Relatively Hyperbolic Groups

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1. Introduction

Let V be a complete, noncompact Riemannian manifold of constant negative curvature with finite volume. Then V has only finitely many ends, $E_1, E_2, \dots, E_k$. The inclusions $E_i \subset V$ induce injections $\pi_1(E_i) \hookrightarrow \pi_1(V)$ $(i = 1, 2, \dots, k)$. The relations between the fundamental group $\pi_1(V)$ and the subgroups $\pi_1(E_i)$ are the main motivation for introducing a theory of hyperbolic groups relatively to the family of subgroups (for short, relatively hyperbolic groups). The idea is similar to the case of the fundamental group of compact hyperbolic manifold. Its geometric and combinatorial structure gives us the definition of word-hyperbolic groups [1; 3; 7].

There are two definitions of relatively hyperbolic groups. The first one proposed in [7] by Gromov (cf. Definition 1) is a generalization of the parabolic properties of the subgroups $\pi_1(E_i)$. The second definition (cf. Definition 2), proposed by Farb in [4] (see also [5]), is expressed by properties of the modification of the Cayley graph (coned-off Cayley graph) of $\pi_1(V)$. According to the first definition, it is obvious that $\pi_1(V)$ is hyperbolic relatively to the family of subgroups $\pi_1(E_i)$ $(i = 1, 2, \dots, k)$. Farb's definition is weaker, and the proof of the hyperbolicity of $\pi_1(V)$ relatively to the family $\pi(E_i)$ $(1 \le i \le k)$ becomes more difficult. However, it is convenient for constructing many illustrative examples. In this note we want to prove (Theorem 1) that the Gromov definition is stronger than the one by Farb, and we give an example (Example 3) of a group that is relatively hyperbolic in the sense of Farb's definition but is *not* relatively hyperbolic in the sense of Gromov's definition.

The paper is organized as follows. In Section 2 we formulate the two definitions of relatively hyperbolic groups and give some examples. This part is based on [7, 8.6] and [4, 1.1]. In Section 3 we prove our main result (Theorem 1) that the Farb definition is more general than the Gromov definition. The main idea of proof, which was proposed to us by Brian Bowditch, is the following proposition.

Proposition. *Let (X, d) be a δ-hyperbolic metric space $(\delta \ge 0)$ with the collection of closed disjoint ε-quasiconvex subsets. Let each subset contract to a point.*

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Then the resulting space is hyperbolic (*in the Gromov sense*) *provided the distance between any two quasiconvex sets is bounded below by some constant* $R = R(\delta, \varepsilon)$ *depending on* δ *and* ε.

Section 3 ends with an example of a relatively hyperbolic group in the sense of Farb that is not a relatively hyperbolic group in the Gromov sense.

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2. Definitions

Let X be a complete hyperbolic locally compact geodesic space with a discrete isometric action of a group Γ such that the quotient space $V = X/\Gamma$ is quasi-isometric to the union of k copies of $[0, \infty)$ joined at zero. To simplify the matter, we assume that the action of Γ on X is free and then lift the k rays in V (corresponding to the k points in $\partial V \approx \{1, 2, \dots, k\}$) to k rays $r_i : [0, \infty) \to X$, $i = 1, 2, \dots, k$. Denote by h_i the corresponding horofunctions and by $r_i(\infty) \in \partial X$ the limit points of r_i. Denote by $\Gamma_i \subset \Gamma$ the isotropy subgroups of $r_i(\infty)$ for the action of Γ on ∂X, and assume that Γ_i preserves h_i for $i = 1, 2, \dots, k$.

Denote by $B_i(\rho)$ the horoballs $h_i^{-1}(-\infty, \rho) \subset X$ and assume that, for a sufficiently small ρ, the intersection $\gamma B_i(\rho) \cap B_j(\rho)$ is empty unless $i = j$ and $\gamma \in \Gamma_i$. Denote by $\Gamma B(\rho) \subset X$ the union $\bigcup_{i,\gamma} \gamma B_i(\rho)$ over $i = 1, 2, \dots, k$ and all $\gamma \in \Gamma$. Let $X(\rho) = X \setminus \Gamma B(\rho)$ and assume that the action of Γ on $X(\rho)$ is co-compact for all $\rho \in (-\infty, \infty)$.

DEFINITION 1 [6]. A group Γ is called *word-hyperbolic* relative to some subgroups $\Gamma_1, \Gamma_2, \dots, \Gamma_k$ in Γ if Γ admits an action on some X with the foregoing properties, where Γ_i denotes the isotropy subgroups of h_i.

EXAMPLE 1. Let Γ be a finite co-volume discrete isometry group of a complete simply connected Riemannian manifold X with *pinched* negative curvature,

$$0 > -a \geq K(X) \geq -b > -\infty.$$

Then Γ is hyperbolic relative to the cuspidal subgroups in the sense of Definition 1.

Let us formulate a Farb definition (see [4; 5]). Let G be a finitely generated group, and let $\{H_1, \dots, H_r\}$ be a finite set of finitely generated subgroups of G. We begin with the Cayley graph Γ of G, and we form a new graph $\hat{\Gamma} = \hat{\Gamma}(\{H_1, \dots, H_r\})$ as follows: for each coset gH_i $(1 \leq i \leq r)$ of H_i in G, add a vertex $\upsilon(gH_i)$ to Γ and add an edge $e(gh_i)$ of length $1/2$ from each element gh_i of gH_i to the vertex $\upsilon(gH_i)$.

We call this new graph the *coned-off Cayley graph* of G with respect to $\{H_1, \dots, H_r\}$. Although $\hat{\Gamma}$ is not a proper metric space (i.e., closed balls are not always compact), it is still a path-metric space.

The graph $\hat{\Gamma}$ is also a geodesic metric space; that is, there exists a geodesic between any two points. Thus it makes sense to talk about geodesic triangles in $\hat{\Gamma}$ and whether or not these triangles are δ-thin.

DEFINITION 2 [4]. The group G is *hyperbolic* relative to $\{H_1, \dots, H_r\}$ if the coned-off Cayley graph $\hat{\Gamma}$ of G with respect to $\{H_1, \dots, H_r\}$ is a negatively curved metric space.

We now give a few examples of relatively hyperbolic groups in the sense of Definition 2. Most of them come from Farb's thesis [4].

EXAMPLE 2. (1) Let M^n be a complete, finite volume Riemannian manifold as in Example 1. A group $\Gamma = \pi_1(M^n)$ is hyperbolic relative to the cuspidal subgroups (see [4, p. 73]).

(2) Let $\mathrm{Mod}(S)$ be a mapping class group. By definition, this is the group of autohomeomorphisms of the surface S, up to isotopy. There are only a finite number of distinct nontrivial, nonperipheral homotopy classes of simple curves in S (distinguished by the topological type of their complement), up to the action of $\mathrm{Mod}(S)$. Let $\{\alpha_1, \alpha_2, \dots, \alpha_N\}$ be a fixed list of representatives of these $\mathrm{Mod}(S)$-orbits, and let H_j be the subgroup of $\mathrm{Mod}(S)$ fixing α_j. Then the group $\mathrm{Mod}(S)$ is hyperbolic relative to $\{H_1, H_2, \dots, H_N\}$ (see [8, Thm. 1.3]).

(3) A group G is hyperbolic relative to the trivial subgroup if and only if G is a word-hyperbolic group.

(4) If H is normal in Γ, then Γ is hyperbolic relative to H if and only if Γ/H is a word-hyperbolic group.

(5) A word-hyperbolic group is hyperbolic relative to any quasiconvex subgroup (cf. [6]).

For commentary and more examples, see [4] and [5].

3. Proof of Main Result

We need the following proposition.

PROPOSITION 1. *Let (X, d) be a δ-hyperbolic metric space $(\delta \geq 0)$ with the collection of closed disjoint ε-quasiconvex subsets. Let each subset contract to a point. Then the resulting space is hyperbolic (in Gromov's sense) provided the distance between any two quasiconvex sets is bounded below by some constant $R = R(\delta, \varepsilon)$ depending on δ and ε.*

Proof. Choose R so that $R > (24\delta + 4\varepsilon)$. Let X' be the space formed by deleting the interiors of all quasiconvex subsets, and give X' the following metric: it is zero at the boundary of quasiconvex subsets, and it agrees with the X-metric outside of these subsets. This makes X' into a pseudometric space, which we give the path metric.

When considering paths γ in X', we will always assume for simplicity that any subsegment of γ lying on a quasiconvex subset A is drawn as a geodesic on A; this does not change any metric properties of paths, since the metric on A is zero. The X'-length of a path γ, denoted by $l_{X'}(\gamma)$, is just the sum of the X-lengths of the subpaths of γ lying outside every quasiconvex subset.

A geodesic between $x, y \in X'$ is a path γ in X' from x to y such that $l_{X'}(\gamma)$ is minimal. It is not hard to see that, for a X' geodesic γ between x and y, the subpaths of γ lying outside every quasiconvex subset qualitatively consist of the following: the shortest X-path from x to some quasiconvex subset, followed by the union of paths that are the shortest X-paths between two quasiconvex subsets, followed by the shortest X-path from some final quasiconvex subset to y (cf. [4, 3.2]).

LEMMA 1. *The assumptions of the preceding proposition are kept. Then there exist constants $K = K(\delta, \varepsilon)$ and $L = L(R)$ with the following property. Let β be any X'-geodesic from x to y, and let γ be the X-geodesic from x to y. Then any subsegment of β that lies outside $Nbhd_X(\gamma, K)$ must have X'-length at most L. In particular, any X' geodesic from x to y stays completely inside $Nbhd_{X'}(\gamma, K + L/2)$.*

Proof of Lemma 1. For one quasiconvex set we have a picture shown as Figure 1, where P is the projection on a quasiconvex set (see [3, p. 108]. We shall give an estimate for $d_X(P(x), P(y))$.

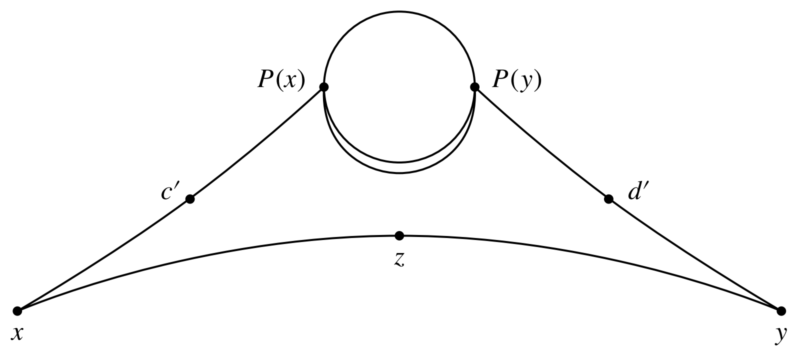

Figure 1

We first recall the inequality

$$d_X(P(x), P(y)) \leq \max(C, C + d_X(x, y) - d_X(x, P(x)) - d_X(y, P(y))),$$

where $C = 2\varepsilon + 12\delta$ (see [7, Lemma 7.3D] or [3, Chap. 10, Prop. 2.1]). Hence, if $d_X(x, y) \leq d_X(x, P(x)) + d_X(y, P(y))$ then $d_X(P(x), P(y)) \leq C$.

We assume that

$$d_X(x, y) \geq d_X(P(x), x) + d_X(P(y), y).$$

From the definition of δ-hyperbolic space, for the points $(x, y, P(x), P(y))$ we have the following inequality:

$$\forall z \in [x, y],\ \ d_X(z, [x, P(x)] \cup [P(x), P(y)] \cup [P(y), y]) \leq 2\delta.$$

We have two cases: either (i)

$$\exists z \in [x, y] \mid d_X(z, [P(x), P(y)]) \leq 2\delta;$$

or (ii)

$$\exists z \in [x, y] \quad \text{and} \quad \exists c' \in [x, P(x)],\ d' \in [y, P(y)]$$

such that

$$d_X(c', z) \leq 2\delta \quad \text{and} \quad d_X(d', z) \leq 2\delta.$$

In the second case, by assumption and from the triangle inequality we have

$$d_X(P(x), x) + d_X(P(y), y) \leq d_X(x, y) \leq 4\delta + d_X(d', y) + d_X(c', x).$$

Since $d_X(P(x), x) = d_X(x, c') + d_X(c', P(x))$ and $d_X(P(y), y) = d_X(y, d') + d_X(d', P(y))$, it follows that

$$d_X(c', P(x)) + d_X(d', P(y)) \leq 4\delta.$$

Hence, in this case we have $d_X(P(x), P(y)) \leq 8\delta$.

Summing up we have proved:

$$d_X(z, [P(x), P(y)]) > 2\delta\ \forall z \in [x, y] \Rightarrow d_X(P(x), P(y)) \leq C. \qquad (\dagger)$$

Let us start to prove a general case. Choose $K \geq 2\delta + \varepsilon$. Suppose that β' is a subsegment of β lying completely outside $Nbhd_X(\gamma, K)$; say $z = \beta'(0)$ satisfies $d_X(z, \gamma) = K$, and let w be the last point of β' with $d_X(w, \gamma) = K$. We can assume that the first and the last point of β' do not lie on one of the given quasiconvex sets. Let z' (resp. w') denote the image of z (resp. w) under a projection onto γ (note that γ is a quasiconvex set).

See Figure 2 and recall that $d_X(s, \gamma) > K$ for all $s \in \beta'$. We have:

$$l_{X'}(\beta') \leq d_X(z, z') + d_X(z', w') + d_X(w', w); \qquad (1)$$

$$l_{X'}(\beta') \leq 2K + (2(\#\text{q.c.s.}) + 1))C = 2K + 2(\#\text{q.c.s.})C + C, \qquad (2)$$

where "(#q.c.s.)" means the number of quasiconvex sets penetrated by β'. The second inequality follows from ($\dagger$). Now quasiconvex sets are separated by an X-distance of at least R, so

$$l_{X'}(\beta') \geq (\#\text{q.c.s.} - 1)R$$

and so

$$(\#\text{q.c.s.}) \leq \frac{l_{X'}(\beta')}{R} + 1.$$

Plugging back into the inequalities (1) and (2) gives

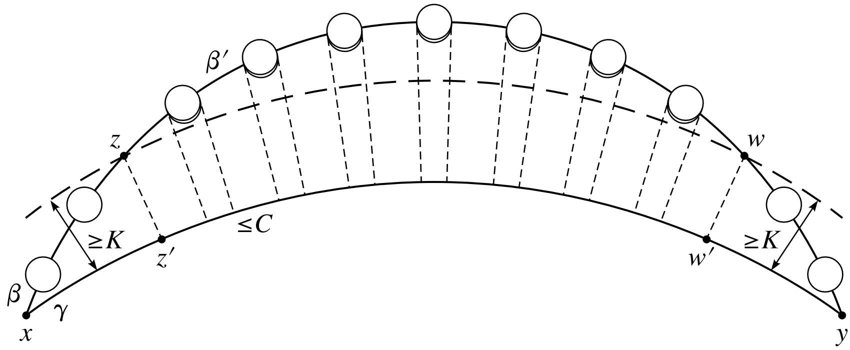

Figure 2

$$l_{X'}(\beta') \leq 2K + 2C\frac{l_{X'}(\beta')}{R} + 2C + C = 2K + 3C + 2C\frac{l_{X'}(\beta')}{R}$$

and so

$$l_{X'}(\beta')\left(\frac{R-2C}{R}\right) \leq 2K + 3C.$$

Now $R > 2C$ by our choice of R, so we can divide:

$$l_{X'}(\beta') \leq \frac{R(2K+3C)}{R-2C}.$$

Now let $L = R(2K+3C)/(R-2C)$, and we are done. □

Now we can finish the proof of Proposition 1. Suppose we are given a triangle $\triangle(x, y, z) \subset X'$, where by "triangle" we mean the union of the three X'-geodesics $\overline{xy}$, $\overline{yz}$, and $\overline{xz}$. Now consider the X-geodesic between each pair of vertices. The resulting triangle in X is δ-thin.

Let K and L be the constants given by Lemma 1. Suppose we are given a point p on $\overline{xy}$. Then there is a point p' on the X-geodesic from x to y so that $d_{X'}(p, p') \leq K + L/2$, and there is a point q' on (say) the X-geodesic from x to z with

$$d_{X'}(p', q') \leq d_X(p', q') \leq \delta.$$

Combining Lemma 1 with the definitions of quasiconvex set and hyperbolic space (see [3, Lemma 7.2, p. 153]), we conclude that there is a point q on $\overline{xz}$ so that $d_{X'}(q, q') \leq 2K + L + 2\delta + \varepsilon$; without changing we may assume that q' does not lie on a quasiconvex set. It follows from these observations that

$$d_{X'}(p, q) \leq K + L/2 + \delta + 2K + L + 2\delta + \varepsilon = 3K + 3/2L + 2\delta + \varepsilon.$$

Hence triangles in X' are $(3K + 3/2L + 2\delta + \varepsilon)$-thin. □

Now we can formulate our theorem.

THEOREM 1. *Let Γ be a finitely generated group, and let $\{H_1, \ldots, H_r\}$ be a finite set of finitely generated subgroups. If Γ is hyperbolic relative to $\{H_1, \ldots, H_r\}$*

in the sense of Definition 1, then Γ *is hyperbolic relative to* H *in the sense of Definition 2.*

Proof. For simplicity, we assume that the family of subgroups has one element; the proof for more than one subgroup is similar (see [4, Apx. A] and [5]). We retain the notation of Definition 1.

Let X be a complete δ-hyperbolic locally compact geodesic space with a discrete isometric action of the group Γ such that H is the isotropy of $r(\infty) \in \partial X$ and preserves a corresponding horofunction h. Because horoballs are (2δ)-quasiconvex sets (see [7, p. 192]), we can apply Proposition 1 for the space X. From Definition 1 we have the quasi-isometry $f : \Gamma \to X(\rho)$. Hence we have the following commutative diagram:

$$\begin{array}{ccc} \Gamma & \xrightarrow{f} & X(\rho) \\ \big\downarrow & & \big\downarrow \\ \hat{\Gamma}(H) & \xrightarrow{\hat{f}} & X(\rho)', \end{array}$$

where $X(\rho)'$ denotes $X(\rho)$ with a metric defined on the beginning of the proof of the Proposition 1. The map $\hat{f}$ is quasi-isometry induced by f (see [4, Sec. 3.2]). By Proposition 1, the space $X(\rho)'$ is $\bar{\delta}$-hyperbolic for some $\bar{\delta}$. Hence the group Γ is hyperbolic relative to H in the sense of Definition 2. □

NOTE. These considerations are a generalization of [4, 3.3]. In particular, Lemma 3.4 and Proposition 3.5 of [4] are corollaries of our Theorem 1.

QUESTIONS. (1) Let Γ be hyperbolic group relative to subgroup H in the sense of Gromov's definition. Does the pair (Γ, H) have a bounded coset penetration property? (See [4] or [5] for a definition of this property.)

(2) Is the mapping class group hyperbolic relative to the family of subgroups in the Gromov sense? (Compare Example 2(2).)

EXAMPLE 3. Let $\Gamma = Z \oplus Z$ and let $H = Z$. The group Γ is hyperbolic relative to H in the sense of Definition 2 but not in the sense of Definition 1. In fact, any $Z \oplus Z$ acting on a hyperbolic space has an ideal fixed point (see [2, p. 86]).

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