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Porteous's Formula for Maps between Coherent Sheaves

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1. Introduction

Recall what the Thom–Porteous formula for vector bundles tells us (see [2, Sec. 14.4] for details). Let X be an n-dimensional variety. Let E and F be vector bundles on X of ranks e and f, respectively, and let $\sigma: E \to F$ be a homomorphism of vector bundles. Choose an integer k with $k \le \min(e, f)$. Define

$$D_k(\sigma) = \{x \in X \mid \operatorname{rank}(\sigma_x) \le k\}.$$

Give $D_k(\sigma)$ the scheme structure defined locally by the vanishing of the $k \times k$ minors of a matrix representing σ. If each component of $D_k(\sigma)$ is either empty or of the expected codimension $(e-k)(f-k)$ in X and if X is Cohen–Macaulay, then the Thom–Porteous formula gives the class of $D_k(\sigma)$ in the Chow group of X as an explicit polynomial in the Chern classes of the vector bundles E and F. When X is not Cohen–Macaulay, one gets a positive cycle whose support is $D_k(\sigma)$.

Now suppose instead that E and F are assumed only to be coherent sheaves. What might one hope for from a formula of Thom–Porteous type in this situation? Consider Question (3.134) from [3].

Question. Is there a Porteous-type formula for maps of torsion-free coherent sheaves? That is, given a map $\varphi: E \to F$ of such sheaves on X, can we give the locus

$$X_r := \text{closure of } \{p : E \text{ and } F \text{ are locally free at } p \text{ and } \operatorname{rank}_p(\varphi) \le r\}$$

a scheme structure and then express its class in terms of the Chern classes of E and F and of local contributions, where E and F aren't locally free?

It is also perhaps useful to consider how excess and residual intersection formulas are used in enumerative geometry. Consider for instance Examples 9.1.8 and 9.1.9 of [2]. When counting the number of plane conics tangent to five given lines or five given conics, one knows that the excess component consists of double lines. Thus we know that one wishes to count not the excess component but instead only the number of points outside the excess component. Similarly, when trying to apply some sort of Porteous formula for coherent sheaves, we will need to determine from the specifics of the given problem whether one wants to count degeneracies that may occur in the nonlocally free locus of the sheaves involved.

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What is presented in this paper cannot really be called a Thom–Porteous formula for coherent sheaves. It is a process or procedure, rather than a formula, which proceeds as follows. Let $Y \subset X$ be the locus over which either E or F fails to be locally free. Restricting to the complement $X - Y$, σ is a homomorphism of vector bundles. We will show that, by blowing up Fitting ideals related to E and F, one can obtain a variety X' with its natural map g to X with the following properties. The map g gives an isomorphism between $X - Y$ and $g^{-1}(X - Y)$, the pullbacks of the restrictions of E and F to $X - Y$ extend to vector bundles on all of X', and the pullback of the restriction of σ to $X - Y$ extends to a homomorphism of these extended vector bundles over all of X'. One then applies the standard Thom–Porteous formula to this homomorphism of vector bundles on X'. Since the map g is given explicitly in terms of blowing up Fitting ideals, the hope is that one can understand what is going on in $X' - g^{-1}(X - Y)$ well enough to decide how degeneracies there (if any) should be counted and how to relate the Chern classes of E and F with those of the extended vector bundles on X'.

In Section 2 we explain the process in detail, constructing X' with its map g to X and showing that the claimed properties are satisfied. In Section 3 we apply the process to a specific question from [3].

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2. The General Setup

We shall always assume that our base scheme X is an n-dimensional variety (by *variety* we mean an integral separated scheme of finite type over an algebraically closed field k). We start with a lemma about coherent sheaves on X. See [1, Sec. 20.2] for some information on Fitting ideals.

LEMMA 1. *Let F be a coherent sheaf on the n-dimensional variety X. Let $Y \subset X$ be the locus where F fails to be locally free. Restricted to $X - Y$, F is a vector bundle; call its rank f. Let $\mathcal{I}$ be the fth Fitting ideal sheaf of F on X. The zero scheme of $\mathcal{I}$ has support equal to Y. Let $h\colon Z \to X$ be the blow-up of X along $\mathcal{I}$. Let h^*F be the pullback of F to Z. Then the double dual $(h^*F)^{**}$ is locally free, that is, a vector bundle.*

Proof. The question is clearly local on Z, so for any point $p \in Z$ we may replace Z by an affine open set containing p and assume that Z is affine equal to Spec A and that h^*F is the sheaf associated to an A-module M. Let M^t be the torsion submodule of M. Because Z was obtained by blowing up the fth Fitting ideal of F, we have that the fth Fitting ideal of M is generated by a single regular element of A. Also, because f was chosen to be the rank of F on its locally free locus, the fth Fitting ideal of M is its smallest nonzero Fitting ideal. Thus Lemma 1 of [4] applies to say that the quotient M/M^t is free. Dualizing the exact sequence

$0 \to M^t \to M \to M/M^t \to 0$, we obtain $0 \to (M/M^t)^* \to M^* \to (M^t)^* = 0$. Since M/M^t is free, its dual is free. Thus, M^* is free, so M^{**} is free. □

Now we may proceed with our general setup. Let X be an n-dimensional variety, let E and F be coherent sheaves on X, let $\sigma : E \to F$ be a homomorphism of coherent sheaves, and let $Y \subset X$ be the locus where either E or F fails to be locally free. On $X - Y$, E and F are vector bundles. Let e and f (respectively) be their ranks.

Let $h_1 \colon Z \to X$ be the blow-up of X along the fth Fitting ideal of F. On Z we have a homomorphism of coherent sheaves $h_1^*\sigma : h_1^*E \to h_1^*F$. Taking double duals yields $(h_1^*\sigma)^{**} \colon (h_1^*E)^{**} \to (h_1^*F)^{**}$, which extends the vector bundle homomorphism $h_1^*\sigma$ on $Z - h_1^{-1}(Y)$; by the lemma, the target sheaf $(h_1^*F)^{**}$ is locally free. Now let $h_2 \colon X' \to Z$ be the blow-up of Z along the eth Fitting ideal of $(h_1^*E)^{**}$. We then (using the lemma once again) have a homomorphism of vector bundles on X':

$$(h_2^*(h_1^*\sigma)^{**})^{**} \colon (h_2^*(h_1^*E)^{**})^{**} \to (h_2^*(h_1^*F)^{**})^{**}. \tag{1}$$

For notational convenience, rename the objects in equation (1) as

$$\sigma' \colon E' \to F'. \tag{2}$$

Also let $g = h_1 \circ h_2 \colon X' \to X$. We have proven the following theorem.

Theorem 1. *The map g gives an isomorphism between $X - Y$ and $g^{-1}(X - Y)$, the pullbacks of the restrictions of E and F to $X - Y$ extend to vector bundles on all of X', and the pullback of the restriction of σ to $X - Y$ extends to a homomorphism of these extended vector bundles over all of X'.*

For each nonnegative integer $k \le \min(e, f)$, set $m = n - (e-k)(f-k)$. Since σ' is a morphism of vector bundles, the standard Porteous formula gives a degeneracy class whose image in the Chow group $A_m(X')$ is

$$\Delta_{f-k}^{(e-k)}(c(F' - E')) \cap [X']$$

(see [2, Sec. 14.4]). The construction between Lemma 1 and Theorem 1 did not involve any unspecified choices. Thus, it is reasonable to make the following definition.

Definition 1. The kth degeneracy class of the morphism of coherent sheaves $\sigma \colon E \to F$ is

$$g_*(\Delta_{f-k}^{(e-k)}(c(F' - E')) \cap [X']).$$

3. An Example

This is the example that led Harris and Morrison to pose the question about Porteous's formula for nonlocally free sheaves mentioned in the Introduction. Their work on the example occurs in Chapter 3 (sections E through H) of [3]. For this example we work over the field of complex numbers.

Let $\bar{\mathcal{M}}_3$ be the Deligne–Mumford compactification of the moduli space of curves of genus 3 with open set $\mathcal{M}_3$ corresponding to nonsingular curves. Inside $\mathcal{M}_3$ we have the codimension-1 locus H of points corresponding to hyperelliptic curves. Harris and Morrison compute the class in the Picard group $\mathrm{Pic}(\bar{\mathcal{M}}_3)$ of the closure $\bar{H}$ of H in $\bar{\mathcal{M}}_3$ in terms of the first Chern class λ of the Hodge bundle and boundary divisors δ_0 (the closure of the locus of points corresponding to irreducible stable curves with one node) and δ_1 (the closure of the locus of points corresponding to an elliptic curve and a curve of genus 2 meeting at one point). They start by working on a generic 1-parameter family of curves of genus 3, $\pi: X \to B$. The plan is to construct a homomorphism of vector bundles $\sigma: E \to F$ on X for which the locus of points where the rank is less than or equal to unity consists of the hyperelliptic Weierstrass points of fibers of π.

Toward this end (see [3, pp. 162–164]), let $\omega_{X/B}$ be the relative dualizing sheaf of the family, let $X_2 := X \times_B X$ be the fiber product of X with itself over B with projection maps π_1 and π_2, and let $\Delta \subset X_2$ be the diagonal with ideal sheaf $\mathcal{I}_\Delta$. On X_2 we have the natural restriction map $\mathcal{O}_{X_2} \to \mathcal{O}_{X_2}/\mathcal{I}_\Delta^2$. Tensor both sides with $\pi_2^*\omega_{X/B}$ and then push down via π_1 to obtain

$$\sigma: (\pi_1)_*(\pi_2^*\omega_{X/B}) \to (\pi_1)_*(\pi_2^*\omega_{X/B} \otimes \mathcal{O}_{X_2}/\mathcal{I}_\Delta^2).$$

This is hoped to be the desired map σ, and E and F are the domain and target (respectively). Away from the singular fibers of π, this works well. The fiber of E over a point x of X is identified with global sections of the canonical bundle of the curve $\pi^{-1}(\pi(x))$, and the map σ takes the constant and linear terms of those sections written as power series centered at x; thus, σ has rank ≤ 1 exactly at the hyperelliptic Weierstrass points of fibers of π. This allows Harris and Morrison to use the standard Thom–Porteous formula to find the coefficient of λ in the expression for the class of $\bar{H}$.

The authors point out [3, p. 169] that, at singular points of fibers of π, F fails to be locally free (this can also be seen from the Fitting ideal computation in the proof of our Lemma 2, which will come up shortly), so one cannot use the standard Thom–Porteous formula to obtain the coefficients of δ_0 and δ_1. Thus they pose Question (3.134), mentioned in our Introduction. Harris and Morrison do obtain these coefficients by other means. In what follows we will use the process described in Sections 1 and 2 in order to use the Thom–Porteous formula to find the coefficients of δ_0.

We do not find the coefficient of δ_1; more than a coherent Thom–Porteous formula is needed for that. At singular curves corresponding to points of δ_1, not only does the sheaf F fail to be locally free at singular points of fibers of π, but also the map σ has rank ≤ 1 at all points of the elliptic component of the fiber. Perhaps someone could combine the coherent Porteous process presented here with an excess Porteous formula such as found in [2, Ex. 14.4.7] to compute the coefficient of δ_1. So, for the rest of this section we will use the coherent Porteous process of Sections 1 and 2 to prove the following proposition.

PROPOSITION 1. *Letting h equal the class of $\bar{H}$ in* $\mathrm{Pic}(\bar{\mathcal{M}}_3)$, *we have*

$$h = 9\lambda - \delta_0 + (??)\delta_1.$$

In order to apply the process of Sections 1 and 2, we must first compute the Fitting ideal of F.

LEMMA 2. *Near a node p of a fiber of π, the second Fitting ideal of F is the maximal ideal of p.*

Proof. Let p be a node of a fiber of π. The construction of F involves taking a product of X with itself. To help keep things organized, think of two copies of the family of curves $\pi : X \to B$. Choose local coordinates x_i and y_i $(i = 1, 2)$ centered at p on each copy of X, with the map π given locally by $x_i y_i = t_i$, so that t_i is a local coordinate on B centered at $\pi(p)$. Recall that $F = (\pi_1)_*(\pi_2^*\omega_{X/B} \otimes \mathcal{O}_{X_2}/\mathcal{I}_\Delta^2)$. Because $\pi_2^*\omega_{X/B}$ is locally free and of rank 1, when finding a presentation of F in a neighborhood of p we really need only find a presentation of $(\pi_1)_*\mathcal{O}_{X_2}/\mathcal{I}_\Delta^2$. Near $p \times p$, $X \times X$ has local coordinates x_1, y_1, x_2, y_2, $X \times_B X$ has local equation $x_1 y_1 = x_2 y_2$, and the diagonal has local equations $x_1 = x_2$ and $y_1 = y_2$. The ideal $\mathcal{I}_\Delta^2$ is locally generated by $(x_1 - x_2)^2$, $(y_1 - y_2)^2$, and $(x_1 - x_2)(y_1 - y_2)$. In a neighborhood of p on the first copy of X, $(\pi_1)_*\mathcal{O}_{X_2}/\mathcal{I}_\Delta^2$ as a module over $\mathcal{O}_X$ is generated by 1, $(x_1 - x_2)$, and $(y_1 - y_2)$ with the relation $y_1(x_1 - x_2) + x_1(y_1 - y_2) = 0$. The Fitting ideal is the maximal ideal (x_1, y_1). □

Let $g\colon X' \to X$ be the blow-up of X at the nodes of fibers of π, let E' and F' be the extended vector bundles on X', and let σ' be the extended map.

There is one issue that may be of concern. Of course, there are many possible extensions of g^*F restricted to $X' - g^{-1}(Y)$ to a vector bundle on all of X'. For instance, tensor any given extension with the line bundle associated to any Cartier divisor supported on $g^{-1}(Y)$. Which is the double dual that appears in Section 2?—the one for which we know the restriction of the map σ extends. The proof of [4, Lemma 1] shows that locally the desired extension is the one obtained by factoring out the exceptional divisor in a local presentation. How do these local pictures fit together globally? In the current example we get lucky. The nonlocally free locus on X is a set of isolated points. The local presentation for F near each point pulls back to give local presentations of g^*F along the exceptional divisor together with transition data on how they patch together. For more complicated nonlocally free loci, this issue could be more difficult to deal with.

As mentioned in the Introduction, when applying the coherent Porteous process to a given problem one must determine (a) how points in the nonlocally free locus need to be counted for the given enumerative problem and (b) how the coherent Porteous process is counting them. One then makes an adjustment if these two are not equal. We are using a generic 1-parameter family, so none of the singular fibers are elements of the closure of H. We do not want to count any singular curves as hyperelliptic curves. That the coherent Porteous process does not in fact count them (at least for fibers corresponding to general points of δ_0) is given by the next two lemmas.

LEMMA 3. *For fibers corresponding to general points of δ_0, the map σ' is surjective at all points of the exceptional divisor.*

Proof. Let C be a singular fiber of the family with node p. Let $\tilde{C}$ be the normalization of C with p_1 and p_2 lying over p. Fibers of E' at points of C are sections of the dualizing sheaf ω_C. From [3, Sec. 3A] we see that these are rational differentials on $\tilde{C}$ whose only allowed poles are simple poles at p_1 and p_2, with the sum of the residues being zero. With coordinates chosen as in the computation of the Fitting ideal in the proof of Lemma 2, x_2 will be a local coordinate on $\tilde{C}$ at p_1 and y_2 will be a local coordinate on $\tilde{C}$ at p_2. Because C is a general point of δ_0, we may assume that p_1 and p_2 are not conjugate under the hyperelliptic involution of $\tilde{C}$. Choose $\{\alpha_1, \alpha_2, \alpha_3\}$ as a basis for $\mathsf{H}^0(\omega_C)$, where α_1 has simple poles and no constant terms at p_1 and p_2 and where α_2 and α_3 are regular differentials on $\tilde{C}$, with α_2 vanishing at p_2 but not p_1 and with α_3 vanishing at p_1 but not p_2. After blowing up p on X and extending the bundles, the relation $y_1(x_1 - x_2) + x_1(y_1 - y_2) = 0$ becomes $\tilde{y}_1(x_1 - x_2) + (y_1 - y_2) = 0$ on one patch and $(x_1 - x_2) + \tilde{x}_1(y_1 - y_2) = 0$ on the other patch. After multiplying all the α by suitable constants, the map σ' becomes:

$$\sigma'(\alpha_1) = 1(1) + 0(x_1 - x_2) + 0(y_1 - y_2),$$

$$\sigma'(\alpha_2) = 0(1) + 1(x_1 - x_2) + 0(y_1 - y_2),$$

$$\sigma'(\alpha_3) = 0(1) + 0(x_1 - x_2) + 1(y_1 - y_2).$$

Thus, we see that σ' is always surjective at points of the exceptional divisor. □

LEMMA 4. *The map σ' is surjective at all nonsingular points of singular fibers corresponding to general points of δ_0.*

Proof. Suppose to the contrary that σ' failed to be surjective at q. Using the same notation as in the proof of Lemma 3, this would be a point q such that $\mathsf{h}^0(\omega_C - 2q) = 2$. Letting K be the canonical bundle on $\tilde{C}$, this becomes $\mathsf{h}^0(K + p_1 + p_2 - 2q) = 2$, which means that q is a hyperelliptic Weierstrass point on $\tilde{C}$ and that p_1 and p_2 are conjugate under the hyperelliptic involution—contrary to the assumption that C corresponds to a general point of δ_0. □

We can now conclude that the first degeneracy class of the map $\sigma : E \to F$ given in Definition 1 counts hyperelliptic curves as we wish (at least away from δ_1). What remains is to compute the Chern classes of E' and F' that come up in Porteous's formula and then to apply the standard Porteous formula. Because E is already a vector bundle, the Chern classes of E' are simply the pullbacks to X' of the Chern classes of E. To compute the Chern classes of F, Harris and Morrison [3, p. 163] use a two-term filtration:

$$0 \to F_2 \to F \to F_1 \to 0.$$

This filtration can be obtained by taking the following exact sequence on the fiber product X_2,

$$0 \to \frac{\mathcal{I}_\Delta}{\mathcal{I}_\Delta^2} \to \frac{\mathcal{O}_{X_2}}{\mathcal{I}_\Delta^2} \to \frac{\mathcal{O}_{X_2}}{\mathcal{I}_\Delta} \to 0,$$

tensoring it with $\pi_2^*\omega_{X/B}$, and then pushing down via $(\pi_1)_*$. The sheaf F_1 is still a vector bundle on all of X, but F_2 (like F) is only a coherent sheaf. Continue with the notation of the proofs of Lemmas 2, 3, and 4. Locally near the node p, $\mathcal{I}_\Delta/\mathcal{I}_\Delta^2$ is generated by (x_1-x_2) and (y_1-y_2) with the relation $y_1(x_1-x_2)+x_1(y_1-y_2) = 0$, just as $\mathcal{O}_{X_2}/\mathcal{I}_\Delta^2$ is generated by 1, (x_1-x_2), and (y_1-y_2) with the same relation. As we saw before, after blowing up p on X and extending the bundles, the relation becomes $\tilde{y}_1(x_1-x_2)+(y_1-y_2) = 0$ on one patch and $(x_1-x_2)+\tilde{x}_1(y_1-y_2) = 0$ on the other patch. Let $g\colon X' \to X$ be the blow-up map. Restricted to the exceptional divisor D, either (x_1-x_2) or (y_1-y_2) becomes a section of the extension of g^*F_2 with one simple zero on D. Harris and Morrison state that, on X (away from singular fibers), F_1 is $\omega_{X/B}$ and F_2 is the square of $\omega_{X/B}$. This continues to be true away from the singular points of singular fibers. The pullback of the square of $\omega_{X/B}$ to X' is trivial along D, whereas the extension of g^*F_2 restricted to D has degree 1. Putting all this together, we have the following lemma.

LEMMA 5. *On X' we have a two-term filtration,*

$$0 \to F_2' \to F' \to F_1' \to 0,$$

where F_1' is $g^*\omega_{X/B}$ and F_2' is $g^*\omega_{X/B}^2 \otimes \mathcal{O}(-D)$.

Thus the Chern class and Porteous computations from [3, pp. 163–164] are modified as follows:

$$c(F') = (1+\gamma)(1+2\gamma-D) = 1+3\gamma-D+2\gamma^2;$$
$$c(E') = 1-\lambda.$$

Let $[\Omega]$ denote the first degeneracy class of $\sigma'\colon E' \to F'$ on X'. Porteous's formula says $[\Omega] = c_2(E'^* - F'^*)$. Putting in our Chern classes for E' and F' yields

$$[\Omega] = 7\omega^2 - 3\omega\lambda + D^2.$$

But there is one component of D for each fiber of π from δ_0, and each component has square -1. Hence

$$(\pi g)_*([\Omega]) = 7\kappa - 12\lambda - \delta_0.$$

Use $\lambda = (\kappa+\delta)/12$ from [3, eq. (3.110)] and ignore δ_1 to obtain

$$(\pi g)_*([\Omega]) = 72\lambda - 8\delta_0 + (??)\delta_1.$$

Then divide by 8, since each hyperelliptic curve has eight hyperelliptic Weierstrass points:

$$h = 9\lambda - \delta_0 + (??)\delta_1.$$

This agrees with the Harris–Morrison result [3, p. 188].

References

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