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A Note on Brieskorn Spheres and the Generalized Smith Conjecture

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1. Introduction

Let $f: \mathbf{C}^n \to \mathbf{C}$ be the complex polynomial function defined by

$$f(z_1, \dots, z_n) = z_1^{a_1} + \dots + z_n^{a_n},$$

where the a_k are integers greater than 1. Then the origin is the only isolated singular point of the hypersurface $f^{-1}(0)$. In [9] Milnor showed that the intersection $\Sigma_{\mathbf{a}} = \Sigma(a_1, \dots, a_n)$ of $f^{-1}(0)$ with a sufficiently small sphere S_ε centered at the origin is a $(2n-3)$-dimensional smooth manifold and also is $(n-3)$-connected, where $\mathbf{a}$ denotes the n-tuple $(a_1, \dots, a_n)$ of the a_k. For each $1 \le k \le n$, $\Sigma_{\mathbf{a}}$ admits a periodic diffeomorphism T_k of period a_k defined by

$$T_k(z_1, \dots, z_k, \dots, z_n) = (z_1, \dots, \omega_k z_k, \dots, z_n)$$

such that the fixed point set of T_k is $\Sigma_{\hat{\mathbf{a}}_k}$, where ω_k is a primitive a_kth root of unity and $\hat{\mathbf{a}}_k = (a_1, \dots, \hat{a}_k, \dots, a_n)$. For the complement $\Sigma_{\mathbf{a}} - \Sigma_{\hat{\mathbf{a}}_k}$ of $\Sigma_{\hat{\mathbf{a}}_k}$ in $\Sigma_{\mathbf{a}}$, we first show the following theorem, which implies that $\Sigma_{\hat{\mathbf{a}}_k}$ is knotted in $\Sigma_{\mathbf{a}}$.

Theorem 1.1. *For each $1 \le k \le n$ and $n \ge 4$, $\Sigma_{\mathbf{a}} - \Sigma_{\hat{\mathbf{a}}_k}$ does not have the same homotopy type as S^1.*

It is well known that, for many suitable $\mathbf{a} = (a_1, \dots, a_n)$ and $n \ne 3$, $\Sigma_{\mathbf{a}}$ is a topological sphere (called a Brieskorn sphere). Milnor [9] and Brieskorn [1] gave the necessary and sufficient condition for $\Sigma_{\mathbf{a}}$ to be a topological sphere. We use a simple method of determining whether $\Sigma_{\mathbf{a}}$ is a topological sphere in terms of $\mathbf{a} = (a_1, \dots, a_n)$ given by Brieskorn. By choosing a special $\mathbf{a} = (a_1, \dots, a_n)$ such that $\Sigma_{\mathbf{a}}$ and $\Sigma_{\hat{\mathbf{a}}_k}$ (for some k) are topological spheres, we obtain a counterexample for the generalized Smith conjecture in the topological category.

Theorem 1.2. *Let m and n be integers such that $m \ge 2$ and $n \ge 5$. For $\mathbf{a} = (\underbrace{2, \dots, 2}_{n-3}, 2m-1, 2m+1, m)$, the Brieskorn manifolds $\Sigma_{\mathbf{a}}$ and $\Sigma_{\hat{\mathbf{a}}_n}$ are the topological spheres of dimensions $2n-3$ and $2n-5$, respectively. The $\mathbf{Z}_m$-action T_n on $\Sigma_{\mathbf{a}}$ has the fixed point set $\Sigma_{\hat{\mathbf{a}}_n}$ that is knotted in $\Sigma_{\mathbf{a}}$.*

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Brieskorn spheres can be exotic spheres or the spheres with standard differentiable structure (here called the standard spheres). By using the Arf invariant and the signature, Brieskorn [1] gave methods for determining whether $\Sigma_{\mathbf{a}}$ is an exotic sphere or a standard sphere. We also obtain counterexamples of the generalized Smith conjecture with fixed point set being a differentiably knotted standard sphere in the standard sphere. Let

$$\sigma_l = 2^{2l+1}(2^{2l-1}-1)\cdot \text{numerator}(4B_l/l),$$

where B_l denotes the lth Bernoulli number. Then our result is stated as follows.

THEOREM 1.3. *Let m and l be integers such that $m \geq 2$ and $l \geq 2$.*

(i) *For $\mathbf{a} = (\underbrace{2,\ldots,2}_{2l-1}, 2m\sigma_l+1, 2m\sigma_l-1, m)$, the Brieskorn manifolds $\Sigma_{\mathbf{a}}$ and $\Sigma_{\hat{\mathbf{a}}_{2l+2}}$ are the standard spheres of dimensions $4l+1$ and $4l-1$, respectively. The $\mathbf{Z}_m$-action T_{2l+2} on $\Sigma_{\mathbf{a}}$ has the fixed point set $\Sigma_{\hat{\mathbf{a}}_{2l+2}}$, which is differentiably knotted in $\Sigma_{\mathbf{a}}$.*

(ii) *For $\mathbf{a} = (\underbrace{2,\ldots,2}_{2l-2}, 2m\sigma_l+1, 2m\sigma_l-1, m)$, the Brieskorn manifolds $\Sigma_{\mathbf{a}}$ and $\Sigma_{\hat{\mathbf{a}}_{2l+1}}$ are the standard spheres of dimensions $4l-1$ and $4l-3$, respectively. The $\mathbf{Z}_m$-action T_{2l+1} on $\Sigma_{\mathbf{a}}$ has the fixed point set $\Sigma_{\hat{\mathbf{a}}_{2l+1}}$, which is differentiably knotted in $\Sigma_{\mathbf{a}}$.*

REMARK. The original Smith conjecture, which states that no periodic transformation of S^3 can have a tame knotted S^1 as its fixed point set, has been solved provided that the transformation is required to be a diffeomorphism (see [10]). However, its higher-dimensional analogs—known collectively as the generalized Smith conjecture (i.e., for all $n > 3$ no periodic transformation of S^n can have the tame knotted S^{n-2} as fixed point set)—are false in either category, as Giffen [3], Sumners [12], and Gordon [4] have shown using different methods. The idea of using Brieskorn manifolds to construct counterexamples is indicated by Davis [2]. The theorems just stated give *explicit counterexamples* of periodic actions on Brieskorn manifolds for any period $m > 1$; these examples are of interest because of their algebraic nature. The Brieskorn manifold $\Sigma(a_1, a_2, a_3)$ is not, in general, simply connected. For example, $\Sigma(2,3,5)$ is the Poincaré dodecahedral space $SO(3)/I$ (see [9]). Hence the counterexample using a Brieskorn manifold is given for odd-dimensional spheres of dimension not less than 7. Of course, our method is different from those methods used in [3], [12], and [4].

Theorem 1.1 is proved in Section 2. In Section 3 we review the work of Brieskorn—that is, the necessary and sufficient condition for $\Sigma_{\mathbf{a}}$ to be a topological sphere and the methods that determine $\Sigma_{\mathbf{a}}$ to be an exotic sphere or a standard sphere—and then give the proofs of Theorems 1.2 and 1.3. Throughout this paper, for a real number d, $[d]$ denotes the greatest integer not greater than d. For integers $a, b > 1$, by (a, b) we mean the greatest common divisor of a and b.

2. Proof of Theorem 1.1

Let $\Xi_{\mathbf{a}}(t) = \{\mathbf{z} \in \mathbf{C}^n \mid z_1^{a_1} + \cdots + z_n^{a_n} = t\}$ and $\Xi_{\mathbf{a}} = \Xi_{\mathbf{a}}(1)$. It is easy to see that, for $t \neq 0$, $\Xi_{\mathbf{a}}(t)$ is diffeomorphic to $\Xi_{\mathbf{a}}$ (also see [11]). In a natural way there exist diffeomorphisms $\xi_k^l\colon \Xi_{\mathbf{a}} \to \Xi_{\mathbf{a}}$ defined by

$$\xi_k^l(z_1, \ldots, z_k, \ldots, z_n) = (z_1, \ldots, \omega_k^l z_k, \ldots, z_n),$$

where $\omega_k = \exp(2\pi\mathbf{i}/a_k)$ and $1 \le l \le a_k$. All such ξ_k^l can generate a group denoted by $\Omega_{\mathbf{a}}$, and we let $\mathbf{Z}_{a_k} = \{\exp(2\pi l\mathbf{i}/a_k) \mid l = 1, \ldots, a_k\}$; then $\Omega_{\mathbf{a}} = \prod_{k=1}^n \mathbf{Z}_{a_k}$, the direct product of cyclic groups. Again let $J_{\mathbf{a}}$ denote the integer groupring on $\Omega_{\mathbf{a}}$, and let $I_{\mathbf{a}}$ be the ideal in $J_{\mathbf{a}}$ generated by elements $1 + \xi_k + \cdots + \xi_k^{a_k - 1}$. The following two results are due to Pham [11] and Brieskorn [1].

LEMMA 2.1 (Pham). *For $i \neq 0$ and $n-1$, $H_i(\Xi_{\mathbf{a}}; \mathbf{Z}) \cong 0$ and $H_{n-1}(\Xi_{\mathbf{a}}; \mathbf{Z}) \cong J_{\mathbf{a}}/I_{\mathbf{a}}$ are nontrivial.*

LEMMA 2.2 (Brieskorn). *For each $\mathbf{a} = (a_1, \ldots, a_n)$ and $n \ge 3$, $\Xi_{\mathbf{a}}$ is $(n-2)$-connected.*

In order to prove the Theorem 1.1, we look at $L = f^{-1}(0) - \{\mathbf{z} \in f^{-1}(0) \mid z_k = 0\}$.

LEMMA 2.3. *$\Sigma_{\mathbf{a}} - \Sigma_{\hat{\mathbf{a}}_k}$ is a deformation retract of L.*

Proof. Consider the diffeomorphism $\varphi\colon (\Sigma_{\mathbf{a}} - \Sigma_{\hat{\mathbf{a}}_k}) \times \mathbf{R}^+ \to L$ defined by

$$\varphi(z_1, \ldots, z_n, t) = (t^{1/a_1} z_1, \ldots, t^{1/a_n} z_n).$$

Given any $\mathbf{z} = (z_1, \ldots, z_n) \in L$, there exists only one $t_{\mathbf{z}}$ determined by $\mathbf{z}$ such that $(t_{\mathbf{z}}^{-1/a_1} z_1, \ldots, t_{\mathbf{z}}^{-1/a_n} z_n) \in \Sigma_{\mathbf{a}} - \Sigma_{\hat{\mathbf{a}}_k}$. In particular, if $\mathbf{z} \in \Sigma_{\mathbf{a}} - \Sigma_{\hat{\mathbf{a}}_k}$ then $t_{\mathbf{z}} = 1$. Now consider the map $F\colon L \times [0,1] \to L$ defined by

$$F(\mathbf{z}, s) = (t_{\mathbf{z}}^{-s/a_1} z_1, \ldots, t_{\mathbf{z}}^{-s/a_n} z_n).$$

Then F satisfies the following three properties: (i) $F(\mathbf{z}, 0) = \mathbf{z}$ for $\mathbf{z} \in L$; (ii) $F(\mathbf{z}, 1) \in \Sigma_{\mathbf{a}} - \Sigma_{\hat{\mathbf{a}}_k}$ for $\mathbf{z} \in L$; (iii) $F(\mathbf{z}, s) = \mathbf{z}$ for $\mathbf{z} \in \Sigma_{\mathbf{a}} - \Sigma_{\hat{\mathbf{a}}_k}$. This means exactly that $\Sigma_{\mathbf{a}} - \Sigma_{\hat{\mathbf{a}}_k}$ is a deformation retract of L. $\square$

Let $p\colon L \to \mathbf{C}^* = C - \{0\}$ be the map defined by

$$p(z_1, \ldots, z_k, \ldots, z_n) = z_k.$$

It is obvious that $p^{-1}(s) = \Xi_{\hat{\mathbf{a}}_k}(-s^{a_k})$ for each $s \in \mathbf{C}^*$.

LEMMA 2.4. *The map $p\colon L \to \mathbf{C}^*$ is a locally trivial fiber bundle over $\mathbf{C}^*$ with typical fiber $p^{-1}(1) = \Xi_{\hat{\mathbf{a}}_k}(-1)$.*

Proof. Consider the diffeomorphism $h_t\colon L \to L$ defined by

$$h_t(z_1, \ldots, z_n) = (t^{1/a_1} z_1, \ldots, t^{1/a_n} z_n),$$

where $t \in \mathbf{C}^* - R^-$ and t^{1/a_i} denotes the single-value branch with $1^{1/a_i} = 1$. Clearly h_t carries each fiber $p^{-1}(s)$ diffeomorphically onto the fiber $p^{-1}(t^{1/a_k}s)$. Given $s_0 \in \mathbf{C}^*$, let U be a small neighborhood of s_0 in $\mathbf{C}^*$. Then the correspondence $\psi\colon U \times p^{-1}(s_0) \to p^{-1}(U)$ defined by

$$\begin{aligned}
&\psi(s, (z_1, \dots, z_{k-1}, s_0, z_{k+1}, \dots, z_n)) \\
&\quad = h_{(s_0^{-1}s)^{a_k}}(z_1, \dots, z_{k-1}, s_0, z_{k+1}, \dots, z_n) \\
&\quad = ((s_0^{-1}s)^{a_k/a_1}z_1, \dots, (s_0^{-1}s)^{a_k/(a_{k-1})}z_{k-1}, s, (s_0^{-1}s)^{a_k/(a_{k+1})}z_{k+1}, \dots, (s_0^{-1}s)^{a_k/a_n}z_n)
\end{aligned}$$

maps the product $U \times p^{-1}(s_0)$ diffeomorphically onto $p^{-1}(U)$. Therefore, $p\colon L \to \mathbf{C}^*$ is a locally trivial fiber bundle over $\mathbf{C}^*$. □

Proof of Theorem 1.1. We look at the fiber homotopy exact sequence (in integer coefficients) for $p\colon L \to \mathbf{C}^*$ in Lemma 2.4:

$$\begin{aligned}
\cdots &\to \pi_{n-1}(\mathbf{C}^*) \to \pi_{n-2}(\Xi_{\hat{\mathbf{a}}_k}(-1)) \to \pi_{n-2}(L) \to \pi_{n-2}(\mathbf{C}^*) \to \cdots \to \pi_2(\mathbf{C}^*) \\
&\to \pi_1(\Xi_{\hat{\mathbf{a}}_k}(-1)) \to \pi_1(L) \to \pi_1(\mathbf{C}^*) \to \pi_0(\Xi_{\hat{\mathbf{a}}_k}(-1)) \to \pi_0(L) \to \pi_0(\mathbf{C}^*).
\end{aligned}$$

Note that S^1 and $\mathbf{C}^*$ have the same homotopy type, and $\Xi_{\hat{\mathbf{a}}_k}(-1)$ is diffeomorphic to $\Xi_{\hat{\mathbf{a}}_k}$. By Lemmas 2.1 and 2.2 and the Hurewicz theorem, we obtain from the above exact sequence that

$$\pi_1(L) \cong \pi_1(\mathbf{C}^*) \cong \mathbf{Z},$$
$$\pi_{n-2}(L) \cong \pi_{n-2}(\Xi_{\hat{\mathbf{a}}_k}(-1)) \cong H_{n-2}(\Xi_{\hat{\mathbf{a}}_k}(-1)) \cong J_{\hat{\mathbf{a}}_k}/I_{\hat{\mathbf{a}}_k} \ncong 0,$$

and $\pi_i(L) \cong \pi_i(\Xi_{\hat{\mathbf{a}}_k}(-1)) \cong 0$ when i is less than $n-2$ and $i \neq 1$. Moreover, by Lemma 2.3 we have that $\pi_i(\Sigma_{\mathbf{a}} - \Sigma_{\hat{\mathbf{a}}_k}) \cong 0$ when i is less than $n-2$ and $i \neq 1$, and

$$\pi_1(\Sigma_{\mathbf{a}} - \Sigma_{\hat{\mathbf{a}}_k}) \cong \mathbf{Z}, \qquad \pi_{n-2}(\Sigma_{\mathbf{a}} - \Sigma_{\hat{\mathbf{a}}_k}) \cong J_{\hat{\mathbf{a}}_k}/I_{\hat{\mathbf{a}}_k} \ncong 0.$$

This implies that $\Sigma_{\mathbf{a}} - \Sigma_{\hat{\mathbf{a}}_k}$ is not homotopy equivalent to S^1. □

3. Brieskorn's Work and Proofs of Theorems 1.2 and 1.3

Following the notation of Brieskorn [1], for $\mathbf{a} = (a_1, \dots, a_n)$ let $G_{\mathbf{a}} = G(a_1, \dots, a_n)$ be the graph with n vertices having weights $a_1, \dots, a_n$ and with edges defined as follows. Two vertices with weights a_i, a_j in $G_{\mathbf{a}}$ are connected by an edge if $\gcd(a_i, a_j) > 1$. The vertex with weight a_i is an isolated point of $G_{\mathbf{a}}$ if $(a_i, a_j) = 1$ for all $a_j \neq a_i$ in $G_{\mathbf{a}}$. Brieskorn proved the following.

PROPOSITION 3.1. *Let n be an integer greater than* 3, *and let $\mathbf{a} = (a_1, \dots, a_n)$ be an n-tuple of integers greater than* 1. *Then $\Sigma_{\mathbf{a}}$ is a topological sphere if and only if the graph $G_{\mathbf{a}}$ satisfies one of the following two conditions*:

(i) *$G_{\mathbf{a}}$ has at least two isolated points*;
(ii) *$G_{\mathbf{a}}$ has one isolated point and one component K consisting of an odd number of vertices, each with even weight, and with $(a_i, a_j) = 2$ for $i \neq j$ in K.*

Now it is very easy to show Theorem 1.2.

Proof of Theorem 1.2. It is obvious that $(m, 2m-1) = 1$, $(m, 2m+1) = 1$, and $(2m-1, 2m+1) = 1$; thus the vertices with weights $2m-1$ and $2m+1$ are two isolated points in the graph $G(\underbrace{2, \dots, 2}_{n-3}, 2m-1, 2m+1, m)$. Hence it follows from Proposition 3.1(i) that

$$\Sigma_{\mathbf{a}} = \Sigma(\underbrace{2, \dots, 2}_{n-3}, 2m-1, 2m+1, m) \quad \text{and} \quad \Sigma_{\hat{\mathbf{a}}_n} = \Sigma(\underbrace{2, \dots, 2}_{n-3}, 2m-1, 2m+1)$$

are topological spheres. As stated in Section 1, in a natural way $\Sigma_{\mathbf{a}}$ admits a periodic diffeomorphism of period m defined by

$$(z_1, \dots, z_{n-1}, z_n) \longrightarrow (z_1, \dots, z_{n-1}, \omega z_n)$$

such that the fixed point set is exactly $\Sigma_{\hat{\mathbf{a}}_n}$, where ω is a primitive mth root of unity. By Theorem 1.1, $\Sigma_{\mathbf{a}} - \Sigma_{\hat{\mathbf{a}}_n}$ does not have the same homotopy type as S^1. This means that the complement does not meet the unknotting criterion (see [4; 6]) and hence $\Sigma_{\hat{\mathbf{a}}_n}$ must be knotted in $\Sigma_{\mathbf{a}}$. This completes the proof. □

Next we review the methods given by Brieskorn for determining whether a topological sphere $\Sigma_{\mathbf{a}}$ is an exotic sphere or a standard sphere.

Let bP_{2k} denote the group (under the connected sum operation) of all $(2k-1)$-dimensional homotopy spheres, each of which bounds a parallelizable manifold. Using the Arf invariant and the signature, Kervaire and Milnor [7] showed that for odd k, $bP_{2k} \cong 0$ or $\mathbf{Z}_2$; for even $k = 2l \neq 2$, bP_{4l} is a cyclic group of order $\sigma_l/8$, where σ_l is the number stated in Section 1. Now let $M_{\mathbf{a}}(t)$ denote the intersection of $\Xi_{\mathbf{a}}(t)$ with a small ball

$$D_\varepsilon = \{\mathbf{z} \in \mathbf{C}^n \mid |z_1|^2 + \cdots + |z_n|^2 \le \varepsilon^2\}$$

and $M_{\mathbf{a}}(1) = M_{\mathbf{a}}$. Then $M_{\mathbf{a}}$ is a parallelizable manifold with boundary $\partial M_{\mathbf{a}}$ diffeomorphic to $\Sigma_{\mathbf{a}}$ (see [1, Lemma 7]). When $\Sigma_{\mathbf{a}}(a_1, \dots, a_n)$ is a topological sphere, Brieskorn calculated the Arf invariant $c(M_{\mathbf{a}})$ and the signature $\sigma(M_{\mathbf{a}})$ of $M_{\mathbf{a}}$, thus showing that each element of bP_{2k} is represented by some $\Sigma_{\mathbf{a}}$. Brieskorn's results are stated as follows.

Proposition 3.2. *Let $\Sigma_{\mathbf{a}} = \Sigma(a_1, \dots, a_n)$ be a topological sphere with $n > 3$.*

(i) *When n is even, $c(M_{\mathbf{a}}) \equiv 1 \pmod 2$ if and only if the graph $G_{\mathbf{a}}$ has only one isolated point with weight $a_k \equiv \pm 3 \pmod 8$ and only one component K consisting of an odd number of vertices, each with even weight, and with $(a_i, a_j) = 2$ for $i \neq j$ in K.*

(ii) *When n is odd, $\sigma(M_{\mathbf{a}}) = \sigma_{\mathbf{a}}^+ - \sigma_{\mathbf{a}}^-$. Here $\sigma_{\mathbf{a}}^+$ denotes the number of all n-tuples $(j_1, \dots, j_n)$ with $0 < j_k < a_k$ and $0 < \sum_{k=1}^n (j_k/a_k) < 1 \pmod 2$; $\sigma_{\mathbf{a}}^-$ denotes the number of all n-tuples $(j_1, \dots, j_n)$ with $0 < j_k < a_k$ and $-1 < \sum_{k=1}^n (j_k/a_k) < 0 \pmod 2$.*

Remark. It should be pointed out that, for even n, if $c(M_{\mathbf{a}}) \equiv 0 \pmod 2$ then $\Sigma_{\mathbf{a}}$ is diffeomorphic to the standard sphere (see [7]), and Levine [8] obtained that $c(M_{\mathbf{a}}) \equiv 0 \pmod 2$ if and only if $\Delta_{\mathbf{a}}(-1) \equiv \pm 1 \pmod 8$ and that

$c(M_{\mathbf{a}}) \equiv 1 \pmod 2$ if and only if $\Delta_{\mathbf{a}}(-1) \equiv \pm 3 \pmod 8$, where $\Delta_{\mathbf{a}}(t) = \prod_{0<l_k<a_k}(t-\omega_1^{l_1}\cdots\omega_n^{l_n})$ with $\omega_k = \exp(2\pi\mathbf{i}/a_k)$. For odd $n = 2l+1$, if $\sigma(M_{\mathbf{a}}) \equiv 0 \pmod{\sigma_l}$ then $\Sigma_{\mathbf{a}}$ is diffeomorphic to the standard sphere (see [7]).

Now let us discuss counterexamples of the generalized Smith conjecture for a knotted standard sphere in a standard sphere. To prove Theorem 1.3, we need the following results.

Let a, b, c be positive integers greater than 1. By $\Gamma(a, b, c)$ we denote the number of all (x, y, z) with $1 \le x \le a-1$, $1 \le y \le b-1$, $1 \le z \le c-1$, and $0 < x/a + y/b + z/c < 1$. From $0 < x/a + y/b + z/c < 1$ we have $0 < bcx + acy + abz < abc$. Furthermore,

$$1 \le x < \frac{abc - acy - abz}{bc} = a\left(1 - \frac{y}{b} - \frac{z}{c}\right).$$

Again from $1 - y/b - z/c > 0$, it follows that $1 \le y < b(1 - z/c)$. Therefore we have the following lemma.

LEMMA 3.3. $\Gamma(a, b, c) = \sum_{1\le z\le c-1}\sum_{1\le y\le[b(1-z/c)]}[a(1 - y/b - z/c)]$.

Hirzebruch and Mayer [5, p. 108, Proposition] gave a computation method of $\sigma(M_{\mathbf{a}})$ for $\mathbf{a} = (\underbrace{2, \dots, 2}_{2l-1}, a, b)$ with the positive odd numbers $a, b > 1$ and $(a, b) = 1$. For our purposes we give a computation formula of $\sigma(M_{\mathbf{a}})$ for $\mathbf{a} = (\underbrace{2, \dots, 2}_{2l-2}, a, b, c)$ such that the positive integer numbers $a, b, c > 1$ are relatively prime.

PROPOSITION 3.4. *Let* $\mathbf{a} = (\underbrace{2, \dots, 2}_{2l-2}, a, b, c)$ *such that the positive integer numbers* $a, b, c > 1$ *are relatively prime. Then*

$$\sigma(M_{\mathbf{a}}) = (-1)^{l-1}\{4\Gamma(a, b, c) - (a-1)(b-1)(c-1)\}.$$

Proof. By the definitions of $\sigma_{\mathbf{a}}^+$ and $\sigma_{\mathbf{a}}^-$ in Proposition 3.2, it is easy to see that $\sigma(M_{\mathbf{a}}) = (-1)^{l-1}\sigma(M_{\mathbf{a}'})$ where $\mathbf{a}' = (a, b, c)$. Now we need only consider $\sigma(M_{\mathbf{a}'})$. Since the equations

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \text{ or } 2$$

have no solutions in $\{(x, y, z) \mid 1 \le x \le a-1, 1 \le y \le b-1, 1 \le z \le c-1\}$, we have

$$\sigma_{\mathbf{a}'}^+ + \sigma_{\mathbf{a}'}^- = (a-1)(b-1)(c-1).$$

Using the mapping defined by $(x, y, z) \to (a-x, b-y, c-z)$, we conclude that the two sets

$$\left\{(x, y, z) \;\middle|\; 0 < \frac{x}{a} + \frac{y}{b} + \frac{z}{c} < 1,\ 1 \le x \le a-1,\ 1 \le y \le b-1,\ 1 \le z \le c-1\right\}$$

and

$$\left\{(x,y,z) \;\middle|\; 2 < \frac{x}{a}+\frac{y}{b}+\frac{z}{c} < 3,\ 1 \le x \le a-1,\ 1 \le y \le b-1,\ 1 \le z \le c-1\right\}$$

have the same number of elements. By Lemma 3.3, it follows that

$$\sigma^+_{\mathbf{a}'} = 2\Gamma(a,b,c)$$

and thus

$$\begin{aligned}\sigma(M_{\mathbf{a}}) &= (-1)^{l-1}\sigma(M_{\mathbf{a}'})\\ &= (-1)^{l-1}(\sigma^+_{\mathbf{a}'} - \sigma^-_{\mathbf{a}'})\\ &= (-1)^{l-1}\{4\Gamma(a,b,c) - (a-1)(b-1)(c-1)\}. \qquad \square\end{aligned}$$

COROLLARY 3.5. *Let* $\mathbf{a} = (\underbrace{2,\ldots,2}_{2l-2}, 2uv+1, 2uv-1, u)$ *with integers* $u \ge 2$ *and* $v \ge 1$. *Then*

$$\sigma(M_{\mathbf{a}}) = (-1)^l \tfrac{4}{3}u(u^2-1)v^2.$$

Proof. By Proposition 3.4, we need only compute $\Gamma(2uv+1, 2uv-1, u)$. Hence, by Lemma 3.3 we have

$$\begin{aligned}&\Gamma(2uv+1, 2uv-1, u)\\ &\quad= \sum_{1\le z\le u-1}\ \sum_{1\le y\le[(2uv-1)(1-z/u)]}\left[(2uv+1)\left(1-\frac{y}{2uv-1}-\frac{z}{u}\right)\right]\\ &\quad= \sum_{1\le z\le u-1}\ \sum_{1\le y\le 2uv-2vz-1}\left[2uv+1-2vz-y-\left(\frac{2y}{2uv-1}+\frac{z}{u}\right)\right].\end{aligned}$$

By direct calculations, we see that $1 \le y \le uv - vz - 1$ implies $0 < \frac{2y}{2uv-1} + \frac{z}{u} < 1$ and that $uv - vz \le y \le 2uv - 2vz - 1$ implies $1 < \frac{2y}{2uv-1} + \frac{z}{u} < 2$. Therefore,

$$\begin{aligned}\Gamma(2uv+1, 2uv-1, u) &= \sum_{1\le z\le u-1}\left\{\sum_{1\le y\le uv-vz-1}(2uv-2vz-y)\right.\\ &\qquad\left. + \sum_{uv-vz\le y\le 2uv-2vz-1}(2uv-2vz-y-1)\right\}\\ &= 2v\sum_{1\le z\le u-1}(u-z)(uv-vz-1)\\ &= \frac{v^2u(u-1)(2u-1)}{3} - vu(u-1).\end{aligned}$$

Furthermore, we have

$$\begin{aligned}\sigma(M_{\mathbf{a}}) &= (-1)^{l-1}\{4\Gamma(2uv+1, 2uv-1, u) - 4uv(uv-1)(u-1)\}\\ &= (-1)^{l-1}4uv(u-1)\left\{\frac{v(2u-1)}{3} - 1 - (uv-1)\right\}\\ &= (-1)^l\tfrac{4}{3}u(u^2-1)v^2. \qquad \square\end{aligned}$$

Proof of Theorem 1.3(i). By Proposition 3.1(i), $\Sigma_{\mathbf{a}}$ and $\Sigma_{\hat{\mathbf{a}}_{2l+2}}$ are topological spheres because the vertices with weights $2m\sigma_l+1$ and $2m\sigma_l-1$ are two isolated points in the graph

$$G(\underbrace{2,\ldots,2}_{2l-1}, 2m\sigma_l+1, 2m\sigma_l-1, m).$$

It follows from Proposition 3.2(i) that $c(M_{\mathbf{a}}) \equiv 0 \pmod 2$ and thus $\Sigma_{\mathbf{a}}$ is a $(4l+1)$-dimensional standard sphere. Now we prove that $\Sigma_{\hat{\mathbf{a}}_{2l+2}}$ is a standard sphere, too. Choose $u=2$ and $v=m\sigma_l/2$ in Corollary 3.5 (note that σ_l is even); we have

$$\sigma(M_{\hat{\mathbf{a}}_{2l+2}}) = (-1)^l 2m^2\sigma_l^2 \equiv 0 \pmod{\sigma_l}$$

and thus $\Sigma_{\hat{\mathbf{a}}_{2l+2}}$ is a standard sphere. Finally, as in the proof of Theorem 1.2, by Theorem 1.1 we conclude that $\Sigma_{\mathbf{a}}$ admits a periodic diffeomorphism of period m with differentiably knotted $\Sigma_{\hat{\mathbf{a}}_{2l+2}}$ in $\Sigma_{\mathbf{a}}$ as fixed point set. □

Proof of Theorem 1.3(ii). Similarly to the proof of Theorem 1.3(i), we need merely to show that $\Sigma_{\mathbf{a}}$ and $\Sigma_{\hat{\mathbf{a}}_{2l+1}}$ are standard spheres for $\mathbf{a} = (\underbrace{2,\ldots,2}_{2l-2}, 2m\sigma_l+1, 2m\sigma_l-1, m)$. First, it is easy to see from Propositions 3.1(i) and 3.2(i) that $\Sigma_{\mathbf{a}}$ and $\Sigma_{\hat{\mathbf{a}}_{2l+1}}$ are topological spheres; in particular, $\Sigma_{\hat{\mathbf{a}}_{2l+1}}$ is a $(4l-3)$-dimensional standard sphere. Taking $u=m$ and $v=\sigma_l$ in Corollary 3.6, it follows that

$$\sigma(M_{\mathbf{a}}) = (-1)^l \tfrac{4}{3} m(m^2-1)\sigma_l^2 \equiv 0 \pmod{\sigma_l}$$

and thus $\Sigma_{\mathbf{a}}$ is a standard sphere. This completes the proof. □

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