# Removable Singularities for Analytic Functions

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## 1. Introduction

The question of removable singularities for analytic functions which are of bounded mean oscillation (in BMO) or uniformly Hölder continuous with exponent $\alpha$, $0<\alpha<1$, (in $\mathrm{Lip}_\alpha$) is well understood (see e.g., [Gn, 4.5; Kr; Ka]). In these cases a more or less complete answer can be given in terms of the Hausdorff dimension: a compact subset $E$ of a domain $G$ in the complex plane $\mathbf{C}$ is removable for analytic functions defined in $G\setminus E$ and belonging to $\mathrm{BMO}(G)$ ($\mathrm{Lip}_\alpha(G)$) if and only if $\mathrm{H}^1(E)=0$ ($\mathrm{H}^{1+\alpha}(E)=0$). Here $\mathrm{H}^\beta$ denotes $\beta$-dimensional Hausdorff measure.

In this note we consider the analogous question for analytic functions defined in $G\setminus E$ and belonging to $\mathrm{BMO}(G\setminus E)$ or $\mathrm{locLip}_\alpha(G\setminus E)$—that is, instead of assuming a regularity condition in all of $G$, we require only that our analytic functions satisfy a regularity condition on $G\setminus E$. Recall that if $U$ is an open set in $\mathbf{C}$, then a complex-valued function $f$ belongs to $\mathrm{BMO}(U)$ if there is a constant $M$ such that

$$|B|^{-1}\int_B |f(z)-f_B|\,dx\,dy \le M$$

for each open disc $B\subset U$, where $f_B=|B|^{-1}\int_B f(z)\,dx\,dy$ and $|B|$ is the area of $B$. Next, suppose that $0<\alpha\le 1$. Following [GM], we say that $f$ belongs to $\mathrm{locLip}_\alpha(U)$ if there is a constant $M$ such that

$$|f(z)-f(w)|\le M|z-w|^\alpha$$

whenever $z,w$ belong to a disc $B$ contained in $U$. Finally, we recall the definition of the Minkowski dimension of a compact set $K\subset\mathbf{C}$. For $\lambda>0$ and $r>0$ write

$$M_r^\lambda(K)=\inf\left\{kr^\lambda : K\subset\bigcup_{i=1}^k B(z_i;r)\right\}$$

and let

$$M^\lambda(K)=\limsup_{r\to 0} M_r^\lambda(K).$$

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The Minkowski dimension of $K$ is then defined as

$$\dim_M(K)=\inf\{\lambda>0: M^{\lambda}(K)<\infty\}.$$

Clearly the Minkowski dimension of $K$ is larger than or equal to the Hausdorff dimension of $K$, and it may happen that $\dim_M(K)>0$ even if $\dim_H(K)=0$. On the other hand, these two dimensions coincide for a large class of compact sets $K$ including self-similar Cantor sets; we refer the reader to [MV] for the basic properties of the Minkowski dimension. For notational clarity we employ the following convention: when $E$ is a relatively closed subset of a domain $G$, we write $\dim_M(E)<\lambda$ $(0<\lambda\leq 2)$, provided $\dim_M(K)<\lambda$ for each compact $K\subset E$. Next, for any Borel set $E$ we denote the projections of $E$ along the real and imaginary axes by $\operatorname{Re} E$ and $\operatorname{Im} E$, respectively.

Now we are able to state our main results. Let $G$ be a subdomain of the complex plane, and suppose that $E$ is a relatively closed subset of $G$. Then we have the following theorem.

THEOREM A. *Let $f$ be analytic in $G\setminus E$, and let $0\leq s<2$. Suppose that for each $z_0\in E$ there exist $r>0$ and a constant $C$ such that*

$$|f'(z)|\leq C\operatorname{dist}(z,E)^{-s} \tag{1}$$

*for all $z\in B(z_0;r)\setminus E$. If*

$$\dim_M(E)<2-s,\quad H^1(\operatorname{Re} E)=H^1(\operatorname{Im} E)=0,$$

*then $f$ has an analytic extension to $G$.*

In the converse direction we establish the next theorem.

THEOREM B. *For each domain $G\subset \mathbf{C}$ and $0<s\leq 2$, there is a compact set $E\subset G$ with*

$$\dim_M(E)=2-s,\qquad \dim_H(E)=0,$$

*for which there exists an analytic function $f$ defined in $G\setminus E$ and satisfying* (1) *in a neighborhood of $E$ such that $f$ does not have an analytic extension to $G$.*

Condition (1) and the assumption on the Minkowski dimension of $E$ ensure that $f'$ is integrable over $U\setminus E$ for each $U\subset\subset G$; see Lemma 2.1. Theorem A will then be established using this integrability condition and the assumption $H^1(\operatorname{Re} E)=H^1(\operatorname{Im} E)=0$. This observation is the core of this note.

From Theorems A and B we obtain Theorem C as follows.

THEOREM C. *Suppose that $f$ is an analytic function defined in $G\setminus E$ which belongs to* $\operatorname{locLip}_\alpha(G\setminus E)$ *(resp., to* $\operatorname{BMO}(G\setminus E)$*). If*

$$\dim_M(E)<1+\alpha \text{ and } H^1(\operatorname{Re} E)=H^1(\operatorname{Im} E)=0\quad (\dim_M(E)<1),$$

*then $f$ has an analytic extension to $G$. Moreover, for each $G$ there is a compact subset $E$ of $G$ with*

$$\dim_M(E) = 1+\alpha \text{ and } \dim_H(E) = 0 \quad (\dim_M(E) = 1, \dim_H(E) = 0),$$

*and an analytic function defined in* $G \backslash E$ *belonging to* $\mathrm{locLip}_\alpha(G\backslash E)$ ($\mathrm{BMO}(G\backslash E)$) *which does not have an analytic extension to G.*

Note that our assumption $H^1(\operatorname{Re} E) = H^1(\operatorname{Im} E) = 0$ is satisfied if $H^1(E) = 0$. On the other hand, it is easy to construct compact sets $E$ with $\dim_H(E) = 2$ and $H^1(\operatorname{Re} E) = H^1(\operatorname{Im} E) = 0$. It will become clear in Section 2 that one needs some assumption on the projections of $E$ in Theorems A and C. We have not stated Theorem C in the ultimate generality. Indeed, our argument of proof shows that one could also establish results for analytic functions in appropriate Campanato spaces; see [Kr] for results in terms of the Hausdorff dimension. We prove Theorems A, B, and C in Section 2. In Section 3 we make some observations concerning quasiregular mappings.

## 2. Proofs of Theorems A, B, and C

We begin with a lemma in which we assume that $E$ is a relatively closed subset of $G$ and that $f$ is analytic in $G\backslash E$.

2.1. Lemma. *Suppose that* $\dim_M(E) < 2-s$ *and there exists a neighborhood U of* $z_0 \in E$ *such that* (1) *holds for all* $z \in U \backslash E$. *Then there exists a neighborhood V of* $z_0$ *such that*

$$\int_{V\backslash E} |f'(z)|\, dx\, dy < \infty.$$

*Proof.* We may assume that $\bar{U} \subset G$ and that $U$ is bounded with $\operatorname{dist}(z, E) < 1$ for each $z \in U$. For $j = 1, 2, \ldots$ set

$$U_j = \{z \in U : 2^{-j} \le \operatorname{dist}(z, E) < 2^{-j+1}\}.$$

Then

$$\int_{U\backslash E} |f'(z)|\, dx\, dy \le \sum \int_{U_j} |f'|\, dx\, dy \le C \sum 2^{sj} |U_j|. \tag{2}$$

Next, from the definition of Minkowski dimension and our assumption $\dim_M(E) < 2-s$, we conclude (see [MV, Lemma 3.9]) that there exist $0 < \dim_M(E) < \lambda < 2-s$ and a constant $C_1$ such that

$$|U_j| \le C_1 2^{-j(2-\lambda)}. \tag{3}$$

From (2) and (3) we conclude that $|f'|$ is integrable over $U\backslash E$. □

Next we prove Theorem A. For the proof recall that $\bar\partial$ is defined by $2\bar\partial = \partial/\partial x + i\partial/\partial y$.

2.2. Proof of Theorem A. Given a function $f$ analytic on $G\backslash E$ and satisfying (1), define a function $g: G \to \mathbf{C}$ by setting $g(z) = f(z)$ for $z \in G\backslash E$ and $g(z) = 0$ for $z \in E$. Then, by Weyl's lemma (cf. [Ga, 10.3]), it suffices to verify that $g$ is locally integrable in $G$ and

$$\int_G g(z)\bar{\partial}\phi(z)\,dx\,dy=0 \tag{4}$$

for all $\phi\in C_0^1(G)$.

Note first that owing to the condition $H^1(\mathrm{Re}\,E)=H^1(\mathrm{Im}\,E)=0$, $g$ is ACL in $G$, and the partial derivatives of $g$ exist a.e. in $G$ and coincide with those of $f$. Hence Lemma 2.1 shows that the partial derivatives of $g$ are locally integrable in $G$. The local integrability of $g$ immediately follows by the Fubini theorem from the absolute continuity of $g$ on almost all lines parallel to the coordinate axes.

Fix $\phi\in C_0^1(G)$. According to Lemma 2.1 there is an open set $V\subset\subset G$ such that the partial derivatives of $g$ are integrable over $V$ and the closure of the support of $\phi$ is contained in $V$. Moreover, from the first part of the proof we observe that $g$ is integrable over $V$. Applying the Fubini theorem and the absolute continuity of $g$, we arrive at

$$\int_G g(z)\bar{\partial}\phi(z)\,dx\,dy=-\int_G \bar{\partial}g(z)\phi(z)\,dx\,dy.$$

Since $\partial\bar{g}(z)=0$ for a.e. $z\in G$, (4) follows and the proof is complete. □

Here we produce an example to prove Theorem B; we emphasize that Theorem A is void for $s>2$. Clearly it suffices to make the construction for the plane $\mathbf{C}$.

2.3. PROOF OF THEOREM B. We show that for each $0<s\le 2$ there is a countable compact subset $E$ of the unit disc $D$, with $\dim_M(E)=2-s$ and such that the only accumulation point of $E$ is the origin, as well as an analytic function $f$ defined in $\mathbf{C}\backslash E$ with $|f'(z)|\le C\,\mathrm{dist}(z,E)^{-s}$ for all $z\in D\backslash E$ such that $f$ does not extend to an analytic function defined in $D$.

We begin with the function $f(z)=z^{-1}$, which is analytic in the punctured disc. Note first that for $s=2$ it suffices to set $E=\{0\}$. Hence we may assume $0<s<2$. We construct our set $E$ as a sequence accumulating at the origin so that

$$|f'(z)|=|z|^{-2}\le C\,\mathrm{dist}(z,E)^{-s}$$

for all $z\in D\backslash E$. To ensure this inequality, we set $r_{j,k}=2^{-j}+k2^{-2j/s}$ for $j=1,2,\dots$ and all positive integers $k\le 2^{j(2/s-1)}$, and then define

$$E=\{0\}\cup\{r_{j,k}\exp\{2^{1-j(2/s-1)}l\pi i\}: k \text{ and } l\le 2^{j(2/s-1)},\ j=1,2,\dots\}.$$

A simple calculation verifies that

$$\mathrm{dist}(z,E)\le C|z|^{2/s},$$

and the desired growth condition for $|f'(z)|$ follows. Moreover, $E$ is a countable, compact subset of $D$ and has the origin as its only accumulation point, and the restriction of $f$ to $\mathbf{C}\backslash E$ does not extend to a function analytic in $D$.

We are left with the estimate on the Minkowski dimension of $E$. Given a small $r>0$, pick $m\ge 1$ such that $2^{-m-1}\le r<2^{-m}$. Then

$$\bigcup_{z\in E} B(z;r) \subset B(0,2^{-ms/2}) \cup A,$$

where

$$A = \bigcup \{B(r_{j,k}\exp\{2^{1-j(2/s-1)} l\pi i\}; 2^{-m}) : k \text{ and } l \le 2^{j(2/s-1)}, j \le ms/2\}.$$

Hence

$$\left|\bigcup_{z\in E} B(z;r)\right| \le \pi 2^{-ms} + \pi 2^{-2m} \sum_{j\le ms/2} 2^{2j(2/s-1)} \le C\pi 2^{-ms}.$$

Therefore $\dim_M(E) \le 2-s$ by [MV, 3.1], as desired. □

It is perhaps worthwhile to point out that we have proved a somewhat stronger result than indicated in Theorems A and B. Indeed, our estimates show that instead of the condition on the Minkowski dimension it suffices to assume that

$$\int_{V\setminus K} \operatorname{dist}(z,K)^{-s}\,dx\,dy < \infty \tag{5}$$

for some neighborhood $V$ of $K$ for each compact $K \subset E$. The measure estimates in the proof of Theorem B then show that the exponent $-s$ is critical.

The proof of Theorem C is now immediate from Theorems A and B and the following well-known lemma. The latter characterization is a consequence of a theorem of Hardy and Littlewood [HL, pp. 426–427], and the first characterization can be found, for example, in [AG] or in [CG, p. 693].

2.4. Lemma. *Let $U$ be a proper subdomain of* $\mathbf{C}$, *and suppose that $f$ is analytic in $U$. Then $f$ belongs to* BMO($U$) *if and only if there is a constant $C$ such that*

$$|f'(z)| \le C\operatorname{dist}(z,\partial U)^{-1}$$

*in $U$. Moreover, $f$ belongs to* $\operatorname{locLip}_\alpha(U)$ *if and only if*

$$|f'(z)| \le C\operatorname{dist}(z,\partial U)^{\alpha-1}.$$

2.5. Proof of Theorem C. Observe that $\dim_M(E) < 1$ ensures that $H^1(E) = 0$, and, in particular, that $H^1(\operatorname{Re} E) = H^1(\operatorname{Im} E) = 0$. Hence the claim follows from Theorems A and B and Lemma 2.4. □

The careful reader notices that we could now state a refined version of Theorem C in the spirit of the aforementioned classical results. Applying Lemma 2.4 and the reasoning preceding it, we see that we could replace the use of the Minkowski dimension with the convergence of the integral in (5) for $s=1$ and $s=1-\alpha$.

Finally, we comment on the assumption $H^1(\operatorname{Re} E) = H^1(\operatorname{Im} E) = 0$.

2.6. Remark. Theorems A, B, and C do not, in general, hold without some assumption on $\operatorname{Re} E$ and $\operatorname{Im} E$ when $\dim_M(E) \ge 1$. To see this, notice that for a line segment $E$, $\dim_M(E) = \dim_H(E) = 1$; hence such a set $E$ may separate $G$, and the removability of $E$ fails even for analytic functions which are piecewise constant functions in $G\backslash E$.

The situation is not any better if, in addition, we require $G\backslash E$ to be connected. A counterexample is provided by the analytic function $f(z)=z^{1/2}$, $z\in(B(0,2)\backslash\bar{B}(0,1))\backslash E$, where $E$ is the interval from 1 to 2 on the real axis.

## 3. Quasiregular Mappings

Recently we established removability theorems for quasiregular mappings belonging to $\mathrm{locLip}_\alpha(G\backslash E)$ in a joint paper [KM] with O. Martio; qualitatively, part of Theorem C is a special case of our previous work. It should be noted that the method of [KM] fails to give the sharp bounds of Theorem C and does not apply to the case of quasiregular mappings in $\mathrm{BMO}(G\backslash E)$. Recent remarkable results of Iwaniec [Iw], (see Theorem 3.1 below) allow us now to produce a removability theorem for quasiregular mappings in $\mathrm{BMO}(G\backslash E)$.

Recall that a continuous mapping $f\colon G\to\mathbf{R}^n$ is $K$-quasiregular if $f\in W^1_{n,\mathrm{loc}}(G)$ and

$$\max_{|h|=1}|f'(x)h|\le K\min_{|h|=1}|f'(x)h| \tag{6}$$

holds for almost every $x\in D$. Here $f\in W^1_{n,\mathrm{loc}}(G)$ means that the coordinate functions of $f$ belong to the local Sobolev space $W^1_{n,\mathrm{loc}}(G)$, and $f'(x)$ denotes the formal derivative of $f$. For the properties of quasiregular mappings, see [BI] or [Re].

We begin by recording Theorem 3 of [Iw].

3.1. THEOREM [Iw]. *For each $K, n$, there exists an exponent $p=p(K,n)<n$ such that if $f\in W^1_{p,\mathrm{loc}}(G)$ satisfies* (6) *then $f$ coincides a.e. in $G$ with a $K$-quasiregular mapping of $G$.*

Let $E$ be a relatively closed subset of a domain $E\subset\mathbf{R}^n$. We prove the following.

3.2. THEOREM. *Suppose that $f$ is $K$-quasiregular in $G\backslash E$ and that $f\in\mathrm{BMO}(G\backslash E)$. There exists $0<\lambda=\lambda(K,n)<n-1$ such that if $\dim_M(E)<\lambda$ then $f$ extends to a $K$-quasiregular mapping of $G$.*

The proof of Theorem 3.2 mimics the proof of [KM, 3.11].

*Proof of Theorem 3.2.* Observe first from Theorem 3.1, the ACL-characterization of Sobolev functions [Re, p. 20], and the argument of the proof of Theorem B that it suffices to construct for each $x\in E$ a neighborhood $V$ of $x$ with

$$\int_{V\backslash E}|f'|^p\,dm<\infty,$$

where $p$ is the exponent from Theorem 3.1.

Fix a cube $Q \in W$, where $W = \{Q\}$ is the Whitney decomposition of $G \setminus E$ into closed cubes with pairwise disjoint interiors; see [St, p. 16]. Then the edge length $l(Q)$ of $Q$ is $2^{-j}$ for some integer $j$, and

$$\sqrt{n}2^{-j} \le \operatorname{dist}(Q, E \cup (\mathbf{R}^n \setminus G)) \le \sqrt{n}2^{2-j}.$$

Now we use our assumption $u \in \mathrm{BMO}(G \setminus E)$. As is well known, the John-Nirenberg lemma yields that

$$l(Q')^{-n} \int_{Q'} |f - f_{Q'}|^n \, dm \le C_1$$

for each cube $Q' \subset G \setminus E$, where $f_{Q'} = l(Q')^{-n} \int_{Q'} f \, dm$ and $C_1$ is independent of $Q'$. In particular, this estimate holds for $Q' = \frac{3}{2}Q$ whenever $Q \in W$. Since $f$ is quasiregular in $G \setminus E$, we infer from the Caccioppoli-type inequality

$$\int_Q |f'|^n \, dm \le C_2 l(Q)^{-n} \int_{\frac{3}{2}Q} |f - f_{\frac{3}{2}Q}|^n \, dm$$

in [BI, 6.1] or in [Re, p. 299] that, for each $Q \in W$,

$$\int_Q |f'|^n \, dm \le C_3$$

for some constant $C_3$ independent of $Q$. From the Hölder inequality we conclude that

$$\int_Q |f'|^p \, dm \le C_3^{p/n} 2^{(n-p)j}. \tag{7}$$

Next, for any fixed $x \in E$ and any $0 < r \le \sqrt{n}/2$, we have that $l(Q) \le 1/2$ for each cube $Q \in W$ with $Q \cap B(x; r) \ne \emptyset$, and thus for each such $x, r$

$$\int_{B(x;r)\setminus E} |f'|^p \, dm \le \sum_{j=1}^{\infty} \sum_{k=1}^{N_j} \int_{Q_{jk}} |f'|^p \, dm, \tag{8}$$

where each $Q_{jk}$ is of edge length $2^{-j}$ and $N_j$ is the number of the cubes $Q_{jk} \in W$ that intersect $B(x; r)$. Now fix $0 < r < \min\{d(x, \partial G)/2, \sqrt{n}/2\}$. Then [MV, 3.9] yields

$$N_j \le C_4 2^{\lambda' j}, \quad j = 1, 2, \ldots \tag{9}$$

for any $\lambda' > \dim_M(E)$ for some $C_1$ independent of $j$. Combining (7)–(9), we obtain

$$\int_{B(x;r)\setminus E} |f'|^p \, dm \le C_5 \sum_{j=1}^{\infty} 2^{(\lambda' + p - n)j},$$

where $C_5 = C_3^{p/n} C_4$. The claim follows with the choice $\lambda(n, K) = n - p$. $\square$

3.3. Remark. Iwaniec [Iw] has proved that for each $K, n$ there exists a $\lambda > 0$ such that compact sets $E \subset G$ of Hausdorff dimension not exceeding $\lambda$ are removable for bounded $K$-quasiregular mappings of $G \setminus E$. It follows from his argument that the same conclusion holds for $K$-quasiregular mappings that are of bounded mean oscillation in $G$. Nevertheless, one can apply the argument of the proof of Theorem B to show that, for each $n$, the correct dimension for the study of removability for quasiregular mappings in BMO($G\setminus E$) is the Minkowski dimension.

*Added in proof.* Riihentaus (*Removable singularities for Bloch and normal functions,* to appear in Czechoslov. Math. J.) has recently established results related to Theorem C.

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