1 - u_1|^{p_1}(dd^c v)^n] \\
&\quad := A^1_{c,j} + A^2_{c,j,t} + A^3_{c,j,t} + A^4_{c,t} + A^5_c.
\end{aligned}
$$

It follows from [X4, Lemma 2] that for all j the signed measures $A^1_{c,j}$ have zero mass on the set $\{v_1 > -c, u_1 > -c, v_j > -c\} \supset \{v > -c, u > -c\}$. So by (b) we obtain that $\|A^1_{c,j}\|_\Omega \to 0$ as $c \to \infty$ uniformly for all j, where $\|A^1_{c,j}\|_\Omega$ denotes the mass of total variation of the $A^1_{c,j}$ on Ω. Similarly, we get that $\|A^5_c\|_\Omega \to 0$ as $c \to \infty$. Hence for any $\varepsilon > 0$ there exists a $c_0 > 0$ such that $\|A^1_{c_0,j}\|_\Omega + \|A^5_{c_0}\|_\Omega < \varepsilon$ for all j. On the other hand, since $v_{1,t} \searrow v$ and $u_{1,t} \searrow u$ we have (see the proof of Theorem 1) that

$$
|\max(v_{1,t}, -c) - \max(u_{1,t}, -c)|^{p_1} \longrightarrow |\max(v_1, -c) - \max(u_1, -c)|^{p_1}
$$

in C_n on each $E \subset\subset \Omega$ as $t \to \infty$. Then, by the definition of C_n, it is easy to show that there exists a t_0 such that $\|A^2_{c_0,j,t_0}\|_\Omega + \|A^4_{c_0,t_0}\|_\Omega < \varepsilon$ for all j. Finally, since the function $|\max(v_{1,t_0}, -c_0) - \max(u_{1,t_0}, -c_0)|^{p_1}$ is continuous in Ω and since $w_{v,u,s} \in \mathrm{PSH}^-(\Omega) \cap L^\infty(\Omega)$, by [X4, Thm. 1] we obtain that $A^3_{c_0,j,t_0} \to 0$ weakly in Ω as $j \to \infty$. Thus, we have proved (d) and so the proof of Lemma 1 is complete. $\square$

Proof of Theorem 2. Let

$$
w^1_{v,u,s} = \max(w_1 - r, s\rho, -s(-v)^{p-p_1}, -s(-u)^{p-p_1})
$$

for $s > 0$, $p_1 > 0$, and $0 < p - p_1 < 1$. Then $w^1_{v,u,s} \in \mathrm{PSH}^-(\Omega) \cap L^\infty(\Omega)$ and $w^1_{v,u,s} = 0$ on $\partial\Omega$.

(1) First we prove that, for all $u, v \in \mathcal{E}_p(\Omega) \cap C(\bar{\Omega})$, the inequality

$$A_{n,p_1} \int_{u<v} (v-u)^{n+p_1}\, dd^c w^1_{v,u,s} \wedge dd^c w_2 \wedge \cdots \wedge dd^c w_n + \int_{u<v} (-w^1_{v,u,s})(v-u)^{p_1}(dd^c v)^n \le \int_{u<v} (-w^1_{v,u,s})(v-u)^{p_1}(dd^c u)^n \quad \text{(A)}$$

holds for all $s > 0$, $p_1 > 0$, and $0 < p - p_1 < 1$. To prove this we begin by noting that $u \in \mathcal{E}_0(\Omega)$ because $\limsup_{z\to\zeta} u(z) = 0$ for all $\zeta \in \partial\Omega$, and similarly we have $v \in \mathcal{E}_0(\Omega)$. Hence, it is no restriction to assume that $u < v$ in Ω and that $u = v$ on $\partial\Omega$. Let $v_\varepsilon = \max(u, v-\varepsilon)$. Then $v_\varepsilon \nearrow v$ in Ω as $\varepsilon \searrow 0$ and $v_\varepsilon = u$ near $\partial\Omega$. Now integration by parts (see [B; X1]) yields that

$$\begin{aligned}
&\int_\Omega (v_\varepsilon - u)^{n+p_1} dd^c w^1_{v,u,s} \wedge dd^c w_2 \wedge \cdots \wedge dd^c w_n \\
&= (n+p_1)(n+p_1-1) \int_\Omega w_n (v_\varepsilon - u)^{n+p_1-2}\, d(v_\varepsilon - u) \\
&\quad \wedge d^c(v_\varepsilon - u) \wedge dd^c w^1_{v,u,s} \wedge \cdots \wedge dd^c w_{n-1} \\
&\quad + (n+p_1) \int_\Omega w_n (v_\varepsilon - u)^{n+p_1-1}\, dd^c(v_\varepsilon - u) \wedge dd^c w^1_{v,u,s} \wedge \cdots \wedge dd^c w_{n-1} \\
&\le (n+p_1) \int_\Omega (v_\varepsilon - u)^{n+p_1-1}\, dd^c(v_\varepsilon + u) \wedge dd^c w^1_{v,u,s} \wedge \cdots \wedge dd^c w_{n-1} \\
&\;\vdots \\
&\le (n+p_1)(n+p_1-1)\cdots(p_1+2) \int_\Omega (v_\varepsilon - u)^{p_1+1}\, dd^c(v_\varepsilon + u)^{n-1} \\
&\quad \wedge dd^c w^1_{v,u,s} \\
&\le n!\,(n+p_1)(n+p_1-1)\cdots(p_1+2) \int_\Omega (v_\varepsilon - u)^{p_1+1}\, dd^c w^1_{v,u,s} \\
&\quad \wedge \sum_{k=0}^{n-1} (dd^c v_\varepsilon)^{n-1-k} \wedge (dd^c u)^k \\
&\le A^{-1}_{n,p_1} \left(\int_\Omega (-w^1_{v,u,s})(v_\varepsilon - u)^{p_1}(dd^c u)^n - \int_\Omega (-w^1_{v,u,s})(v_\varepsilon - u)^{p_1}(dd^c v_\varepsilon)^n \right).
\end{aligned}$$

Therefore, we get the inequality

$$A_{n,p_1} \int_\Omega (v_\varepsilon - u)^{n+p_1}\, dd^c w^1_{v,u,s} \wedge dd^c w_2 \wedge \cdots \wedge dd^c w_n + \int_\Omega (-w^1_{v,u,s})(v_\varepsilon - u)^{p_1}(dd^c v_\varepsilon)^n \le \int_\Omega (-w^1_{v,u,s})(v-u)^{p_1}(dd^c u)^n.$$

Since $(v_\varepsilon - u)^{p_1} \to (v-u)^{p_1}$ uniformly in Ω as $\varepsilon \to 0$, it follows from [X4, Thm. 1] and Lemma 1(a) that

$$\begin{aligned}(-w^1_{v,u,s})(v_\varepsilon - u)^{p_1}(dd^c v_\varepsilon)^n \\ &= (-w^1_{v,u,s})[(v_\varepsilon-u)^{p_1} - (v-u)^{p_1}](dd^c v_\varepsilon)^n + (-w^1_{v,u,s})(v-u)^{p_1}(dd^c v_\varepsilon)^n \\ &\longrightarrow (-w^1_{v,u,s})(v-u)^{p_1}(dd^c v)^n\end{aligned}$$

weakly in Ω as $\varepsilon \searrow 0$, where we have used continuity of the $(v-u)^{p_1}$. Letting $\varepsilon \searrow 0$ in the last inequality and using the Lebesgue monotone convergence theorem, we obtain the required inequality for all $u, v \in \mathcal{E}_0(\Omega) \cap C(\bar{\Omega})$.

(2) Secondly, we prove that inequality (A) holds for all $u, v \in \mathcal{E}_p(\Omega)$. Toward this end, by [Ce2, Thm. 2.1] we know that there exist sequences $\{u_k\}$ and $\{v_j\}$ in $\mathcal{E}_0(\Omega) \cap C(\bar{\Omega})$ such that $u_k \searrow u$ and $v_j \searrow v$ in Ω. From Corollary 1 it follows that all $u_k, v_j \in \mathcal{E}_p(\Omega)$; then, by inequality (A), for all $s > 0$ and all $p_1 > 0$ with $0 < p - p_1 < 1$ we have

$$\begin{aligned}A_{n,p_1}\int_\Omega \chi_{\{u_k<v_j\}}(v_j-u_k)^{n+p_1}\, dd^c w^1_{v,u,s}\wedge\cdots\wedge dd^c w_n \\ + \int_{u_k<v_j}(-w^1_{v,u,s})(v_j-u_k)^{p_1}(dd^c v_j)^n \\ \le \int_{u_k<v_j}(-w^1_{v,u,s})(v_j-u_k)^{p_1}(dd^c u_k)^n,\end{aligned}$$

where $\chi_{\{u_k<v_j\}}$ is the characteristic function of the set $\{u_k < v_j\}$. Letting $k \to \infty$ and $j \to \infty$, by Fatou's lemma we get that the limit inferior of the first term in the left-hand side exceeds $A_{n,p_1}\int_{u<v}(v-u)^{n+p_1}\, dd^c w^1_{v,u,s}\wedge\cdots\wedge dd^c w_n$. Hence we have

$$\begin{aligned}A_{n,p_1}\int_{u<v}(v-u)^{n+p_1}\, dd^c w^1_{v,u,s}\wedge\cdots\wedge dd^c w_n \\ + \liminf_{j\to\infty}\liminf_{k\to\infty}\int_{u_k<v_j}(-w^1_{v,u,s})(v_j-u_k)^{p_1}(dd^c v_j)^n \\ \le \limsup_{j\to\infty}\limsup_{k\to\infty}\int_{u_k<v_j}(-w^1_{v,u,s})(v_j-u_k)^{p_1}(dd^c u_k)^n.\end{aligned}$$

It follows again from Fatou's lemma that

$$\begin{aligned}\liminf_{j\to\infty}\limsup_{k\to\infty}\int_{u_k<v_j}(-w^1_{v,u,s})(v_j-u_k)^{p_1}(dd^c v_j)^n \\ \ge \liminf_{j\to\infty}\limsup_{k\to\infty}\int_{u_k<v}(-w^1_{v,u,s})(v-u_k)^{p_1}(dd^c v_j)^n \\ \ge \liminf_{j\to\infty}\int_{u<v}(-w^1_{v,u,s})(v-u)^{p_1}(dd^c v_j)^n.\end{aligned}$$

Given $\varepsilon > 0$, it follows from Lemma 1(b) and the quasicontinuity of psh functions in [BeT] that there exist an open subset $O_\varepsilon \subset \Omega$ and a $v^1 \in C(\Omega)$ such that $\{v \neq v^1\} \subset O_\varepsilon$ and

$$\int_{O_\varepsilon}(-w^1_{v,u,s})|v-u|^{p_1}[(dd^cv_j)^n+(dd^cv)^n]<\varepsilon \quad \text{for all } j.$$

Since $\{u<v^1\}$ is open, by Lemma 1(d) we get that

$$\begin{aligned}\liminf_{j\to\infty}&\int_{u<v}(-w^1_{v,u,s})(v-u)^{p_1}(dd^cv_j)^n\\ &\geq \liminf_{j\to\infty}\int_{u<v^1}(-w^1_{v,u,s})|v-u|^{p_1}(dd^cv_j)^n-\varepsilon\\ &\geq \int_{u<v}(-w^1_{v,u,s})|v-u|^{p_1}(dd^cv)^n-2\varepsilon.\end{aligned}$$

Hence we have obtained that

$$\begin{aligned}\liminf_{j\to\infty}\limsup_{k\to\infty}&\int_{u_k<v_j}(-w^1_{v,u,s})(v_j-u_k)^{p_1}(dd^cv_j)^n\\ &\geq \int_{u<v}(-w^1_{v,u,s})(v-u)^{p_1}(dd^cv)^n-2\varepsilon.\end{aligned}$$

On the other hand, by (b) and (c) of Lemma 1 there exist $v_j^1, u^1\in C(\Omega)$, a closed subset F^ε, and an open subset O^ε in Ω such that $\{v_j\neq v_j^1\}\cup\{u\neq u^1\}\subset O^\varepsilon$ and

$$\begin{aligned}\int_{\Omega\setminus F^\varepsilon}&(-w^1_{v,u,s})|v_j-u|^{p_1}[(dd^cu_k)^n+(dd^cu)]\\ &+\int_{O^\varepsilon}(-w^1_{v,u,s})|v_j-u|^{p_1}(dd^cu_k)^n<\varepsilon\end{aligned}$$

for all j and k. Hence using $u_k\geq u$ yields

$$\begin{aligned}\limsup_{j\to\infty}\limsup_{k\to\infty}&\int_{u_k<v_j}(-w^1_{v,u,s})(v_j-u_k)^{p_1}(dd^cu_k)^n\\ &\leq \limsup_{j\to\infty}\limsup_{k\to\infty}\int_{u\leq v_j}(-w^1_{v,u,s})(v_j-u)^{p_1}(dd^cu_k)^n\\ &\leq \limsup_{j\to\infty}\limsup_{k\to\infty}\int_{\{u^1\leq v_j^1\}\cap F^\varepsilon}(-w^1_{v,u,s})(v_j-u)^{p_1}(dd^cu_k)^n+2\varepsilon,\end{aligned}$$

which by Lemma 1(d) does not exceed

$$\begin{aligned}\limsup_{j\to\infty}&\int_{\{u^1\leq v_j^1\}\cap F^\varepsilon}(-w^1_{v,u,s})(v_j-u)^{p_1}(dd^cu)^n+2\varepsilon\\ &\leq \limsup_{j\to\infty}\int_{u\leq v_j}(-w^1_{v,u,s})(v_j-u)^{p_1}(dd^cu)^n+4\varepsilon\\ &\leq \int_{u\leq v}(-w^1_{v,u,s})(v-u)^{p_1}(dd^cu)^n+4\varepsilon,\end{aligned}$$

where the last inequality follows from the Lebesgue monotone convergence theorem. Therefore, we have obtained that the inequality

$$A_{n,p_1}\int_{u<v}(v-u)^{n+p_1}\,dd^cw^1_{v,u,s}\wedge dd^cw_2\wedge\cdots\wedge dd^cw_n$$
$$+\int_{u<v}(-w^1_{v,u,s})(v-u)^{p_1}(dd^cv)^n$$
$$\leq\int_{u<v}(-w^1_{v,u,s})(v-u)^{p_1}(dd^cu)^n+6\varepsilon$$

holds for all $\varepsilon>0$, which yields inequality (A) for all $u,v\in\mathcal{E}_p(\Omega)$.

(3) Finally, let $u,v\in\mathcal{E}_p(\Omega)$. Then we have inequality (A) for u and v. Because $(-w^1_{v,u,s})(v-u)^{p_1}\leq s(-u)^p$, if we let $p_1\nearrow p$ then, by Corollary 1 and the Lebesgue dominated convergence theorem,

$$A_{n,p}\limsup_{p_1\nearrow p}\int_{u<v}(v-u)^{n+p_1}\,dd^cw^1_{v,u,s}\wedge dd^cw_2\wedge\cdots\wedge dd^cw_n$$
$$+\int_{u<v}(-w^1_{1,1,s})(v-u)^{p}(dd^cv)^n$$
$$\leq\int_{u<v}(-w^1_{1,1,s})(v-u)^{p}(dd^cu)^n.$$

Since $w^1_{1,1,s}\searrow w_1-r$ as $s\nearrow\infty$, it follows from the Lebesgue monotone convergence theorem that

$$A_{n,p}\limsup_{s\to\infty}\limsup_{p_1\nearrow p}\int_{u<v}(v-u)^{n+p_1}\,dd^cw^1_{v,u,s}\wedge dd^cw_2\wedge\cdots\wedge dd^cw_n$$
$$+\int_{u<v}(r-w_1)(v-u)^{p}(dd^cv)^n$$
$$\leq\int_{u<v}(r-w_1)(v-u)^{p}(dd^cu)^n.$$

It remains for us to prove that the limit in the last inequality exceeds

$$A_{n,p}\int_{u<v}(v-u)^{n+p}\,dd^cw_1\wedge dd^cw_2\wedge\cdots\wedge dd^cw_n.$$

By [X4, Lemma 2], for any $t>0$ we have that

$$\int_{u<v}(v-u)^{n+p_1}\,dd^cw^1_{v,u,s}\wedge dd^cw_2\wedge\cdots\wedge dd^cw_n$$
$$\geq\int_{-t<u<v}(v-u)^{n+p_1}\,dd^cw^1_{v,u,s}\wedge dd^cw_2\wedge\cdots\wedge dd^cw_n$$
$$=\int_{-t<\bar{u}<\bar{v}}(\bar{v}-\bar{u})^{n+p_1}\,dd^cw^1_{\bar{v},\bar{u},s}\wedge dd^cw_2\wedge\cdots\wedge dd^cw_n,$$

where $\bar{v}=\max(v,-t)$ and $\bar{u}=\max(u,-t)$. For any $\varepsilon>0$ we take an open subset $M_\varepsilon\subset\Omega$ and $v^2,u^2\in C(\Omega)$ such that $-t\leq v^2,u^2\leq 0$ in Ω, $\{\bar{v}\neq v^2\}\cup\{\bar{u}\neq u^2\}\subset M_\varepsilon$, and $C_n(M_\varepsilon)<\varepsilon$. Hence, by the definition of C_n, we get that the last integral exceeds

$$\begin{aligned}
&\int_{\{-t<u^2<v^2\}\setminus M_\varepsilon} (v^2-u^2)^{n+p_1}\, dd^c w^1_{\bar{v},\bar{u},s} \\
&\qquad \wedge dd^c w_2 \wedge \cdots \wedge dd^c w_n - \varepsilon t^{n+p_1} \sup_\Omega |w_1| \\
&\quad \geq \int_{\{-t<u^2<v^2\}} (v^2-u^2)^{n+p_1}\, dd^c w^1_{\bar{v},\bar{u},s} \\
&\qquad \wedge dd^c w_2 \wedge \cdots \wedge dd^c w_n - 2\varepsilon t^{n+p_1} \sup_\Omega |w_1|.
\end{aligned}$$

It is easy to see that $(v^2-u^2)^{n+p_1} \to (v^2-u^2)^{n+p}$ uniformly in Ω as $p_1 \nearrow p$ and that $w^1_{\bar{v},\bar{u},s} \to w^1_{1,1,s}$ uniformly in Ω as $p_1 \nearrow p$. Hence, by continuity of $(v^2-u^2)^{n+p}$ and [X1, Thm. 1], we get that

$$\begin{aligned}
&(v^2-u^2)^{n+p_1} dd^c w^1_{\bar{v},\bar{u},s} \wedge dd^c w_2 \wedge \cdots \wedge dd^c w_n \\
&\qquad\qquad \longrightarrow (v^2-u^2)^{n+p} dd^c w^1_{1,1,s} \wedge dd^c w_2 \wedge \cdots \wedge dd^c w_n
\end{aligned}$$

weakly in Ω as $p_1 \nearrow p$ and that

$$\begin{aligned}
&(v^2-u^2)^{n+p} dd^c w^1_{1,1,s} \wedge dd^c w_2 \wedge \cdots \wedge dd^c w_n \\
&\qquad\qquad \longrightarrow (v^2-u^2)^{n+p} dd^c w_1 \wedge dd^c w_2 \wedge \cdots \wedge dd^c w_n
\end{aligned}$$

weakly in Ω as $s \to \infty$. Therefore,

$$\begin{aligned}
&\limsup_{s\to\infty} \limsup_{p_1 \nearrow p} \int_{u<v} (v-u)^{n+p_1}\, dd^c w^1_{\bar{v},\bar{u},s} \wedge dd^c w_2 \wedge \cdots \wedge dd^c w_n \\
&\quad \geq \limsup_{s\to\infty} \limsup_{p_1 \nearrow p} \int_{\{-t<u^2<v^2\}} (v^2-u^2)^{n+p_1}\, dd^c w^1_{\bar{v},\bar{u},s} \\
&\qquad \wedge dd^c w_2 \wedge \cdots \wedge dd^c w_n - 2\varepsilon t^{n+p_1} \sup_\Omega |w_1| \\
&\quad \geq \int_{\{-t<u^2<v^2\}} (v^2-u^2)^{n+p}\, dd^c w_1 \wedge dd^c w_2 \wedge \cdots \wedge dd^c w_n - 2\varepsilon t^{n+p_1} \sup_\Omega |w_1| \\
&\quad \geq \int_{\{-t<u<v\}} (v-u)^{n+p}\, dd^c w_1 \wedge dd^c w_2 \wedge \cdots \wedge dd^c w_n - 4\varepsilon t^{n+p_1} \sup_\Omega |w_1|.
\end{aligned}$$

Letting $\varepsilon \to 0$ and then $t \to \infty$, we get the required inequality and so the proof of Theorem 2 is complete. □

As an application of Theorem 2 we obtain a comparison theorem in $\mathcal{E}_p(\Omega)$, which is a modification of the Bedford and Taylor comparison principle.

COROLLARY 4. *If $u, v \in \mathcal{E}_p(\Omega)$ with $p > 0$, then*

$$\int_{u<v} (v-u)^p (dd^c v)^n \leq \int_{u<v} (v-u)^p (dd^c u)^n.$$

Proof. Dividing both sides of the inequality in Theorem 2 by r and then letting $r \to \infty$, we obtain the required inequality. □

Moreover, we have the following statement.

Corollary 5. *If $u, v \in \mathcal{E}_p(\Omega)$ with $u \le v$ in Ω, then the inequality*

$$\int_\Omega -w(v-u)^p (dd^c v)^n \le \int_\Omega -w(v-u)^p (dd^c u)^n$$

holds for any $w \in \mathrm{PSH}^-(\Omega)$.

Proof. For $p = 0$ this is the Cegrell inequality. Assume $p > 0$. Applying Theorem 2 for $u, \delta v$ with $\delta < 1$, $w_1 = \max(w/s, -1)$, and $r = 0$ and then letting $\delta \nearrow 1$, we get the inequality

$$\int_\Omega -\max(w, -s)(v-u)^p (dd^c v)^n \le \int_\Omega -\max(w, -s)(v-u)^p (dd^c u)^n.$$

Then, letting $s \to \infty$ finishes the proof. □

Recall that a function $u \in \mathrm{PSH}^-(\Omega)$ is said to be in $\mathcal{E}(\Omega)$ if for each $z \in \Omega$ there exists a neighborhood O_z of z in Ω and a sequence $u_j \in \mathcal{E}_0(\Omega)$ such that $u_j \searrow u$ on O_z and $\sup_j \int_\Omega (dd^c u_j)^n < \infty$; see [Ce2].

Corollary 6. *If $v \in \mathcal{E}(\Omega)$ and $u \in \mathcal{E}_p(\Omega)$, $p > 0$, are such that $(dd^c u)^n \le (dd^c v)^n$ in Ω, then $u \ge v$ in Ω.*

Proof. Since $v \in \mathcal{E}_p(\Omega)$, so is $\max(u, v)$. By Demailly's inequality, we get that

$$(dd^c \max(u, v))^n \ge \chi_{\{u \ge v\}} (dd^c u)^n + \chi_{\{u<v\}} (dd^c v)^n \ge (dd^c u)^n$$

in Ω. Hence, using Theorem 2 for the functions $\max(u, v)$, u, and $w_1 = w_2 = \dots = w_n = |z|^2 (\sup_\Omega |z|^2)^{-1} - 1$, we get that $\int_{u<v} (v-u)^{n+p}\, d\lambda = 0$, where $d\lambda$ denotes the Lebesgue measure. Therefore, $v \ge u$ in Ω and the proof is complete. □

Remark. Following the proof of the Demailly inequality in [D], one can extend that inequality,

$$(dd^c \max(u, v))^n \ge \chi_{\{u \ge v\}} (dd^c u)^n + \chi_{\{u<v\}} (dd^c v)^n,$$

to $\mathcal{E}_p(\Omega)$ with $p > 0$.

Remark. See [Ce3, Thm. 3.12] for a more general version of Corollary 6.

As another direct consequence of Theorem 2, we obtain an estimate of the sublevel of functions in $\mathcal{E}_p(\Omega)$ that is slightly stronger than [CeKZ, Prop. 3.1].

Corollary 7. *If $u \in \mathcal{E}_p(\Omega)$ with $p > 0$, then*

$$C_n(\{z \in \Omega : u(z) < -s\}) \le 2^{n+p} A_{n,p}^{-1} s^{-n-p} \int_{u<-s/2} (-u)^p (dd^c u)^n,$$

where the integral tends to zero as $s \to \infty$.

Proof. For any $w \in \mathrm{PSH}(\Omega)$ with $-1 \le w \le 0$, we have that

$$\begin{aligned}\int_{u<-s}(dd^c w)^n &\le 2^{n+p}s^{-n-p}\int_{u<-s/2}\left(-\frac{s}{2}-u\right)^{n+p}(dd^c w)^n\\ &= 2^{n+p}s^{-n-p}\int_{u<\max(u,-s/2)}\left(\max\left(u,-\frac{s}{2}\right)-u\right)^{n+p}(dd^c w)^n.\end{aligned}$$

Since u and $\max(u,-s/2)$ are both in $\mathcal{E}_p(\Omega)$, by Theorem 2 we have that the last integral does not exceed

$$A_{n,p}^{-1}\int_{u<\max(u,-s/2)}\left(\max\left(u,-\frac{s}{2}\right)-u\right)^p(dd^c u)^n \le A_{n,p}^{-1}\int_{u<-s/2}(-u)^p(dd^c u)^n.$$

Hence we have proved the required inequality, so the proof of Corollary 7 is complete. □

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