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The Zeros of Flat Gaussian Random Holomorphic Functions on $\mathbb{C}^n$, and Hole Probability

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1. Introduction

Random polynomials and random holomorphic functions are studied as a way of gaining insight into difficult problems such as string theory and analytic number theory. A particularly interesting case of random holomorphic functions is when the functions can be defined so that they are invariant with respect to the natural isometries of the space in question. The class of functions that we will study are the unique Gaussian random holomorphic functions, up to multiplication by a nonzero holomorphic function, whose expected zero set is uniformly distributed on $\mathbb{C}^n$. Such functions are also known as the flat Gaussian random holomorphic functions. For a random holomorphic function of this class, we will determine the expected value of the unintegrated counting function for a ball of large radius as well as the chance that there are no zeros present—a pathological event that is known as a "hole". In so doing we generalize a result of Sodin and Tsirelison to n dimensions in order to give the first nontrivial example where the hole probability is computed in more than one complex variable.

The topic of random holomorphic functions is an old one, with many results from the first half of the twentieth century, that is recently experiencing a renaissance. In particular, Kac determined a formula for the expected distribution of zeros of real polynomials in a certain case [5], and this work was subsequently generalized throughout the years [3]. A series of papers by Offord [7; 8] is particularly relevant to questions involving the hole probability of random holomorphic functions and the distribution of values of random holomorphic functions. There has been a flurry of recent interest in the zero sets of random polynomials and holomorphic functions, which are much more natural objects than they may initially appear. For example, Bleher, Shiffman, and Zelditch [1] show that, for any positive line bundle over a compact complex manifold, the random holomorphic sections to L^N (defined intrinsically) have universal high N correlation functions.

In addition to results describing the typical behavior, there have also been several results in one (real or complex) dimension for Gaussian random holomorphic functions where the hole probability has been determined. For a specific class of

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real Gaussian polynomials of even degree $2n$, Dembo et al. [2] have shown that, for the event where there are no real zeros, E_n,

$$\lim_{n\to\infty} \frac{\mathrm{Prob}(E_n)}{\log(n)} n^{-b} = -b, \quad b \in [0.4, 2].$$

Hole probability for the complex zeros of a Gaussian random holomorphic function is a quite different problem. Let $\mathrm{Hole}_r = \{f$ in a class of random holomorphic functions such that, $\forall z \in B(0,r), f(z) \neq 0\}$. For the complex zeros in one complex dimension, there is a general upper bound for the hole probability: $\mathrm{Prob}(\mathrm{Hole}_r) \leq \exp\{-c\mu(B(0,r))\}$, where $\mu(z) = E[Z_{\psi_\omega}]$ (cf. [13, Thm. 2.2]). In one case this estimate was shown by Peres and Virag [9] to be sharp: $\mathrm{Prob}(\mathrm{Hole}_r) = \exp\{-\mu(B(0,r))/24 + o(\mu(B(0,r)))\}$. These last two results on hole probability might lead one to suppose that, if the random holomorphic functions are invariant with respect to the local isometries (thus ensuring that $E[Z_\omega]$ is uniformly distributed on the manifold), then the rate of decay of the hole probability would be the same as if the zeros were distributed according to a Poisson process with the same expected distribution. But the zeros repel in dimension 1 [4], so one might expect a quicker decay for the hole probability of a random holomorphic function. This is the case (see [14]) for random holomorphic functions whose expected zero set is uniformly distributed on $\mathbb{C}^1$:

$$\mathrm{Prob}(\mathrm{Hole}_r) \leq \exp\{-c_1 r^4\} = \exp\{-c\mu(B(0,r))^2\};$$

$$\mathrm{Prob}(\mathrm{Hole}_r) \geq \exp\{-c_2 r^4\} = \exp\{-c\mu(B(0,r))^2\}.$$

The two main results of this paper, Theorems 1.1 and 1.2, concern the distribution of zeros of flat Gaussian random holomorphic functions in n variables.

THEOREM 1.1. *Let*

$$\psi_\omega(z_1, z_2, \ldots, z_n) = \sum_j \omega_j \frac{z_1^{j_1} z_2^{j_2} \cdots z_n^{j_n}}{\sqrt{j_1! \cdot j_n!}},$$

where the ω_j are independent and identically distributed complex Gaussian random variables. Then, for all $\delta > 0$, there exist $c_{3,\delta} > 0$ and $R_{n,\delta}$ such that, for all $r > R_{n,\delta}$,

$$\mathrm{Prob}(\{|n_{\psi_\omega}(r) - \tfrac{1}{2}r^2| \geq \delta r^2\}) \leq \exp\{-c_{3,\delta} r^{2n+2}\},$$

where $n_{\psi_\omega}(r)$ is the unintegrated counting function for ψ_ω.

THEOREM 1.2. *If*

$$\mathrm{Hole}_r = \{\psi_\omega(z) \neq 0 \ \forall z \in B(0,r)\},$$

then there exist $R_n, c_1, c_2 > 0$ such that, for all $r > R_n$,

$$\exp\{-c_2 r^{2n+2}\} \leq \mathrm{Prob}(\mathrm{Hole}_r) \leq \exp\{-c_1 r^{2n+2}\}.$$

The proofs of these two theorems will use techniques from probability theory and several complex variables as well as an invariance rule for flat Gaussian random holomorphic functions.

This paper is based on the work of Sodin and Tsirelison [14] and generalizes their methods to higher dimensions. In order to do this, new results for estimating the values of flat Gaussian random holomorphic functions on polydisks are derived. Additionally, many technical changes must be made regarding computations of surface integrals of flat Gaussian random holomorphic functions. These include Lemma 4.2, which is needed to appropriately partition a sphere, and significant changes in the proof of Lemma 4.4 that are caused by the dependence of the formula for the Poisson kernel on the number of dimensions n.

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2. Defining Flat Gaussian Random Holomorphic Functions, and Common Results Concerning These Functions

Gaussian random holomorphic functions are defined as

$$\psi_\omega(z) = \sum_j \omega_j \psi_j(z),$$

where $\{\omega_j\}$ is a sequence of independent and identically distributed (i.i.d.) real or complex Gaussian random variables and $\{\psi_j(z)\}_j$ is a sequence of holomorphic functions. Since $\lim_j \sup|\omega_j|^{1/j} = 1$ a.s., it follows that ψ_ω is a.s. a holomorphic function on a domain Ω provided that for all compact $K \subset \Omega$,

$$\sum_{j\in\mathbb{N}} \max_{z\in K} |\psi_j(z)|^2 < \infty.$$

For this paper we will be concerned with a particular class of random holomorphic functions: flat Gaussian random holomorphic functions.

DEFINITION 2.1. A *flat Gaussian random holomorphic function* is a random holomorphic function that may be written in the form

$$\psi_\omega(z) = \sum_{j\in\mathbb{N}^n} \omega_j \left(\frac{z_1^{j_1} z_2^{j_2} \cdots z_n^{j_n}}{\sqrt{j_1!\, j_2! \cdots j_n!}} \right), \tag{1}$$

where $\{\omega_j\}$ is a sequence of i.i.d. standard complex Gaussian random variables.

Additionally, the set of flat Gaussian random holomorphic functions that do not converge on $\mathbb{C}^n$ is null.

Let us briefly review properties of the zeros of random holomorphic functions. We will view the zero set of a holomorphic function f as Z_f: a divisor and a (1,1) current. The regular points of Z_f are a manifold, and we identify forms in $D_M^{(n-1,n-1)}$ with ones in $D_{Z_f}^{(n-1,n-1)}$ by restricting their domain. If $f \in \mathcal{O}(M^n)$ for M an n-complex manifold, then $Z_f = \frac{i}{2\pi}\partial\bar\partial \log|f|^2$ as (1,1) currents on M. Starting with this, a standard result may be proved on the expected zero distribution.

THEOREM 2.2. *If $E[|\psi_\omega|^2] = \sum |\psi_j(z)|^2$ converges locally uniformly in Ω, then* $E[Z_\omega] = \frac{i}{2\pi}\partial\bar{\partial}\log E[|\psi_\omega|^2]$.

Many forms of this theorem have been proven (see [3; 5; 13]). Theorem 2.2 is more general than those in the literature, but the standard method of proof still works. In particular, a concise proof can be given by simplifying the argument in [13]. Specializing this result to flat Gaussian random holomorphic functions yields the following corollary.

COROLLARY 2.3. *For a flat Gaussian random holomorphic function ψ_ω,*

$$E[Z_{\psi_\omega}] = \frac{i}{2\pi}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 + \cdots + dz_n \wedge d\bar{z}_n).$$

More can be said about this because specifying the expected zero set of a Gaussian random holomorphic function defines an almost unique class of Gaussian random holomorphic functions, as follows.

THEOREM 2.4. *For Gaussian random holomorphic functions, the expected zero set determines the process uniquely* (*up to multiplication by nonzero holomorphic functions*) *for a simply connected domain.*

This theorem is proven in one dimension by Sodin [13], and the same proof works in n dimensions. A much stronger refinement of this result for flat Gaussian random holomorphic functions is the following translational law.

PROPOSITION 2.5. *For all $z \in \mathbb{C}^n$ and for all sequences of i.i.d. standard Gaussian random variables $\{\omega_j\}_{j\in\mathbb{N}}$, there exist ω_j' i.i.d. standard complex Gaussian random variables such that*

$$\psi_\omega(z) = \exp\{-\tfrac{1}{2}|\zeta|^2 - z\bar{\zeta}\}\psi_{\omega'}(z+\zeta).$$

This proposition is proven in [6] and is especially useful because statements may be proved in one particular region and then translated to another. In particular, we have the following result.

COROLLARY 2.6. *For all $z \in \mathbb{C}^n$ and for all sequences of i.i.d. standard Gaussian random variables $\{\omega_j\}_{j\in\mathbb{N}}$, there exist ω_j' i.i.d. standard complex Gaussian random variables such that*

$$\max_{z\in\partial B(0,r)}(\log(|\psi_\omega(z)|) - \tfrac{1}{2}|z|^2) = \max_{z\in\partial B(\zeta,r)}(\log(|\psi_{\omega'}(z)|) - \tfrac{1}{2}|z|^2).$$

Here, $\psi_{\omega'}$ is itself a flat Gaussian random holomorphic function.

Proof of Corollary 2.6. We simply apply Proposition 2.5 to obtain

$$\begin{aligned}
&\max_{z\in\partial B(0,r)}(\log|\psi_\omega(z)| - \tfrac{1}{2}|z|^2)\\
&\quad= \max_{z\in\partial B(0,r)}(\log|\exp\{-\tfrac{1}{2}|z|^2\}\psi_\omega(z)|)\\
&\quad= \max_{z\in\partial B(0,r)}\log(|\exp\{-\tfrac{1}{2}|\zeta|^2 - |z|^2 - z\bar{\zeta}\}\psi_{\omega'}(z+\zeta)|)\\
&\quad= \max_{z\in\partial B(\zeta,r)}(\log|\psi_{\omega'}(z)| - \tfrac{1}{2}|z|^2). \qquad \square
\end{aligned}$$

3. An Estimate for the Growth Rate of Flat Gaussian Random Holomorphic Functions

In this section we begin working toward our main results. Lemma 3.2 is interesting in its own right because it proves that flat Gaussian random functions are of finite order 2 a.s. Let $M_{r,\omega} = \max_{\partial B(0,r)} \log|\psi_\omega(z)|$. We will be need several elementary estimates.

PROPOSITION 3.1. (a) *If ω is a standard complex Gaussian random variable, then*

(i) $\text{Prob}(\{|\omega| \geq \lambda\}) = \exp\{-\lambda^2\}$ *and*
(ii) $\text{Prob}(\{|\omega| \leq \lambda\}) = 1 - \exp\{-\lambda^2\} \in [\frac{1}{2}\lambda^2, \lambda^2]$ *if* $\lambda \leq 1$.

(b) *If $\{\omega_j\}_{j\in\mathbb{N}^n}$ is a set of i.i.d. standard Gaussian random variables, then* $\text{Prob}(\{|\omega_j| < (1+\varepsilon)^{|j|}\}) = c > 0$.

(c) *If $j \in \mathbb{N}^{+,n}$ then $|j|^{|j|}/j^j \leq n^{|j|}$.*

Here and throughout this paper, $|j| := \sum j_i$ for $j \in \mathbb{N}^n$. Let

$$M_{r,\omega} = \max_{z\in B(0,r)} |\psi_\omega(z)|.$$

LEMMA 3.2. *For all $\delta > 0$ and all $r > R$, there exists a $c_\delta > 0$ such that*

$$\text{Prob}\left(\left\{\left|\frac{\log(M_{r,\omega})}{r^2} - \frac{1}{2}\right| \geq \delta\right\}\right) \leq \exp\{-c_\delta r^{2n+2}\}.$$

Proof. We will first prove that $\text{Prob}(\{\log(M_{r,\omega})/r^2 \geq \frac{1}{2}+\delta\}) \leq \exp\{-c_{\delta,1}r^{2n+2}\}$ by specifying an event Ω_r of measure almost 1 where $M_{r,\omega} \leq \exp\{(\frac{1}{2}+\delta)r^2\}$. Let Ω_r be the event where

$$|\omega_j| \leq \begin{cases} e^{\delta r^2/4} & \text{if } |j| \leq 2e\cdot n\cdot r^2, \\ 2^{|j|/2} & \text{if } |j| > 2e\cdot n\cdot r^2. \end{cases}$$

Then

$$\begin{aligned} \text{Prob}(\Omega_r^c) &\leq \sum_{|j|\leq 2e\cdot nr^2} \text{Prob}(\{|\omega_j| > e^{\delta r^2/4}\}) + \sum_{|j|>2e\cdot nr^2} \text{Prob}(\{|\omega_j| > 2^{|j|/2}\}) \\ &\leq c_n r^{2n} \exp\{-e^{\delta r^2/2}\} + \sum_{|j|>2e\cdot nr^2} \exp\{-2^{|j|}\} \\ &\leq \exp\{-e^{cr^2}\} + c\exp\{-2^{cr^2}\} \quad \forall r > R_0 \\ &\leq \exp\{-e^{cr^2}\}. \end{aligned}$$

We now have that $\text{Prob}(\Omega_r^c) \leq \exp\{-e^{cr^2}\} < \exp\{-cr^{2n+2}\}$, so it remains for us to show that, for all $\omega \in \Omega_r$, $\log|M_{r,\omega}|/r^2 \leq \frac{1}{2} + \frac{1}{2}\delta$. For all $\omega \in \Omega_r$ and all $z \in B(0,r)$, we have

$$\begin{aligned} M_{r,\omega} &\leq \max_{z\in B(0,r)} \left(\sum_{|j|=0}^{|j|\leq 4e\cdot n(r^2/2)} |\omega_j| \frac{|z|^j}{\sqrt{j!}} + \sum_{|j|>4e\cdot n(r^2/2)} |\omega_j| \frac{|z|^j}{\sqrt{j!}} \right) \\ &= \max_{z\in B(0,r)} \sum^{1} + \max_{z\in B(0,r)} \sum^{2}. \end{aligned}$$

Using the Cauchy–Schwartz inequality then yields

$$\begin{aligned}\max_{z\in B(0,r)} \overset{1}{\sum} &\le (\exp\{\tfrac{1}{4}\delta r^2\})\sqrt{c(r^2)^n}\max_{z\in B(0,r)}\Bigl(\sum_j \frac{|z^{2j}|}{j!}\Bigr)^{1/2}\\ &\le c_n \exp\{\tfrac{1}{4}\delta r^2\} r^n \exp\{\tfrac{1}{2}r^2\}\\ &\le \exp\{(r^2)(\tfrac{1}{2}+\tfrac{1}{3}\delta)\} \quad \forall r > R_{n,\delta}.\end{aligned}$$

Then, by Sterling's formula $(j! \approx \sqrt{2\pi}\sqrt{j}\, j^j e^{-j})$, we have

$$\begin{aligned}\max_{z\in B(0,r)} \overset{2}{\sum} &\le \max_{z\in B(0,r)} \sum_{|j|>4e\cdot nr^2} (2)^{|j|/2}\frac{|z^j|}{\sqrt{j!}}\\ &\le \sum_{|j|>4e\cdot nr^2} (2)^{|j|/2}\Bigl(\frac{|j|}{4en}\Bigr)^{|j|/2}\prod_k\Bigl(\frac{e}{j_k}\Bigr)^{j_k/2}\\ &\le C. \quad \text{(by Proposition 3.1(c))}\end{aligned}$$

Hence $\log(M_{r,\omega}) \le (\frac{1}{2}+\frac{1}{2}\delta)r^2$ for all $\omega \in \Omega_r$, proving half of the claim.

Let $M'_{r,\omega} = \max_{z\in P(0,r)}|\psi_\omega(z)|$, where $P(0,r) := \{z \in \mathbb{C}^n : |z_i| < r\ \forall i\}$. Then we need only show that

$$\forall\delta > 0,\ \forall r > R,\quad \text{Prob}\biggl(\biggl\{\frac{\log(M_{r,\omega})}{r^2} \le \frac{1}{2}-\delta\biggr\}\biggr) \le \exp\{-c_{\delta,2}r^{2n+2}\}.$$

It suffices to prove this result for small δ only. However, we will actually prove a stronger claim: for all δ and all $r > R$, there exists a c such that

$$\text{Prob}(M'_{r,\omega} \le \tfrac{n}{2}r^2 - \delta r^2) < \exp\{-cr^{2n+2}\}.$$

This is a stronger claim because

$$M_{r,\omega} \ge M'_{r/\sqrt{n},\omega} \implies \{M_{r,\omega} < (\tfrac{1}{2}-\delta)r^2\} \subset \{M'_{r,\omega} < (\tfrac{1}{2}-\delta)r^2\},$$

and the desired probability estimate therefore holds by monotonicity.

We begin this second half of the proof by considering what we know given that ω belongs to the event where

$$\log(M'_{r,\omega}) \le (\tfrac{n}{2}-\delta)r^2$$

has occurred. By Cauchy's integral formula, $|\partial^j\psi_\omega/\partial z^j|(0) \le j!\, M'_{r,\omega}r^{-|j|}$. Furthermore, we may use equation (1) to directly compute

$$\left|\frac{\partial^j\psi_\omega}{\partial z^j}\right|(0) = |\omega_j|\sqrt{j!}.$$

Hence $|\omega_j| \le cM'_{r,\omega}\sqrt{j!}\,r^{-|j|}$, and using Sterling's formula then yields, for all k,

$$|\omega_j| \le (2\pi)^{n/2}\Bigl(\prod_k j_k^{1/4}\Bigr)\exp\biggl\{\Bigl(\frac{n}{2}-\delta\Bigr)r^2 + \sum\frac{j_k}{2}\log(j_k) - (|j|)\log r - \frac{|j|}{2}\biggr\},$$

where $j_k \neq 0$. The $(2\pi)^{n/2}j^{1/4}$ term will not matter in the end and so we focus instead on the exponent, which we will now call A:

$$A = \left(\frac{n}{2} - \delta\right) r^2 - \frac{|j|}{2} + \sum_k \left(\frac{j_k}{2} \log(j_k)\right) - (|j|) \log(r)$$
$$= \sum_{k=1}^{k=n} \left(\frac{j_k}{2}\right) \left(\left(1 - \frac{2\delta}{n}\right) \frac{r^2}{j_k} - 1 + \log(j_k) - 2\log(r)\right).$$

Let $j_k = \gamma_k r^2$. Then:

$$A = \sum_{k=1}^{k=n} \left(\frac{\gamma_k r^2}{2}\right) \left(\left(1 - \frac{2\delta}{n}\right) \frac{1}{\gamma_k} - 1 + \log(\gamma_k)\right)$$
$$= -\delta r^2 + n f(\gamma_k) \frac{r^2}{2}, \quad \text{where } f(\gamma_k) = 1 - \gamma_k + \gamma_k \log(\gamma_k);$$
$$f(\gamma_k) = (1-\gamma_k)^2 - (1-\gamma_k)^3 + o((1-\gamma)^4) \text{ near 1.}$$

As a result, there exists a Δ such that, for all $\delta \leq \Delta$, if $\gamma_k \in [1 - \sqrt{\delta/n}, 1 + \sqrt{\delta/n}\,]$ then $A \leq -\frac{1}{2}\delta r^2$.

Hence, for j as before, we have

$$|\omega_j| \leq (2\pi)^{n/2} \left(\prod_k j_k^{1/4}\right) e^{-\delta r^2/2} \leq c r^{n/2} e^{-\delta r^2/2}.$$

Specializing our work for large r, we have that for all $\varepsilon > 0$, there exists an R such that, for all $r > R$, $|\omega_j| \leq \exp\{-\frac{1}{2}(\delta - \varepsilon) r^2\}$. Here the factor of ε is used to compensate for the $\sqrt{2\pi}\, j_k^{1/4}$ terms, which had previously been ignored.

Therefore, $E_{\delta,r}$ has the desired decay rate in terms of r:

$$\begin{aligned}
&\mathrm{Prob}(\{\log M'_{r,\omega} \leq (\tfrac{1}{2} - \delta) r^2\}) \\
&\quad \leq \mathrm{Prob}(\{|\omega_j| \leq \exp\{-\tfrac{1}{2}(\delta - \varepsilon) r^2\} \text{ and } j_k \in [(1 - \sqrt{\delta/n}\,) r^2, (1 + \sqrt{\delta/n}\,) r^2]\}) \\
&\quad \leq (\exp\{-(\delta - \varepsilon) r^2\})^{(2\sqrt{\delta/n} r^2)^n} = \exp\{-2^n (1 + o(\delta)) \delta^{(n+2)/2} r^{2n+2}\} \\
&\quad = \exp\{-c_{1,\delta} r^{2n+2}\},
\end{aligned}$$

where we have used Proposition 3.1 and the independence of ω_j. □

COROLLARY 3.3. *Let $z_0 \in \overline{B(0,r)} \backslash B(0, \frac{1}{2} r)$. Then, for all $\delta > 0$ and all $r > R$,*

$$\mathrm{Prob}(\{\nexists \zeta \in B(z_0, \delta r) \text{ s.t. } \log|\psi_\omega(\zeta)| > (\tfrac{1}{2} - 3\delta)|z_0|^2\}) \leq \exp\{-c r^{2n+2}\}.$$

Proof. Without loss of generality, assume that $\delta < \frac{1}{4}$. By Lemma 3.2 we have

$$\mathrm{Prob}\left(\left\{\max_{z \in \partial B(0,r)} \log|\psi_\omega(z)| - \tfrac{1}{2}|z|^2 \leq -\delta r^2\right\}\right) \leq \exp\{-c r^{2n+2}\}.$$

By Proposition 2.5 it follows that, for $z_0 \in B(0,r) \backslash B(0, \frac{1}{2} r)$ and $z \in B(z_0, \delta r)$,

$$\mathrm{Prob}\left(\left\{\max_{z \in \partial B(0,\delta r)} \log|\psi_\omega(z - z_0)| - \tfrac{1}{2}|z - z_0|^2 \leq -\delta(\delta r)^2\right\}\right) \leq \exp\{-c r^{2n+2}\}.$$

Hence there exists a $z \in B(z_0, \delta r)$ such that $\log|\psi_\omega(z - z_0)| - \frac{1}{2}|z - z_0|^2 \geq -\delta(\delta r)^2$, except for an event of probability less than $\exp\{-cr^{2n+2}\}$.

By hypothesis, $|z_0| \in [\frac{1}{2}r, r)$ and so $|z - z_0| \leq \delta r \leq \frac{1}{4}r = \frac{r}{2}\frac{1}{2} \leq \frac{1}{2}|z_0|$. Therefore, $|z_0 - z|^2 \geq |z_0|^2 - \delta r^2 \geq |z_0|^2(1 - 2\delta)$, which implies that

$$\begin{aligned}\log|\psi_\omega(z - z_0)| &\geq \tfrac{1}{2}|z - z_0|^2 - \delta^3 r^2 \geq |z_0|^2\tfrac{1}{2}(1 - 2\delta)^2 - 4\delta^3|z_0|^2\\ &\geq \tfrac{1}{2}|z_0|^2 - 2\delta|z_0|^2 - \tfrac{1}{4}\delta|z_0|^2\\ &\geq \tfrac{1}{2}|z_0|^2 - 3\delta|z_0|^2.\end{aligned}$$

Setting $\zeta = z - z_0$, we have shown what we set out to prove. □

Using that $\log \max_{B(0,r)}|\psi_\omega|$ is an increasing function in terms of r, we may prove the following corollary.

COROLLARY 3.4. *For all $\delta > 0$, we have*:

(a) $\operatorname{Prob}\left(\left\{\lim_{r\to\infty} \dfrac{(\log \max_{z\in B(0,r)}|\psi_\omega(z)|) - \frac{1}{2}r^2}{r^2} \notin [-\delta, \delta]\right\}\right) = 0;$

(b) $\operatorname{Prob}\left(\left\{\lim_{r\to\infty} \dfrac{(\log \max_{z\in B(0,r)}|\psi_\omega(z)|) - \frac{1}{2}r^2}{r^2} \neq 0\right\}\right) = 0.$

This corollary has been proven by more direct methods; see [14].

Proof of Corollary 3.4. Part (b) follows immediately from part (a), which we now prove. Let

$$E_{\delta,R} = \left\{\frac{\log \max_{B(0,R)}|\psi_\omega(z)| - \frac{1}{2}R^2}{R^2} \notin [-\delta, \delta]\right\}.$$

Let $R_m = r + \delta(m+1)r$ for $r > 0$, and let $s_m \in [R_{m-1}, R_m]$. We claim that, for all s_m with $m > M_\delta$, $E_{\delta,s_m} \subset E_{\delta/3,R_m} \cup E_{\delta/3,R_{m-1}}$.

Let $M_\delta = \max\{M_{1,\delta}, M_{2,\delta}\}$, which may be specifically determined.

Case (i): for $\omega \in E_{\delta,s_m}$, $\log \max_{B(0,s_m)}|\psi_\omega| \geq \frac{1}{2}s_m^2 + \delta s_m^2$. Then

$$\begin{aligned}\log \max_{B(0,R_m)}|\psi_\omega| &\geq \tfrac{1}{2}s_m^2 + \delta s_m^2,\\ &\geq \tfrac{1}{2}(1 + m\delta)^2 r^2 + \delta(1 + m\delta)^2 r^2\\ &> \tfrac{1}{2}R_m^2 + \tfrac{1}{3}\delta R_m^2 \quad \forall m > M_{1,\delta}.\end{aligned}$$

Therefore, $\omega \in E_{\delta/3,R_m}$.

Case (ii): for $\omega \in E_{\delta,s_m}$, $\log \max_{B(0,s_m)} \psi_\omega \leq \frac{1}{2}s_m^2 - \delta s_m^2$. Then

$$\begin{aligned}\log \max_{B(0,R_{m-1})}|\psi_\omega| &\leq \tfrac{1}{2}s_m^2 - \delta s_m^2\\ &\leq \tfrac{1}{2}(1 + (m-1)\delta)^2 r^2 - \delta(1 + m\delta)^2 r^2\\ &\leq \tfrac{1}{2}R_{m-1}^2 - \tfrac{1}{3}\delta R_{m-1}^2 \quad \forall m > M_{2,\delta}.\end{aligned}$$

Therefore, $\omega \in E_{\delta/3,R_{m-1}}$.

We have shown that, for all $m > M_\delta$ and for all $s \in [R_{m-1}, R_m]$, $E_{\delta,s} \subset E_{\delta/3,R_{m-1}} \cup E_{\delta/3,R_m}$. Hence, $\operatorname{Prob}\left(\bigcup_{s\in[R_{m-1},R_m]} E_{\delta,s}\right) \leq 2\exp\{-c_\delta r^{2n+2} m^{2n+2}\}$ and so

$$\sum_{m\in\mathbb{N}} \mathrm{Prob}\left(\bigcup_{s\in[R_{m-1},R_m]} E_{\delta,s}\right) = \sum_{m\in\mathbb{N}} \exp\{-c_\delta m^{2n+2}\} < \infty.$$

The result follows. □

4. The Second Main Lemma

In order to prove Theorem 1 we need another key lemma, Lemma 4.4, in which we will give an estimate for $\int \log|\psi_\omega|$. The lemma will be proved by approximating a surface integral using Riemann integration.

In order to establish notation, I state the Poisson integral formula: for $\zeta \in B(0,r)$ and h a harmonic function,

$$h(\zeta) = \int_{S_r} P_r(\zeta, z) h(z)\, d\sigma_r(z),$$

where $d\sigma_r$ is the Haar measure of the sphere $S_r = \partial B(0,r)$ and P_r is the Poisson kernel for $B(0,r)$. In this notation, the Poisson kernel is given by

$$P_r(\zeta, z) = r^{2n-2}\frac{r^2 - |\zeta|^2}{|\zeta - z|^{2n}}.$$

LEMMA 4.1. *For all $r > R_n$, there exists a $c > 0$ such that*

$$\mathrm{Prob}\left(\left\{\int_{S_r} |\log(|\psi_\omega|)|\, d\sigma_r(z) > (3^{2n}+1)r^2\right\}\right) \le \exp\{-cr^{2n+2}\}.$$

Proof. With the exception of an event whose probability is less than $\exp\{-cr^{2n+2}\}$, by Lemma 3.2 there exists a $\zeta_0 \in \partial B(0, \frac{1}{2}r)$ such that $\log(|\psi_\omega(\zeta_0)|) > 0$. Hence

$$\int_{\partial B(0,r)} P_r(\zeta_0, z)\log(|\psi_\omega(z)|)\, d\sigma_r(z) \ge \log(|\psi(\zeta_0)|) \ge 0.$$

Alternatively,

$$\int_{\partial(B(0,r))} P_r(\zeta_0, z)\log^-(|\psi_\omega(z)|) \le \int_{\partial(B(0,r))} P_r(\zeta_0, z)\log^+(|\psi_\omega(z)|).$$

Since $\zeta \in \partial B(0, \frac{1}{2}r)$ and $z \in \partial B(0,r)$, it follows that $\frac{1}{2}r \le |z-\zeta| \le \frac{3}{2}r$. Estimating the values of the Poisson kernel for these values of z and ζ yields

$$\frac{1}{3}\left(\frac{2}{3}\right)^{2n-2} \le P_r(\zeta, z) \le (2)^{2n-2}3.$$

Therefore, by Lemma 3.2,

$$\int_{\partial B(0,r)} \log^+(|\psi_\omega(z)|)\, d\sigma_r(z) \le \log M_r \le \left(\frac{1}{2}+\delta\right)r^2 \le r^2,$$

except for an event whose probability is less than $\exp\{-cr^{2n+2}\}$.

It remains to compute $\int \log^-|\psi_\omega|$:

$$\begin{aligned}\int_{\partial(B(0,r))} P(\zeta_0, z)\log^+(|\psi_\omega(z)|) &\le \sigma_r(S_r)\log(M_r)3(2)^{2n-2}\\ &\le 3(2)^{2n-2}r^2;\end{aligned}$$

$$\begin{aligned}\int_{\partial(B(0,r))} \log^-(|\psi_\omega(z)|)\,d\sigma_r(z) &\le \frac{1}{\min_z P(\zeta_0,z)} \int_{\partial(B(0,r))} P(\zeta_0,z)\log^+(|\psi_\omega(z)|)\\ &\le 3\left(\frac{3}{2}\right)^{2n-2} \int_{\partial(B(0,r))} P(\zeta_0,z)\log^+(|\psi_\omega(z)|)\\ &\le 9\left(\frac{3}{2}\right)^{2n-2} (2)^{2n-2} r^2\\ &\le 3^{2n} r^2.\end{aligned}$$

The result now follows immediately. □

In order to use Reimann integration to prove Lemma 4.4, we will need the ability to choose "evenly" spaced points on the sphere. This choice will be made according to the next lemma.

LEMMA 4.2 (Partition of a Sphere). *For all $m \in \mathbb{N}^+$, let $N = (2n)m^{2n-1}$. Then $S_r^{2n} \subseteq \mathbb{R}^{2n}$ can be partitioned into N disjoint measurable sets $\{I_1^r, I_2^r, \dots, I_N^r\}$ such that*

$$\operatorname{diam}(I_j^r) \le \frac{\sqrt{2n-1}}{m} r = \frac{c_n}{N^{1/(2n-1)}} r.$$

Proof. Surround S_r with $2n$ pieces of planes: $P_{+,1}, P_{+,2}, \dots, P_{+,n}, P_{-,1}, \dots, P_{-,n}$, where

$$P_{+,j} = \{x \in \mathbb{R}^{2n+1} : \|x\|_{L^\infty} = r,\ x_j = r\},$$

$$P_{-,j} = \{x \in \mathbb{R}^{2n+1} : \|x\|_{L^\infty} = r,\ x_j = -r\}.$$

Subdivide each piece into m^{2n-1} identical closed $2n-1$ cubes in the usual way, and denote these sets $R_1', \dots, R_N'$. In order to remove intersections on the boundary, we then put $R_j = R_j' \setminus \bigcup_{k<j} R_k'$.

Let $I_j^r = \{x \in S_r : \lambda x \in R_j,\ \lambda > 0\}$. Clearly, if $x \in S_r$ then there exists a j such that $x \in I_j^r$. By design, $\lambda \ge 1$ and so $x, y \in I_j^r$ implies that

$$d(x,y) \le d(\lambda_1 x, \lambda_2 y) \le \frac{2}{m} r = \operatorname{diam}(R_j).$$ □

For the following result, note that the integration is with respect to w, which is not the same variable of integration that is usually used in the Poisson integral formula.

LEMMA 4.3. *For $\kappa < 1$ and $z \in \partial B(0,r)$,*

$$\int_{w \in S_{\kappa r}^n} P_r(w,z)\,d\sigma_{\kappa r}(w) = 1.$$

Proof. If $w \in S_{\kappa r}^{2n} \subseteq \mathbb{R}^{2n}$ then the Poisson kernel can be rewritten as a function of $|z-w|$; as such, $P_r(\Upsilon w, \Upsilon z) = P_r(w,z)$ for all $\Upsilon \in U_n(\mathbb{R}^n)$.

Let $f(z) = \int_{w \in S_{\kappa r}^n} P_r(w,z)\,d\sigma_{\kappa r}(w)$. Then

$$\begin{aligned} f(z) &= \int_{w \in S_{\kappa r}^n} P_r(w,z)\,d\sigma_{\kappa r}(w)\\ &= \int_{w \in S_{\kappa r}^n} P_r(\Upsilon w, \Upsilon z)\,d\sigma_{\kappa r}(w) \quad \text{(by our previous results)}\end{aligned}$$

$$= \int_{w \in S^n_{\kappa r}} P_r(\Upsilon w, \Upsilon z)\, d\sigma_{\kappa r}(\Upsilon w) \quad \text{(since } d\sigma_{\kappa r} \text{ is invariant under rotations)}$$
$$= \int_{w \in S^n_{\kappa r}} P_r(w, \Upsilon z)\, d\sigma_{\kappa r}(w) \quad \text{(by a change of coordinates)}$$
$$= f(\Upsilon z).$$

As a result, $f(z) = c$ for all $z \in S^n_r$.

Switching the order of integration, we compute that

$$1 = \int_{w \in S^n_{\kappa r}} \int_{z \in S^n_r} P_r(w, z)\, d\sigma_r(z)\, d\sigma_{\kappa r}(w) = c. \qquad \square$$

Now we are able to prove our final lemma.

LEMMA 4.4. *For all* $\Delta > 0$, *there exists a* $c > 0$ *such that, for all* $r > R$,

$$\operatorname{Prob}\left(\left\{\frac{1}{r^2}\int_{z \in \partial B(0,r)} \log|\psi_\omega|\, d\sigma_r(z) \le \frac{1}{2} - \Delta\right\}\right) \le \exp\{-cr^{2n+2}\}.$$

Proof. It suffices to prove the result for small Δ. Let $a_n = 1/2(2n+2)(2n-1)$. Set $\delta = (\Delta/\lambda)^{1/a_n} < \frac{1}{6}$, with $\lambda > 0$ to be determined later. Choose $m \in \mathbb{N}$ such that $1/(2n)m^{2n-1} \le \delta$, and for notational purposes let $N = (2n)m^{2n-1}$. Also, let $\kappa = 1 - \delta^{a_n}$.

Form a disjoint partition $\{I^{\kappa r}_j\}$ of $S_{\kappa r}$ as in Lemma 4.2. In particular,

$$\operatorname{diam}(I^{\kappa r}_j) \le c\delta^{1/(2n-1)} r.$$

Let $\sigma_j = \sigma_{\kappa r}(I^{\kappa r}_j)$, which does not depend on r, and for all j fix a point $x_j \in I^{\kappa r}_j$.

By Corollary 3.3, for each j there exists a $\zeta_j \in B(x_j, \delta r)$ such that

$$\log(|\psi_\omega(\zeta_j)|) > \left(\frac{1}{2} - 3\delta\right)|x_j|^2 = \left(\frac{1}{2} - 3\delta\right)\kappa^2 r^2,$$

except for N different events each of probability less than $\exp\{-c' r^{2n+2}\}$ (and thus the union of these N events also has probability less than $\exp\{-cr^{2n+2}\}$).

Since for each j we have the same estimate for $|\psi_\omega(\zeta_j)|$ and since $\sum \sigma_j = 1$, it follows that

$$\begin{aligned}\left(\frac{1}{2} - 3\delta\right)(1-\delta^{a_n})^2 r^2 &\le \sum_{j=1}^{N} \sigma_j \log(|\psi_\omega(\zeta_j)|)\\ &\le \int_{\partial B(0,r)} \left(\sum_j \sigma_j P_r(\zeta_j, z) \log(|\psi_\omega(z)|)\, d\sigma_r(z)\right)\\ &= \int_{\partial(B(0,r))} \left(\sum_j \sigma_j (P_r(\zeta_j, z) - 1)\right) \log(|\psi_\omega(z)|)\, d\sigma_r(z)\\ &\quad + \int_{\partial(B(0,r))} \log(|\psi_\omega(z)|)\, d\sigma_r(z).\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_{\partial B(0,r)} \log(|\psi_\omega|)\, d\sigma_r \\
&\qquad \geq \left(\frac{1}{2} - 3\delta\right)(1-\delta^{a_n})^2 r^2 - \int |\log|\psi_\omega||\, d\sigma_r \cdot \max_z \left|\sum_j \sigma_j (P_r(\zeta_j, z) - 1)\right| \\
&\qquad \geq \left(\frac{1}{2} - 3\delta\right)(1-\delta^{a_n}) r^2 - (3^{2n}+1) r^2 \cdot C_n \delta^{1/2(2n-1)} \geq \frac{1}{2} r^2 - \lambda \delta^{a_n} r^2
\end{aligned}$$

by Lemma 4.1 and the following claim, so proving this claim will establish Lemma 4.4.

Claim:

$$\max_{z \in \partial(B(0,r))} \left|\sum_j \sigma_j (P_r(\zeta_j, z) - 1)\right| \leq C_n \delta^{1/2(2n-1)}.$$

Proof of Claim: By Lemma 4.3 we know that, for all $z \in \partial B(0,r)$,

$$\int_{\zeta \in \partial B(0,\kappa r)} P_r(\zeta, z)\, d\sigma_{\kappa r}(\zeta) = 1.$$

Hence

$$1 = \sum_{j=1}^{j=N} \sigma_j P_r(\zeta_j, z) + \sum_{j=1}^{j=N} \int_{\zeta \in I_j^{\kappa r}} (P_r(\zeta, z) - P_r(\zeta_j, z))\, d\sigma_{\kappa r}(\zeta)$$

and

$$\begin{aligned}
\left|\sum_{j=1}^{j=N} \sigma_j (P_r(\zeta_j, z) - 1)\right| &= \left|\sum_{j=1}^{j=N} \int_{\zeta \in I_j^{\kappa r}} (P_r(\zeta, z) - P_r(\zeta_j, z))\, d\sigma_{\kappa r}(\zeta)\right| \\
&\leq \max_{j, \zeta \in I_j^{\kappa r}} |\zeta - \zeta_j| \cdot \max_{w \in B(0,(\kappa+\delta)r) \setminus B(0,(\kappa-\delta)r)} \left|\frac{\partial P_r(w,z)}{\partial w}\right|.
\end{aligned}$$

Then

$$\frac{\partial P_r(w,z)}{\partial w} = -r^{2n-2} \frac{\bar{w}|z-w|^2 + (r^2 - |w|^2) n (\bar{z} - \bar{w})}{|z-w|^{2n+2}}.$$

Since $|z| = r$ and since $|w| = (1-\varepsilon) r \in [(\kappa - \delta) r, (\kappa + \delta) r]$, it follows that

$$\left|\frac{\partial P_r(w,z)}{\partial w}\right| \leq \frac{2 + 4\varepsilon n}{r \varepsilon^{2n+2}} \leq \frac{c_n}{r \varepsilon^{2n+2}} = \frac{c_n}{r} \delta^{-1/2(2n-1)}.$$

Moreover, $\max_\zeta |\zeta - \zeta_j| \leq \operatorname{diam}(I_j) + \delta r \leq c \delta^{1/(2n-1)} r + \delta r \leq c' r \delta^{1/(2n-1)}$ and so

$$\left|\sum_{j=1}^{j=N} \sigma_j (P_r(\zeta_j, z) - 1)\right| \leq C \delta^{1/(2n-1)} \cdot \delta^{-1/2(2n-1)} = C \delta^{1/2(2n-1)},$$

which proves the claim and hence the lemma. □

Lemma 4.4 gives an alternate proof for the growth rate of the characteristic function. Let $T(f, r) = \int_{S_r} \log^+ |f(z)|\, d\sigma_r(z)$, the Nevanlinna characteristic function.

COROLLARY 4.5. *For all $\delta \in (0, \frac{1}{3}]$:*

(a) $$\mathrm{Prob}\left(\left\{\lim_{r\to\infty} \frac{\left(\int_{S_r} \log|\psi_\omega|\, d\sigma_r\right) - \frac{1}{2}r^2}{r^2} \notin [-\delta, \delta]\right\}\right) = 0;$$

(b) $$\mathrm{Prob}\left(\left\{\lim_{r\to\infty} \frac{\left(\int_{S_r} \log|\psi_\omega|\, d\sigma_r\right) - \frac{1}{2}r^2}{r^2} \neq 0\right\}\right) = 0;$$

(c) $$\mathrm{Prob}\left(\left\{\lim_{r\to\infty} \frac{T(\psi_\omega, r) - \frac{1}{2}r^2}{r^2} \neq 0\right\}\right) = 0.$$

Because $\int_{S_r} \log|\psi_\omega|\, d\sigma_r$ is increasing, the proof of Corollary 3.4 can be used in conjunction with Lemma 4.4 to prove that $\psi_\omega(z)$ is a.s. of finite order 2. A more elementary proof of this is already available [14].

5. Proof of Main Results

We can now put the pieces together and estimate the number of zeros in a large ball for a random holomorphic function $\psi_\omega(z)$.

DEFINITION 5.1. For $f \in \mathcal{O}(B(0,r))$ and $B(0,r) \subset \mathbb{C}^n$, the *unintegrated counting function*

$$\begin{aligned} n_f(r) &:= \int_{B(0,t)\cap Z_f} \left(\frac{i}{2\pi}\partial\bar\partial \log|z|^2\right)^{n-1} \\ &= \int_{B(0,t)} \left(\frac{i}{2\pi}\partial\bar\partial \log|z|^2\right)^{n-1} \wedge \frac{i}{2\pi}\partial\bar\partial \log|f|. \end{aligned}$$

The equivalence of these two definitions follows by the Poincaré–Lelong formula. The form $(\frac{i}{2\pi}\partial\bar\partial \log|z|^2)^{n-1}$ gives a projective volume, which is more convenient for measuring the zero set of a random function. The Euclidean volume may be recovered as

$$\int_{B(0,t)\cap Z_f} \left(\frac{i}{2\pi}\partial\bar\partial \log|z|^2\right)^{n-1} = \int_{B(0,t)\cap Z_f} \left(\frac{i}{2\pi t^2}\partial\bar\partial |z|^2\right)^{n-1}.$$

LEMMA 5.2. *If $u \in L^1(\bar B_r)$ and $\partial\bar\partial u$ is a measure, then*

$$\int_{t=r\neq 0}^{t=R} \frac{dt}{t} \int_{B_t} \frac{i}{2\pi}\partial\bar\partial u \wedge \left(\frac{i}{2\pi}\partial\bar\partial \log|z|^2\right)^{m-1} = \frac{1}{2}\int_{S_R} u\, d\sigma_R - \frac{1}{2}\int_{S_r} u\, d\sigma_r.$$

A proof of this result is available in the literature (see [13]).

We may now prove one of our two main theorems.

Proof of Theorem 1.1. It suffices to prove the result for small δ.

Since $n_{\psi_\omega}(r)$ is increasing, it follows that

$$n_{\psi_\omega}(r)\log(\kappa) \leq \int_{t=r}^{t=\kappa r} n_{\psi_\omega}(t)\frac{dt}{t} \leq n_{\psi_\omega}(\kappa r)\log(\kappa). \tag{2}$$

Let $\kappa = 1+\sqrt{\delta}$. Except for an event whose probability is less than $\exp\{-cr^{2n+2}\}$, we have:

$$\begin{aligned}
n_{\psi_\omega}(r)\log(\kappa) &\le \int_{t=r}^{t=\kappa r} n_{\psi_\omega}(t)\frac{dt}{t} \\
&= \int_{t=r}^{t=\kappa r}\int_{B(0,t)} \frac{i}{2\pi}\partial\bar{\partial}\log|\psi_\omega(z)| \wedge \left(\frac{i}{2\pi}\partial\bar{\partial}\log|z|^2\right)^{n-1}\frac{dt}{t} \\
&= \frac{1}{2}\int_{S_{\kappa r}} \log|\psi_\omega(z)|\,d\sigma - \frac{1}{2}\int_{S_r}\log|\psi_\omega(z)|\,d\sigma \quad \text{(by Lemma 5.2)} \\
&\le \frac{1}{2}\left(\left(\frac{1}{2}+\delta\right)\kappa^2 r^2 - \int_{S_r}\log|\psi_\omega(z)|\,d\sigma\right) \quad \text{(by Lemma 3.2)} \\
&\le \frac{1}{2}\left(\left(\frac{1}{2}+\delta\right)r^2\kappa^2 - \left(\frac{1}{2}-\delta\right)r^2\right) \quad \text{(by Lemma 4.4);} \\
2\frac{n_{\psi_\omega}(r)}{r^2} &\le \frac{1}{\log(\kappa)}\left(\kappa^2\left(\frac{1}{2}+\delta\right) - \left(\frac{1}{2}-\delta\right)\right) \\
&= \frac{\kappa^2-1}{2\log(\kappa)} + \delta\frac{\kappa^2+1}{\log(\kappa)} \le 1 + c\sqrt{\delta}.
\end{aligned}$$

We have just shown that

$$\mathrm{Prob}\left(\left\{\frac{n_{\psi_\omega}(r)}{r^2} \ge \frac{1}{2}+\delta\right\}\right) \le \exp\{-c_\delta r^{2n+2}\},$$

so now only half of the result remains to be proven. We complete the proof by observing that (except for an event whose probability is less than $\exp\{-cr^{2n+2}\}$):

$$\begin{aligned}
n_{\psi_\omega}(r)\log(\kappa) &\ge \int_{t=r/\kappa}^{t=r} n_{\psi_\omega}(t)\frac{dt}{t} \quad \text{(by (2))} \\
&= \int_{t=r/\kappa}^{t=r}\int_{B(0,t)} \frac{i}{2\pi}\partial\bar{\partial}\log|\psi_\omega(z)| \wedge \left(\frac{i}{2\pi}\partial\bar{\partial}\log|z|^2\right)^{n-1}\frac{dt}{t} \\
&= \frac{1}{2}\int_{S_r}\log|\psi_\omega(z)|\,d\sigma - \frac{1}{2}\int_{S_{r/\kappa}}\log|\psi_\omega(z)|\,d\sigma \quad \text{(by Lemma 5.2)} \\
&\ge \frac{1}{2}\left[\left(\frac{1}{2}-\delta\right)r^2 - \int_{S_{r/\kappa}}\log|\psi_\omega(z)|\,d\sigma\right] \quad \text{(by Lemma 4.4)} \\
&\ge \frac{1}{2}\left[\left(\frac{1}{2}-\delta\right)r^2 - \left(\frac{1}{2}+\delta\right)r^2\kappa^{-2}\right] \quad \text{(by Lemma 3.2);} \\
2\frac{n_{\psi_\omega}(r)}{r^2} &\ge \frac{1}{\log(\kappa)}\left(\left(\frac{1}{2}-\delta\right) - \left(\frac{1}{2}+\delta\right)\kappa^{-2}\right) \\
&= \frac{1-\kappa^{-2}}{2\log(\kappa)} - \delta\frac{1+\kappa^{-2}}{\log(\kappa)} \ge 1 - 2\sqrt{\delta}.
\end{aligned}$$

We have now finished the proof by showing that:

$$\mathrm{Prob}\left(\left\{\frac{n_{\psi_\omega}(r)}{r^2} \le \frac{1}{2}-\delta\right\}\right) \le \exp\{-c_\delta r^{2n+2}\}. \qquad \square$$

Using this estimate for the typical measure of the zero set of a random function, we obtain an upper bound for the hole probability. A lower bound with the same order of decay is easy to prove, as follows.

Proof of Theorem 1.2. The upper estimate is a consequence of the previous theorem: if there is a hole in a ball of radius r then $n_{\psi_\omega}(r) = 0$, and this can occur only for an event with probability less than $\exp\{-cr^{2n+2}\}$. Hence it suffices to show that the event where there is a hole in the ball of radius r contains an event whose probability is larger than $\exp\{-cr^{2n+2}\}$. We now design such a set. Let Ω_r be the event where $|\omega_0| \ge E_n + 1$ and

$$|\omega_j| \le \begin{cases} e^{-(1+n/2)r^2} & \text{if } 1 \le |j| \le \lceil 24nr^2 \rceil = \lceil (n\cdot 2\cdot 12)r^2 \rceil, \\ 2^{|j|/2} & \text{if } |j| > \lceil 24nr^2 \rceil \ge 24nr^2. \end{cases}$$

Then, by Proposition 3.1, we have

$$\begin{aligned} \mathrm{Prob}(\{|\omega_j| \le \exp\{-(1+n/2)r^2\}\}) &\ge \tfrac{1}{2}(\exp\{-(1+n/2)r^2\})^2 \\ &= \tfrac{1}{2}\exp\{-(2+n)r^2\} \end{aligned}$$

and so

$$\#\{j \in \mathbb{N}^n : 1 \le |j| \le \lceil 24nr^2 \rceil\} = \left(\!\binom{\lceil 24nr^2\rceil + n}{n}\!\right) \approx cr^{2n}.$$

Therefore, $\mathrm{Prob}(\Omega_r) \ge C(\exp\{-c_n r^{2n+2}\}) \ge \exp\{-cr^{2n+2}\}$ by independence and Proposition 3.1. It now suffices to show that, for all $\omega \in \Omega_r$ and all $z \in B(0,r)$, $\psi_\omega(z) \ne 0$. We proceed as follows:

$$|\psi_\omega(z)| \ge |\omega_0| - \sum_{|j|=1}^{|j|\le\lceil 24nr^2\rceil} |\omega_j| \frac{r^{|j|}}{\sqrt{j!}} - \sum_{|j|>\lceil 24nr^2\rceil} |\omega_j| \frac{r^{|j|}}{\sqrt{j!}} = |\omega_0| - \sum^1 - \sum^2,$$

where

$$\begin{aligned} \sum^1 &\le \exp\{-(1+n/2)r^2\} \sum_{|j|=1}^{|j|\le\lceil 24nr^2\rceil} \frac{r^{|j|}}{\sqrt{j!}} \\ &\le \exp\{-(1+n/2)r^2\}\sqrt{(24nr^2+1)^n}\sqrt{(\exp\{r^n\})} \\ &\qquad\qquad \text{(by the Cauchy–Schwarz inequality)} \\ &\le C_n r^n \exp\{-r^2\} \le c\exp\{-0.9r^2\} < \frac{1}{2} \quad \forall r > R_n \end{aligned}$$

and

$$\begin{aligned} \sum^2 &\le \sum_{|j|>24nr^2} 2^{|j|/2}\left(\frac{|j|}{24n}\right)^{|j|/2} \frac{1}{\sqrt{j!}} \quad \left(\text{since } r < \sqrt{|j|/24n}\right) \\ &\le c \sum_{|j|>24nr^2} 2^{|j|/2}\left(\frac{|j|}{24n}\right)^{|j|/2} \prod_{k=1}^{k=n}\left(\frac{e}{j_k}\right)^{j_k/2} \quad \text{(by Sterling's formula)} \\ &= c \sum_{|j|>24nr^2} \frac{(|j|)^{|j|/2}}{(\prod_{k=1}^{k=n} j_k^{j_k/2})n^{|j|/2}} \left(\frac{e}{12}\right)^{|j|/2} \le \end{aligned}$$

$$\leq c \sum_{|j|>1} \left(\frac{1}{4}\right)^{|j|/2} \quad \text{(by Proposition 3.1)}$$

$$\leq c \sum_{l>1} \left(\frac{1}{2}\right)^{l} l^n \leq E_n.$$

Hence $|\psi_\omega(z)| \geq E_n + 1 - \sum^1 - \sum^2 \geq \frac{1}{2}$. □

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