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Borel Images and Analytic Functions

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1. Introduction

The σ-algebra $\mathcal{B}$ of Borel sets in $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is, by definition, the smallest σ-algebra that contains all open sets.

It is well known that the preimage of a Borel set under a continuous mapping is again a Borel set, whereas the image of a Borel set need not be a Borel set. See [13; 25; 26] for a discussion from the point of view of descriptive set theory.

We say that the mapping *f preserves Borel sets on A* if f is defined on A and if

$$B \subset A,\ B \in \mathcal{B} \implies f(B) \in \mathcal{B}.$$

There seems to be no standard name for this property.

The notion of injectivity plays an important role. Lusin and Suslin showed that any injective Borel measurable map $f : B \to \mathbb{C}$ ($B \in \mathcal{B}$) preserves Borel sets. In [7] it was shown that conformal maps of $\mathbb{D}$ into $\hat{\mathbb{C}}$ preserve Borel sets for radial limits.

Lusin and Purves have characterized the functions that preserve Borel sets in terms of the number of preimages at the points in their domain. All that we shall prove is based on the following result (see [14, p. 406; 21]).

Theorem (Lusin–Purves). *Let $A \in \mathcal{B}$ and let $f : A \to \hat{\mathbb{C}}$ have the property that $f^{-1}(E) \in \mathcal{B}$ for every $E \in \mathcal{B}$. Then f preserves Borel sets on A if and only if the set*

$$\{w : w = f(z) \text{ for uncountably many } z \in A\}$$

is countable.

We shall be interested in functions that are analytic (holomorphic) in the unit disk $\mathbb{D}$ and are continuous in $\bar{\mathbb{D}}$ (see Section 2) or have radial limits on subsets of $\mathbb{T} = \partial\mathbb{D}$ (see Sections 3 and 4). In Section 5 we characterize the plane domains whose

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universal covering maps preserve Borel sets. The remarkable Lusin–Purves theorem now allows us to apply results on value distribution to the problem of Borel images.

A *Suslin set* (or analytic set) is, by definition, the continuous image of some Borel set (see e.g. [13, pp. 85–87]). The images of Borel sets that we consider are always Suslin sets [16, Thm. 7(i)].

There are several results ([6; 11; 23], among others) showing that essentially any Suslin analytic set can occur as the image of a Borel set for various classes of meromorphic functions. Berman and Nishiura [5] proved that, for any nowhere dense perfect set $B \subset \mathbb{T}$ and any Suslin set $A \subset \hat{\mathbb{C}}$, there is an analytic function in $\mathbb{D}$ with radial limits on B such that $f(B) = A$. We want to thank the two authors for their valuable information.

2. Continuous Functions and Lacunary Series

Belov [4, Cor. 3.1] has proved an interesting result on lacunary series that improves an earlier result of Kahane, Weiss, and Weiss [12].

Theorem 1 (Belov). *Let $q = 3, 4, \dots$ and let*

$$f(z) = \sum_{k=1}^{\infty} a_k z^{q^k}.$$

If β and γ satisfy $\gamma(1+\beta) < 1$ and if

$$2\pi \frac{q-1}{q-2} \sum_{k=1}^{m} |a_k| q^k \le \gamma |a_{m+1}| q^{m+1} \quad \text{for } m \ge 0, \tag{1}$$

$$|a_{m+1}| \le \beta \sum_{k=m+2}^{\infty} |a_k| \quad \text{for } m \ge 0, \tag{2}$$

then f is continuous in $\bar{\mathbb{D}}$ and assumes every value in some disk uncountably often on $\mathbb{T}$. Hence f does not preserve Borel sets on $\mathbb{T}$.

The last statement follows from the Lusin–Purves theorem.

Example 1 (Kahane–Weiss–Weiss–Baranski). The lacunary series

$$f(z) = \sum_{k=1}^{\infty} \frac{1}{k^2} z^{q^k}$$

is continuous in $\bar{\mathbb{D}}$ and does not preserve Borel sets on $\mathbb{T}$.

This is perhaps the simplest example of this kind. The fact that f does not preserve Borel sets follows from [12, Thm. II′] (which, however, shows this only for the sum starting at some m) and more explicitly from a recent paper of Baranski [3].

Example 2 (Belov). Let $q = 5, 6, \dots$ and

$$f(z) = \sum_{k=1}^{\infty} q^{-\alpha k} z^{q^k}. \tag{3}$$

Now, by [4, Cor. 3.4], if $\alpha \in (0, 1/2)$ is small enough then conditions (1) and (2) are satisfied, so that f does not preserve Borel sets on $\mathbb{T}$. Furthermore, if $|z| = r \to 1$ then

$$\frac{r|f'(z)|}{1-r} \le \frac{1}{1-r}\sum_{k=1}^{\infty} q^{(1-\alpha)k} r^{q^k} \le \sum_{n=1}^{\infty}\Biggl(\sum_{q^k \le n} q^{(1-\alpha)k}\Biggr) r^n$$
$$= \sum_{n=1}^{\infty} O(n^{1-\alpha}) r^n = O((1-r)^{\alpha-2}).$$

Hence, f belongs to the class Λ_α of functions that are Hölder-continuous with exponent α (see [8, Thm. 5.1]).

In order to apply Theorem 1, we must have $\alpha \le 1/2$ because every function with Hölder exponent greater than $1/2$ maps $\mathbb{T}$ into a set of zero area (see [24]). But we only need some uncountable set (and not a disk) to be assumed uncountably often.

PROBLEM 1. Does the function (3) fail to preserve Borel sets for any $\alpha < 1$?

This, in turn, raises to the following general question.

PROBLEM 2. If an analytic function f preserves Borel sets, is the same true for $f + P$ for any polynomial P?

Suppose the answer is yes; then it would follow that any analytic function g with $|g'(z)| < M$ for $z \in \mathbb{D}$ also preserves Borel sets. Indeed, let $f(z) = Mz + g(z)$ and observe that it satisfies $\operatorname{Re}(f'(z)) > 0$, so that f is univalent (see [9, Thm. 2.16]). Hence f preserves Borel sets and so would g.

PROBLEM 3. Does f preserve Borel sets on $\mathbb{T}$ if f' is bounded?

As we have just seen, this would be true if Problem 2 had a positive answer. Compare also Corollary 2.

3. Radial Limits and Injectivity

Let $f: \mathbb{D} \to \hat{\mathbb{C}}$ be continuous; we do not assume that f is analytic. We consider the *radial limit*

$$f(\zeta) := \lim_{r \to 1} f(r\zeta) \in \hat{\mathbb{C}}, \quad \zeta \in \mathbb{T}, \tag{4}$$

wherever it exists. The set where the limit exists is a Borel set, and $f^{-1}(E) \in \mathcal{B}$ for $E \subset \mathcal{B}$ (see [20, Prop. 6.5]). Our positive results will be based on the following rather technical theorem, where we make essential use of the fact that we are working in two dimensions.

THEOREM 2. *Let $f: \mathbb{D} \to \hat{\mathbb{C}}$ be continuous and let $B \subset \mathbb{T}$ be a Borel set such that $f(\zeta)$ exists for $\zeta \in B$. If there are sets $U_n \subset \mathbb{D}$ $(n \in \mathbb{N})$ such that f is injective in each U_n and if*

$$[r\zeta,\zeta) \in U_n \quad \textit{for some } n = n(\zeta) \textit{ and } r = r(\zeta) < 1\ \forall \zeta \in B, \tag{5}$$

then $f(B)$ is a Borel set.

Proof. Let $Z(w) = \{\zeta \in B : f(\zeta) = w\}$ and

$$X = \{w \in \mathbb{C} : Z(w) \text{ is uncountable}\}. \tag{6}$$

If $w \in X$ then, by (5), there exist an $m \in \mathbb{N}$ and three distinct $\zeta_k = \zeta_k(w) \in B$ such that $f(\zeta_k) = w$ and $n(\zeta_k) = m$ for $k = 1,2,3$. Let X_m denote the set of these $w \in X$. Thus $X \subset \bigcup_{m=1}^{\infty} X_m$.

Now let $w \in X_m$. Since f is continuous and injective in U_m and since the radial limits $f(\zeta_k(w)) = w$ exist, the set

$$T(w) := \{w\} \cup \bigcup_{k=1}^{3} \{f(\rho\zeta_k(w)) : r(\zeta_k(w)) \le \rho < 1\}$$

is a *triod* (in the sense of R. L. Moore)—that is, the union of three Jordan arcs that meet only at their junction point w. Since f is injective in U_m, we see from (5) that $T(w) \cap T(w') = \emptyset$ for distinct $w, w' \in X_m$. Hence it follows from the Moore triod theorem ([17]; see also [20, Prop. 2.18]) that X_m is countable.

Therefore, X is also countable. Applying now the Lusin–Purves theorem to the restriction of f to the Borel set $\{\zeta \in B : f(\zeta) \in \mathbb{C}\}$, we deduce from (6) that $f(B)$ is a Borel set. □

COROLLARY 1. *Let f be a homeomorphism of $\mathbb{D}$ into $\hat{\mathbb{C}}$ extended to the subset A of $\mathbb{T}$ where the radial limit exists. Then f preserves Borel sets on $\mathbb{D} \cup A$.*

This result was essentially proved in [7]. Since f is a homeomorphism of $\mathbb{D}$ we have $f(B \cap \mathbb{D}) \in \mathcal{B}$, and $f(B \cap A) \in \mathcal{B}$ follows from Theorem 1 because we can choose $U_1 = \mathbb{D}$.

4. The Angular Derivative and Inner Functions

Let f be analytic in $\mathbb{D}$. This assumption guarantees that Borel sets in $\mathbb{D}$ are mapped to Borel sets. Hence only sets on $\mathbb{T}$ are of interest. We say that f has the *angular derivative* $f'(\zeta)$ at $\zeta \in \mathbb{T}$ if the radial limit $f(\zeta) \neq \infty$ exists and

$$\frac{f(z) - f(\zeta)}{z - \zeta} \to f'(\zeta) \text{ as } z \to \zeta,\ z \in \Delta, \tag{7}$$

for every Stolz angle Δ at ζ. The function f has a finite angular derivative if and only if (see [20, Prop. 4.7])

$$f'(z) \to f'(\zeta) \text{ as } z \to \zeta,\ z \in \Delta. \tag{8}$$

THEOREM 3. *Let f be analytic in $\mathbb{D}$ and let $B \in \mathbb{T}$ be a Borel set such that f has a finite nonzero angular derivative for $\zeta \in B$. Then $f(B)$ is a Borel set.*

Proof. Let $\mathcal{L}$ be the countable collection of all oriented lines $L \subset \mathbb{C}$ that pass through two points with rational coordinates. Since $f'(\zeta) \neq 0, \infty$, it follows (see e.g. [19, p. 291]) from (7) and (8) that f is injective in

$$\Delta(\zeta) = \{z : |\arg(1 - \bar{\zeta} z)| < 3\pi/4,\ r < |z| < 1\}$$

for suitable $r = r(\zeta)$ and that f is angle-preserving at ζ in $\Delta(\zeta)$. Hence there exist an $L \in \mathcal{L}$ and an open isosceles triangle $T(\zeta)$, with base in L and angle $\pi/2$ at its vertex $f(\zeta)$ to the left of L, such that f maps some domain $G(\zeta)$ one-to-one onto $T(\zeta)$ and

$$(r\zeta, \zeta) \subset G(\zeta) \quad \text{for some } \ r = r(\zeta) < 1. \tag{9}$$

Given $L \in \mathcal{L}$ and $k \in \mathbb{N}$, let $B(L, k)$ denote the set of all $\zeta \in B$ for which there is such a triangle $T(\zeta)$ of height $\geq 1/k$. Each of the countably many open sets

$$\bigcup_{\zeta \in B(L,k)} G(\zeta) \quad (L \in \mathcal{L},\ k \in \mathbb{N})$$

has countably many components. Altogether we obtain countably many domains in $\mathbb{D}$, which we arrange in a sequence (U_n).

The triangles $T(\zeta)$ for $\zeta \in B(L, k)$ are all congruent, have their base on L, and lie to the left of L. Furthermore, U_n is connected. Hence $V_n = f(U_n)$ is a Jordan domain; see [20, p. 146] for a diagram.

Fix $\zeta_n \in B \cap \partial U_n$. Now f has an inverse function g_n in $T(\zeta_n)$ because f maps $G(\zeta_n)$ one-to-one onto $T(\zeta_n)$. Since f is locally univalent in U_n, we can continue g_n analytically throughout V_n with values in U_n. Hence we conclude from the monodromy theorem that g_n is well-defined in V_n and $g_n(V_n) \subset U_n$. By the identity theorem, we have $g(f(z)) = z$ for $z \in U_n$.

Hence f is injective in U_n, and in view of (9) it follows from Theorem 2 that $f(B)$ is a Borel set. □

COROLLARY 2. *Let f be analytic in $\mathbb{D}$ and with a finite angular derivative f' for all $\zeta \in \mathbb{T}$. If f has only countably many critical values, then f preserves Borel sets on $\mathbb{T}$.*

A *critical value* is an angular limit $f(\zeta)$ such that $f'(\zeta) = 0$. If f' has only countably many zeros then there are only countably many critical values. Compare Problem 3.

Proof of Corollary 2. Let C be the set of critical values. Then $B \setminus f^{-1}(C) \in \mathcal{B}$, so that $f(B \setminus f^{-1}(C)) \in \mathcal{B}$ by Theorem 3. Since $f(B \cap f^{-1}(C)) \subset C$ is countable by assumption, we conclude that $f(B) \in \mathcal{B}$. □

By [20, Prop. 4.12] there exist conformal maps of $\mathbb{D}$ onto a Jordan domain that do not have a finite nonzero angular derivative at any point, whereas Borel sets are mapped onto Borel sets by [7] or Corollary 1. In this situation the angular derivative has no bearing on the problem of Borel images.

Now we discuss a situation where the angular derivative is very important. Let f be analytic in $\mathbb{D}$ and let $f(\mathbb{D}) \subset \mathbb{D}$. Then the radial limit $f(\zeta)$ exists for almost all $\zeta \in \mathbb{T}$. Let

$$E = \{\zeta \in \mathbb{T} : f(\zeta) \text{ exists and } f(\zeta) \in \mathbb{T}\}. \tag{10}$$

The Julia–Wolff lemma (see [20, Prop. 4.13]) states that $f'(\zeta)$ exists for $\zeta \in E$ and

$$0 < |f'(\zeta)| = \sup_{z \in \mathbb{D}} \frac{1 - |z|^2}{|\zeta - z|^2} \frac{|f(\zeta) - f(z)|^2}{1 - |f(z)|^2} \leq +\infty. \tag{11}$$

Let $\operatorname{mes}(E)$ denote the Lebesgue measure of the set E. Aleksandrov [2, Thm. 2] has proved that

$$\operatorname{mes}\{w \in f(E) : w = f(\zeta) \text{ for uncountably many } \zeta\} = 0 \tag{12}$$

holds if and only if the set $E_\infty = \{\zeta \in E : |f'(\zeta)| = \infty\}$ has measure 0.

By the Lusin–Purves theorem, it follows that f does not preserve Borel sets on E if $\operatorname{mes}(E_\infty) > 0$, whereas by Theorem 3 the function preserves Borel sets on E if E_∞ is countable. There is no conclusion if E_∞ is uncountable and of measure 0.

We call f an *inner function* if $\operatorname{mes}(E) = 2\pi$; see (10). Every Blaschke product is an inner function, and if f is an inner function then $(f - a)/(1 - \bar{a}f)$ is a Blaschke product for almost all $a \in \mathbb{D}$; (see e.g. [10, p. 79]). If $\mathbb{T} \setminus E$ is countable and nowhere dense on $\mathbb{T}$ then f is analytic on E, so f preserves Borel sets on $\mathbb{T}$.

EXAMPLE 3. We consider the Blaschke product

$$f(z) = \prod_{n=1}^{\infty} \frac{r_n^{2^n} - z^{2^n}}{1 - (r_n z)^{2^n}}, \quad r_n = 1 - 1/3^n. \tag{13}$$

Given t, we choose $k = k_n$ such that $|t - 2\pi k_n/2^n| \leq \pi/2^n$. Now

$$f(r_n \exp(2\pi i k/2^n)) = 0$$

by (13) and thus, by (11),

$$|f'(e^{it})| \geq \frac{1 - r_n^2}{(1 - r_n + |t - 2\pi k_n/2^n|)^2} \geq \text{const} \cdot (4/3)^n.$$

Hence $|f'(\zeta)| = \infty$ for all $\zeta \in \mathbb{T}$, so that (12) is false and f does not preserve Borel sets on E.

If the inner function f satisfies $(1 - |z|)f'(z) \to 0$ as $|z| \to 1$ but is not a finite Blaschke product, then $f^{-1}(w)$ has Hausdorff dimension 1 for every $w \in \mathbb{D}$ (by a theorem of Rohde [22]) and is therefore uncountable. Hence f does not preserve Borel sets on $\mathbb{T} \setminus E$.

5. Universal Covering Maps

A *universal covering map* f from $\mathbb{D}$ onto the domain $G \subset \hat{\mathbb{C}}$ is a locally univalent meromorphic function with $f(\mathbb{D}) = G$ such that every branch of f^{-1} has an analytic continuation throughout G with values in $\mathbb{D}$. For every domain G with at least three boundary points, there are infinitely many universal covering maps from $\mathbb{D}$ onto G. If G is simply connected then f is a Riemann map; otherwise, f assumes every value in G infinitely often. See, for instance, [1, Thm. 10.3].

THEOREM 4. *Let f be a universal covering map from $\mathbb{D}$ onto G and let E be the set of $\zeta \in \mathbb{T}$ where the radial limit $f(\zeta)$ exists. Then f preserves Borel sets on E if and only if there are only countably many $w \in \partial G$ with the following property*:

> *There exist an arc $C \subset G \cup \{w\}$ with endpoint w and Jordan domains H_n $(n \in \mathbb{N})$ with*
> $$\partial H_n \subset G, \quad \overline{H_n} \cap C \neq \emptyset, \quad H_n \cap \partial G \neq \emptyset, \quad w \notin H_n, \tag{14}$$
> *such that $H_n \to \{w\}$ as $n \to \infty$.*

Proof. (a) We may assume that $f(0) = \infty$ so that ∂G is a compact subset of $\mathbb{C}$. First we prove: If there are only countably many $w \in \partial G$ that satisfy (14), then $f(B) \in \mathcal{B}$ for every Borel set $B \subset E$. We may assume that no point $w = f(\zeta)$ $(\zeta \in B)$ satisfies (14) nor is an isolated point of ∂G. Indeed, there are only countably many such points w, and $f^{-1}(w)$ is a Borel set.

We consider the collection $\mathcal{D}$ of all disks $D \subset \mathbb{C}$ with rational center and rational radius. The collection $\mathcal{V}$ of all components V of all $D \cap G$ $(D \in \mathcal{D})$ is countable, and so is the collection $\mathcal{U}$ of all components U of all $f^{-1}(V)$ $(V \in \mathcal{V})$.

Let $\zeta \in B$. Then $C = \{f(\rho\zeta) : 0 \le \rho < 1\}$ is a half-open arc in G ending at $w = f(\zeta)$. For every disk $D \in \mathcal{D}$ with $w \in D$, there is a unique component $V \in \mathcal{V}$ that contains a subarc of C ending at w. We claim: If $\mathrm{diam}(D)$ is sufficiently small, then V is simply connected.

Suppose this claim is false. Then we can find $D_n \in \mathcal{D}$ with $w \in D_n$ and $D_n \to \{w\}$ such that V_n is multiply connected. Thus there exists a Jordan curve J_n with $J_n \subset V_n \subset G$ such that the inner domain H_n of J_n contains a point of $\partial V_n \subset \partial D_n \cup \partial G$ and hence a point of ∂G. Since w is not an isolated point of ∂G, we can choose H_n such that $w \notin H_n$. Furthermore, we can modify J_n so that $J_n \cap C \neq \emptyset$. Hence w satisfies (14) because $\mathrm{diam}(H_n) \le \mathrm{diam}(D_n) \to 0$. But this contradicts our assumption.

We have $L = \{f(\rho\zeta) : r \le \rho < 1\} \subset V$ for suitable $r < 1$. Let $U \in \mathcal{U}$ be the component of $f^{-1}(V)$ with $[r\zeta, \zeta) \subset U$, and let g be the branch of f^{-1} that maps L onto $[r\zeta, \zeta)$. Since $V \subset G$, it follows from the definition of a universal covering map that we can continue g analytically throughout V with values in $\mathbb{D}$; and since V is simply connected, it follows from the monodromy theorem that g is well-defined in V. Hence f is injective in U by the identity theorem. We can therefore apply Theorem 2 to the collection $\mathcal{U}$ to conclude that $f(B)$ is a Borel set.

(b) Let $w \in \partial G$ have the property (14). We may assume that C begins at $a_0 = f(0) = \infty$. Let a_n and b_n be the first and last points where C intersects $\overline{H_n}$. Replacing H_n by a suitable subsequence, we may assume that $\overline{H_n} \cap \overline{H_m} = \emptyset$ for $n \neq m$ and that b_n comes before a_{n+1}. Let B_n be an arc of C between b_n and a_{n+1} and let A_{nk} $(k = 0, 1)$ be the two arcs of $\partial H_n \setminus \{a_n, b_n\}$.

Let x be an irrational number in $(0, 1)$; it has the unique representation

$$x = \sum_{n=1}^{\infty} k_n 2^{-n} \quad \text{with } k_n \in \{0, 1\}. \tag{15}$$

We consider the half-open arc

$$C(x) = B_0 \cup A_{1k_1} \cup B_1 \cup A_{2k_2} \cup B_2 \cup \cdots \subset G \tag{16}$$

that begins at ∞ and satisfies $\overline{C(x)} \setminus C(x) = \{w\}$ because $\operatorname{diam}(H_n) \to 0$. The component $\Gamma(x)$ of $f^{-1}(C(x))$ that begins at 0 thus satisfies $\overline{\Gamma(x)} \setminus \Gamma(x) = \{\zeta(x)\}$ for some $\zeta(x) \in \mathbb{T}$. This follows (see [18, Cor. 9.2]) because f omits three values and is thus a normal function. Furthermore, it follows [15; 18, Thm. 9.3] that f has the radial limit $f(\zeta(x)) = w$ for every x.

Now let $x \neq x'$. By (15) there exists an n such that $k_n \neq k'_n$. But $H_n \cap \partial G \neq \emptyset$ and $w \notin H_n$ by (14). Since $\overline{H_n} \cap \overline{H_m} = \emptyset$ for $n \neq m$, we conclude from (16) that $C(x)$ and $C(x')$ are not homotopic in G. Hence we have $\zeta(x) \neq \zeta(x')$.

Thus there are uncountably many $\zeta \in \mathbb{T}$ with $f(\zeta) = w$ for every w satisfying (14). If f preserves Borel sets on E, then it follows from the Lusin–Purves theorem that there are only countably many such w. □

Example 4. Let F be a Cantor set on $\mathbb{R}$ and let

$$\partial G = F \cup \{x + i/n : x \in F,\ n \in \mathbb{N}\}.$$

Then every point of F satisfies (14) and so f does not preserve Borel sets.

Example 5. Let F be a Cantor set on $\mathbb{R}$ and let

$$\partial G = \bigcup_{x \in F} [x, x + i].$$

Then (14) is not satisfied by any point of ∂G because all components of F have diameter 1, whereas $\operatorname{diam}(H_n) \to 0$ as $n \to \infty$ in (14). Hence f preserves Borel sets on E.

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