Interpolating Sets in the Maximal Ideal Space of H^∞

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1. Introduction

Let H^∞ denote the algebra of all bounded analytic functions on the open unit disc $\mathbf{D}$. Let (z_n) be a sequence of points in $\mathbf{D}$. We shall call (z_n) an *interpolating sequence* if, for each bounded sequence of complex numbers (w_n), there exists a function $f \in H^\infty$ such that $f(z_n) = w_n$ for every $n \in \mathbf{N}$. Let

$$\delta = \inf_k \prod_{\substack{j=1 \\ j \neq k}}^{\infty} \left| \frac{z_k - z_j}{1 - \overline{z_j} z_k} \right|. \tag{1}$$

We call δ the *separating constant* of (z_n). Carleson's theorem [2] states that a sequence (z_n) in $\mathbf{D}$ is an interpolating sequence if and only if its separating constant δ fulfills $\delta > 0$. The purpose of this paper is to study a natural generalization of this interpolation problem. Let $M(H^\infty)$ denote the maximal ideal space of H^∞, that is, the space of all complex homomorphisms of H^∞, provided with the Gelfand topology. The corona theorem states that $\mathbf{D}$ is dense in $M(H^\infty)$. For $f \in H^\infty$, its Gelfand transform $\hat{f}$ is a continuous function on $M(H^\infty)$ which extends f. When it cannot cause any confusion we will usually omit the distinction between a function and its Gelfand transform. As usual, the pseudohyperbolic distance $\rho(m_1, m_2)$ for $m_1, m_2 \in M(H^\infty)$ is defined by

$$\rho(m_1, m_2) = \sup\{|f(m_1)| : f \in H^\infty,\ \|f\|_\infty \leq 1,\ f(m_2) = 0\}.$$

Let $m \in M(H^\infty)$. The set

$$P(m) = \{m' \in M(H^\infty) : \rho(m, m') < 1\}$$

is called the *Gleason part* of m. If $P(m)$ contains at least two points, m is called a *nontrivial* point.

NOTATION. We will denote the set of all nontrivial points by G.

For $E \subset M(H^\infty)$, let $C(E)$ denote the set of all continuous functions on E. To generalize the interpolating problem we introduce the following concept.

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DEFINITION. Let $E \subset M(H^\infty)$ be a closed set. We shall call E an *interpolating set* if for each function $f \in C(E)$ there exists a function $g \in H^\infty$ such that the restriction $g|_E \equiv f$.

One of the earliest results on interpolating sets in $M(H^\infty)$ is the following one due to Homer, Colwell, and Earl [8].

THEOREM 1.1. *Let (z_n) be a sequence of distinct points in $\mathbf{D}$. Define E to be the closure of $\{z_n: n \in \mathbf{N}\}$ in the Gelfand topology of $M(H^\infty)$; that is, $E = \overline{\{z_n: n \in \mathbf{N}\}}$. Then E is an interpolating set if and only if (z_n) is an interpolating sequence.*

By using a well-known theorem due to Hoffman [7, p. 101, Thm. 5.5] which states that the point $m \in M(H^\infty)$ belongs to the closure of an interpolating sequence (z_n) if and only if $m \in G$, we immediately obtain the result that every interpolating set E of the form $E = \overline{\{z_n: n \in \mathbf{N}\}}$ is contained in G. Of course, every closed subset of E is also an interpolating set. The main result of this paper is that these sets describe all the interpolating sets in G. An important tool in our proofs will be the Blaschke products

$$B(z) = \prod_{j=1}^{\infty} \frac{\overline{z_j}}{|z_j|}\left(\frac{z_j - z}{1 - \overline{z_j} z}\right)$$

associated with interpolating sequences (z_n). These are commonly called *interpolating* Blaschke products. We set $\delta(B) = \delta$, where δ is the separating constant of (z_n). Let $Z(f)$ denote the zero set in $M(H^\infty)$ of a function $f \in H^\infty$. Then we have the following well-known results.

LEMMA 1.2 (cf. Garnett [4, p. 379, Lemma 3.3]). *Let B be an interpolating Blaschke product with the zero sequence (z_n). Then $Z(B) = \overline{\{z_n: n \in \mathbf{N}\}}$.*

LEMMA 1.3. *The product $B_1 B_2$ of two interpolating Blaschke products B_1 and B_2 is interpolating if and only if $Z(B_1) \cap Z(B_2) = \emptyset$.*

Proof. Let (a_n) and (b_n) be the zero sequences of B_1 and B_2, respectively. First assume that $B_1 B_2$ is interpolating. By solving the interpolation problem $f(a_n) = 0$, $f(b_n) = 1$ we see that $\overline{\{a_n: n \in \mathbf{N}\}} \cap \overline{\{b_n: n \in \mathbf{N}\}} = \emptyset$. With Lemma 1.2 we obtain $Z(B_1) \cap Z(B_2) = \emptyset$. We will now prove the converse direction. By compactness of $Z(B_2)$ we see that there is $\rho > 0$ such that $|B_1(x)| \geq \rho > 0$ for all $x \in Z(B_2)$. Hence for every $y \in Z(B_1)$ and every $x \in Z(B_2)$ we have

$$\rho(x, y) = \sup\{|f(x)|: f \in H^\infty,\ \|f\|_\infty \leq 1,\ f(y) = 0\} \geq |B_1(x)| \geq \rho.$$

In particular, we have $\rho(a_n, b_m) \geq \rho$ for all $n, m \in \mathbf{N}$. By Lemma 2 in [9, p. 338], it follows that $B_1 B_2$ is interpolating. □

2. Totally Disconnected Sets in G

We need some topological results concerning interpolating sets in G.

DEFINITION. Let $E \subset G$. If there exists $\rho > 0$ such that $\rho(x, y) > \rho$ for all $x, y \in E$ $(x \neq y)$, we will say E is *ρ-separated.*

The following well-known lemma is an easy consequence of the open mapping theorem.

LEMMA 2.1. *Every interpolating set E is ρ-separated.*

The aim of this section is to prove the following theorem.

THEOREM 2.2. *Let $E \subset G$ be a closed ρ-separated set. Then E is totally disconnected* (*in the Gelfand topology*).

To prove this theorem we first need some lemmas.

LEMMA 2.3 (Hoffman [7, pp. 86, 106]). *Let $\omega > 0$. For every $\epsilon > 0$ there exists $\delta > 0$ depending on ϵ and ω such that*

(2) $$\{m \in M(H^\infty) : |B(m)| < \delta\} \subset \{m \in M(H^\infty) : \rho(m, Z(B)) \leq \epsilon\}$$

for every interpolating Blaschke product B with $\delta(B) \geq \omega$.

LEMMA 2.4. *Let $E \subset G$ be a closed ρ-separated set and let B be an interpolating Blaschke product. Then there exists $\sigma > 0$ with the following property. Let $E_\sigma = \{x \in E : |B(x)| \leq \sigma\}$. Then there exists a homeomorphism $\Phi : E_\sigma \to \Phi(E_\sigma) \subset Z(B)$ with $\Phi(x) = x$ for every $x \in E_\sigma \cap Z(B)$.*

Proof. Since $Z(B)$ is an interpolating set, it is τ-separated by Lemma 2.1. Hence we may assume that E and $Z(B)$ are ρ-separated. By Lemma 2.3 there is $\delta > 0$ such that

(3) $$\{m \in M(H^\infty) : |B(m)| < \delta\} \subset \{m \in M(H^\infty) : \rho(m, Z(B)) \leq \rho/3\}.$$

Let $0 < \sigma < \delta$ and define $E_\sigma = \{x \in E : |B(x)| \leq \sigma\}$. If $E_\sigma = \emptyset$ then there is nothing to do. Otherwise, by (3) for every $x \in E_\sigma$ there exists $y \in Z(B)$ such that $\rho(x, y) \leq \rho/3$. The point y is unique because $Z(B)$ is ρ-separated. Hence we have a mapping $\Phi : E_\sigma \to Z(B)$ with $\rho(x, \Phi(x)) \leq \rho/3$ for every $x \in E_\sigma$. Since E_σ is ρ-separated, $\Phi : E_\sigma \to \Phi(E_\sigma)$ is bijective. We claim that Φ is a homeomorphism. Let x_α be a net in E_σ converging to x. We want to show that $\lim_\alpha \Phi(x_\alpha) = \Phi(x)$. Let x_β be an arbitrary subnet of x_α. Since $Z(B)$ is compact, there is a subnet x_γ of x_β such that $\Phi(x_\gamma)$ converges; its limit y is a point in $Z(B)$. Since $\rho(x_\gamma, \Phi(x_\gamma)) \leq \rho/3$ for all γ, by the lower semi-continuity of ρ we have that $\rho(x, y) \leq \rho/3$. This means that $\lim_\gamma \Phi(x_\gamma) = y = \Phi(x)$. Therefore every subnet $\Phi(x_\beta)$ of $\Phi(x_\alpha)$ contains a subnet $\Phi(x_\gamma)$ converging to $\Phi(x)$. Hence $\lim_\alpha \Phi(x_\alpha) = \Phi(x)$ and Φ is continuous. Since E_σ is compact, the image $\Phi(K)$ of every closed set $K \subset E_\sigma$ is closed. Thus Φ^{-1} is continuous. □

Now we are able to prove Theorem 2.2.

Proof of Theorem 2.2. Let E be a closed ρ-separated subset of G. We must show that E is totally disconnected. To do this let $a \in E$ and $W \subset E$ be an open neighbourhood of a. We must prove that there is an open and closed subset V of E such that $a \in V \subset W$. Because $a \in G$, there exists an interpolating Blaschke product B that satisfies $B(a) = 0$. Applying Lemma 2.4 we get $\sigma > 0$ and a homeomorphism $\Phi : E_\sigma \to \Phi(E_\sigma) \subset Z(B)$, where $E_\sigma = \{x \in E : |B(x)| \leq \sigma\}$. Since a is an interior point of E_σ with respect to the topology of E, we may assume that $W \subset E_\sigma$. Observe now that $Z(B)$ is totally disconnected because it is homeomorphic to the Stone–Cěch compactification $\beta\mathbf{N}$ of $\mathbf{N}$, which is totally disconnected by [5, p. 96, 6M.1]. Therefore there exists a subset U of $\Phi(W)$ which is open and closed in $\Phi(E_\sigma)$ and contains $a = \Phi(a)$. Hence $V = \Phi^{-1}(U)$ is open and closed in E_σ and $a \in V$. Since $V \subset W \subset E_\sigma$ and W is open and E_σ is closed in E, we conclude that V is open and closed with respect to the topology of E. □

By Lemma 2.1 we may apply Theorem 2.2 to interpolating sets in G. Hence we obtain the corollary.

COROLLARY 2.5. *Every interpolating set $E \subset G$ is totally disconnected.*

3. Interpolating Sets in G

We want to characterize the interpolating sets in G. Izuchi [9, p. 338, Thm. 1] has shown the following result. Let E be an interpolating set of the form $E = Z(f) \setminus \mathbf{D}$ with $f \in H^\infty$. Then there exists an interpolating Blaschke product B with

$$Z(B) \setminus \mathbf{D} = E. \tag{4}$$

We want to obtain a similar result without the hypothesis $E = Z(f) \setminus \mathbf{D}$. Note that (4) implies $E \subset G$; hence all interpolating sets characterized by Izuchi fulfill $E \subset G$. We choose therefore $E \subset G$ as an hypothesis. This will lead to the following result (Theorem 3.3): Every interpolating set $E \subset G$ is a closed subset of the zero set of an interpolating Blaschke product B.

DEFINITION. Let $E \subset M(H^\infty)$. The hull $h(E)$ of E is defined by

$$h(E) = \bigcap_{\substack{f \in H^\infty \\ Z(f) \supset E}} Z(f).$$

Let $I(E) = \{f \in H^\infty : f|_E \equiv 0\}$. By a well-known general result (see, e.g., [10, p. 174, Thm. 7.3.1(iv)]) the maximal ideal space M of the factor algebra

$H^\infty/I(E)$ is homeomorphic to $h(E)$. If E is an interpolating set we know that $H^\infty/I(E)$ is isomorphic to $C(E)$. Therefore M is homeomorphic to E. Now it is easy to see the following lemma.

LEMMA 3.1. *If E is an interpolating set then $h(E)=E$.*

DEFINITION. Let x be a zero of $f\in H^\infty$. The supremum of all $n\in N$ for which there exists a factorization $f=f_1 f_2\cdots f_n$ $(f_k\in H^\infty)$ with $f_k(x)=0$ for $k=1,\dots,n$ is called the *order* of this zero.

THEOREM 3.2. *Let E be closed and $h(E)\subset G$. Then there exist finitely many interpolating Blaschke products $B_1, B_2, \dots, B_n$ such that $Z(B_1\cdots B_n)\supset E$.*

Theorem 3.2 is a corollary of a more general result of Tolokonnikov [11, p. 94, Thm. 2]. We will, however, present a different proof of Theorem 3.2.

Proof. Let $x\in E$. Define

$$I(E)=\{f\in H^\infty : f|_E\equiv 0\}.$$

There is a function $f\in I(E)$ that has a zero of finite order at x, because otherwise we would have $\overline{P(x)}\subset h(E)\subset G$ by Lemma 1.2 in [4, p. 403]. But by a result due to Budde [1, p. 11, Cor. 2.10] there is no part P with $\bar{P}\subset G$. We factorize f in $f=B_f S_\mu F$, where B_f is a Blaschke product, S_μ a singular inner function, and F an outer function. Because $S_\mu F$ does not vanish on $\mathbf{D}$, we have for every $n\in\mathbf{N}$ the factorization

$$S_\mu F=\sqrt[n]{S_\mu F}\cdots\sqrt[n]{S_\mu F}.$$

Hence every zero of $S_\mu F$ in $M(H^\infty)$ is of infinite order. This shows $S_\mu F(x)\neq 0$ and $B_f(x)=0$. By Hoffman [7, p. 100, Thm. 5.3] there is an interpolating Blaschke product B_1 such that $B_f=B_{f,1}B_1$ and $B_1(x)=0$, where $B_{1,f}$ is a Blaschke product (this is trivial for $x\in\mathbf{D}$). By repeating this factorization, we obtain interpolating Blaschke products $B_1,\dots,B_n$ such that $B_f=B_{f,n}\cdot B_1\cdots B_n$ and $B_1(x)=\cdots=B_n(x)=0$ and $B_{f,n}\neq 0$. We now have

$$f=B_{f,n}S_\mu F B_1\cdots B_n.$$

Since $B_{f,n}S_\mu F(x)\neq 0$ we can choose an open neighbourhood U_x of x, where $B_{f,n}S_\mu F$ does not vanish. We obtain

$$E\cap U_x\subset Z(f)\cap U_x=Z(B_1\cdots B_n)\cap U_x\subset Z(B_1\cdots B_n).$$

For every $x\in E$ we get a neighbourhood U_x and finitely many interpolating Blaschke products with this property. Since E is compact, there are finitely many neighbourhoods $U_1,\dots,U_N$ which cover E. Therefore there are finitely many interpolating Blaschke products whose zero sets also cover E. □

I thank Raymond Mortini for showing to me the factorization argument that is used in this proof.

It is not possible to find a bound for the number of interpolating Blaschke products in this theorem. This will be shown by the following remark.

REMARK. *Let $N \in \mathbf{N}$ and let (z_n) be an interpolating sequence with the separating constant δ. Let*

$$K_n = \{z \in \mathbf{D}: \rho(z, z_n) \leq r_n\}$$

be a sequence of discs, where the radius sequence (r_n) tends to zero. We suppose that

$$(5) \qquad r_n \leq \delta/3 \quad \text{for all } n \in \mathbf{N}.$$

In every disc K_n we choose N different points $w_{1,n}, \ldots, w_{N,n}$. Let

$$E = \overline{\{w_{k,n}: k \in \{1, \ldots, N\}, n \in \mathbf{N}\}}.$$

Then we have $h(E) \subset G$, and it is not possible to cover E by the union of zero sets of $N-1$ interpolating Blaschke products.

Proof. By Lemma 5.3 in Garnett [4, p. 310] and condition (5), we see that the sequences $(w_{1,n}), \ldots, (w_{N,n})$ are interpolating. Hence the hull of the set $\overline{\{w_{k,n}: n \in \mathbf{N}\}}$ is a subset of G for $k = 1, \ldots, N$ and consequently $h(E) \subset G$. Suppose that there are $N-1$ interpolating Blaschke products $B_1, \ldots, B_{N-1}$ such that $E \subset Z(B_1 \cdots B_{N-1})$. Then there exists $\rho > 0$ such that each of the sets $Z(B_1), \ldots, Z(B_{N-1})$ is ρ-separated. If $n_0 \in \mathbf{N}$ is large enough, we conclude that $r_{n_0} < \rho/2$. From this it is clear that each interpolating Blaschke product B_k $(k = 1, \ldots, N-1)$ has at most one zero in K_{n_0}. But $E \cap K_{n_0}$ has N elements, which is a contradiction. □

It is possible to reduce the number of interpolating Blaschke products in Theorem 3.2 under the additional assumption that E is an interpolating set. This leads to a characterization of the interpolating sets in G, which is the main result of this paper.

THEOREM 3.3. *Let $E \subset G$. Then the following statements are equivalent.*

(i) *E is an interpolating set.*
(ii) *E is a closed subset of the zero set of an interpolating Blaschke product.*

To prove this theorem we need the following lemma.

LEMMA 3.4. *Let B_1 and B_2 be two interpolating Blaschke products and let $F = Z(B_2) \setminus Z(B_1)$. Then there exist disjoint sets $A_1, A_2, \ldots$ such that $F = \bigcup_{k=1}^{\infty} A_k$. These sets are open and closed in $Z(B_2)$.*

Proof. Let $E_n = \{x \in Z(B_2): |B_1(x)| \geq 1/n\}$. Then $F = \bigcup_{n=1}^{\infty} E_n$. Since $Z(B_2)$ is homeomorphic to the Stone–Cěch compactification $\beta\mathbf{N}$ of $\mathbf{N}$, $Z(B_2)$ is extremally disconnected [5, p. 96, 6M.1]; this means that the closure of every open subset of $Z(B_2)$ is open in $Z(B_2)$. Hence the interior E_n° of the closed sets E_n is open and closed in $Z(B_2)$. Observing that $E_n \subset E_{n+1}^\circ$, we see that $A_n = E_{n+1}^\circ \setminus E_n^\circ$ $(n=1,2,\dots)$ are the desired sets. $\square$

Now we are able to prove Theorem 3.3.

Proof of Theorem 3.3. The zero set $Z(B)$ of an interpolating Blaschke product B is an interpolating set. If E is a closed subset of $Z(B)$, it is easy to see by the Tietze extension theorem that E is interpolating. Hence (ii) implies (i). The nontrivial direction is the converse. Let $E \subset G$ be an interpolating set. By Lemma 2.1 there is $\rho > 0$ such that E is ρ-separated. We want to construct an interpolating Blaschke product B with $Z(B) \supset E$.

Step 1. We show in the first step the existence of B using the additional assumption that there are two interpolating Blaschke products B_1, B_2 with $Z(B_1) \cup Z(B_2) \supset E$. Since $Z(B_1)$, $Z(B_2)$ and E are interpolating sets we may assume that they are ρ-separated. Let $E_1 = E \setminus Z(B_2)$ and $E_2 = E \setminus Z(B_1)$. We claim $\overline{E_1} \cap \overline{E_2} = \emptyset$. Applying Lemma 2.4 we get $\sigma > 0$ and a homeomorphism $\Phi\colon Z(B_1)_\sigma \to \Phi(Z(B_1)_\sigma) \subset Z(B_2)$, where $Z(B_1)_\sigma = \{x \in Z(B_1) : |B_2(x)| \le \sigma\}$. By construction we have $\rho(x, \Phi(x)) \le \rho/3$ for every $x \in Z(B_1)_\sigma$. Since $\overline{E_2} \subset Z(B_2)$ it is clear that the closure of $E_1 \setminus Z(B_1)_\sigma$ is disjoint to $\overline{E_2}$. Hence we may assume that $E_1 \subset Z(B_1)_\sigma$. Let $x \in E_1$ and $y \in E_2$. Since E is ρ-separated we observe that $\rho(\Phi(x), y) \ge \rho(x, y) - \rho(x, \Phi(x)) \ge \rho - \rho/3$. Therefore we have that

$$\Phi(E_1) \cap E_2 = \emptyset. \tag{6}$$

Define $F = Z(B_2) \setminus Z(B_1)$. Since exactly the points in $Z(B_1) \cap Z(B_2)$ are invariant by Φ, we get

$$\Phi(E_1) = F \cap \Phi(E \cap Z(B_1)). \tag{7}$$

Clearly,

$$E_2 = F \cap E. \tag{8}$$

Choose, according to Lemma 3.4, sets A_k such that $F = \bigcup_{k=1}^\infty A_k$. By (6) the sets $A_k \cap \Phi(E_1)$ and $A_k \cap E_2$ are disjoint. We obtain from (7) and (8) that $A_k \cap \Phi(E_1) = A_k \cap \Phi(E \cap Z(B_1))$ and $A_k \cap E_2 = A_k \cap E$ are closed. Since $Z(B_2)$ is totally disconnected, we can choose a partition of A_k into disjoint (in $Z(B_2)$) open and closed sets $A_{k,1}$ and $A_{k,2}$, with $A_k = A_{k,1} \cup A_{k,2}$ and $A_k \cap \Phi(E_1) \subset A_{k,1}$ and $A_k \cap E_2 \subset A_{k,2}$. Consequently,

$$\Phi(E_1) = \bigcup_{k=1}^\infty A_k \cap \Phi(E_1) \subset \bigcup_{k=1}^\infty A_{k,1}$$

and

$$E_2 = \bigcup_{k=1}^\infty A_k \cap E_2 \subset \bigcup_{k=1}^\infty A_{k,2}.$$

The sets $A_{k,1}$ and $A_{j,2}$ are disjoint for $k, j \in \mathbf{N}$. Since $Z(B_2)$ is extremally disconnected we obtain

$$\overline{\bigcup_{k=1}^{\infty} A_{k,1}} \cap \overline{\bigcup_{k=1}^{\infty} A_{k,2}} = \emptyset.$$

Hence

$$\overline{\Phi(E_1)} \cap \overline{E_2} = \emptyset. \tag{9}$$

Assume now that there exists $x \in \overline{E_1} \cap \overline{E_2}$. Then we have $x \in Z(B_1) \cap Z(B_2)$. This yields $x = \Phi(x)$. Since Φ is a homeomorphism we have

$$x = \Phi(x) \in \overline{\Phi(E_1)},$$

which contradicts (9). This proves our claim $\overline{E_1} \cap \overline{E_2} = \emptyset$.

We are now able to construct the interpolating Blaschke product we are looking for. By Corollary 2.5 we know that E is totally disconnected. Hence there are disjoint open and closed sets $H_1, H_2 \subset E$ such that $H_1 \cup H_2 = E$ and $\overline{E_1} \subset H_1$, $\overline{E_2} \subset H_2$. Since H_1, H_2 are also closed in $M(H^\infty)$ there are (in $M(H^\infty)$) open sets U_1, U_2 with

$$U_1 \supset H_1, \quad U_2 \supset H_2, \quad \text{and} \quad \overline{U_1} \cap \overline{U_2} = \emptyset. \tag{10}$$

Let b_1 be the interpolating Blaschke product whose zero sequence consists of the points of the zero sequence of B_1 that lie in $U_1 \cap \mathbf{D}$. Construct b_2 analogously with B_2 and U_2. Then by Lemma 1.2 we have $Z(b_1) \cap Z(b_2) = \emptyset$. Hence by Lemma 1.3 the Blaschke product $B = b_1 b_2$ is interpolating. The set H_1 is disjoint from $E_2 = E \setminus Z(B_1)$. Hence $H_1 \subset Z(b_1)$ and analogously $H_2 \subset Z(b_2)$. Consequently, $Z(B) \supset H_1 \cup H_2 = E$.

Step 2. Now we show the general case. By Lemma 3.1 we have $h(E) = E \subset G$. Hence by Theorem 3.2 we have interpolating Blaschke products $B_1, \ldots, B_n$ such that

$$Z(B_1 \cdots B_n) \supset E. \tag{11}$$

Clearly, $Z(B_1 B_2) \supset E \cap Z(B_1 B_2)$. By Step 1 there is an interpolating Blaschke product b such that $Z(b) \supset E \cap Z(B_1 B_2)$. Doing the same with bB_3, it is possible to reduce successively the number of interpolating Blaschke products in (11) until one is left. □

An analysis of the proof of the implication (i) ⇒ (ii) shows that we did not need the hypothesis that E is interpolating. In fact, we proved the following: Let E be a closed, ρ-separated, and totally disconnected subset of G such that $h(E) \subset G$. Then $E \subset Z(B)$, where B is an interpolating Blaschke product. On the other hand, it follows by Theorem 2.2 that any closed, ρ-separated set $E \subset G$ is indeed totally disconnected. Hence we obtain the following corollary.

COROLLARY 3.5. *Let $E \subset M(H^\infty)$ be a closed and ρ-separated set and let $h(E) \subset G$. Then E is an interpolating set.*

If (z_n) is an interpolating sequence and $(w_n) \in l^\infty$, then (by a result due to Earl [3]) there is a constant $c \in \mathbf{C}$ and an interpolating Blaschke product B

such that $cB(z_n) = w_n$ for all $n \in \mathbf{N}$. So it is possible to solve the interpolation problem by a multiple of an interpolating Blaschke product. As a corollary of Theorem 3.3 we get a characterization of the interpolating sets E for which the interpolation problem $f|_E \equiv g$ for $g \in C(E)$ is also solvable by a multiple of an interpolating Blaschke product B. We will show that these are the sets which we characterized in Theorem 3.3.

COROLLARY 3.6. *Let E be an interpolating set that contains at least two points. Then the interpolation problem*

$$f|_E = g \tag{12}$$

is solvable for every $g \in C(E)$ by $f = cB$, where B is an interpolating Blaschke product and c a complex number, if and only $E \subset G$.

Proof. Let $E \not\subset G$. In other words, there is $x \in E \setminus G$. Let $g \in C(E)$ such that $g(x) = 0$ but $g \not\equiv 0$. Since no interpolating Blaschke product vanishes outside G, there is no solution of (12) such that $f = cB$.

Now let $E \subset G$. Let $g \in C(E)$ be arbitrary. By Theorem 3.3 there is an interpolating sequence (z_n) in $\mathbf{D}$ such that

$$E \subset \overline{\{z_n : n \in \mathbf{N}\}}. \tag{13}$$

By the Tietze extension theorem it is possible to extend g continuously to $\overline{\{z_n : n \in \mathbf{N}\}}$; that is, there is $\tilde{g} \in C\,(\overline{\{z_n : n \in \mathbf{N}\}})$ with $\tilde{g}|_E \equiv g$. Using the result of Earl [3] we get an interpolating Blaschke product B and $c \in \mathbf{C}$ such that $cB(z_n) = \tilde{g}(z_n)$. By (13) we obtain $cB|_E = g$. □

In the last part of this paper we study the interpolation sets that lie in a special subset of G. First we give a definition.

DEFINITION. An interpolating sequence (z_n) is called *thin* if

$$\lim_{k \to \infty} \prod_{\substack{j=1 \\ j \neq k}}^{\infty} \rho(z_k, z_j) = 1.$$

We define G^* to be the set of all points in $M(H^\infty)$ that are in the closure of a thin interpolating sequence. G^* is open. A thin interpolating Blaschke product is a Blaschke product whose zero sequence is a thin interpolating sequence. For the interpolating sets in G^* we get the following refinements of the previous results.

THEOREM 3.7. *Let $E \subset G^*$. Then the following statements are equivalent.*

(i) *E is an interpolating set.*
(ii) *E is a closed subset of the zero set of an interpolating Blaschke product which is a finite product of thin interpolating Blaschke products.*

Proof. In view of Theorem 3.3 it remains only to show the following. Let $E \subset G^*$ be an interpolating set. By Theorem 3.2 there exists an interpolating

Blaschke product $\tilde{B}$ with $Z(\tilde{B}) \supset E$. Let U be an open neighbourhood of E with $\bar{U} \subset G^*$. By factorization we obtain an interpolating Blaschke product B with $Z(B) \subset \bar{U} \subset G^*$. A result due to Hedenmalm and Izuchi [6, p. 493, Thm. 2.6] shows that $B = B_1 \cdots B_N$ where $B_1, \dots, B_N$ are thin. □

It is not possible to find a bound for the number of thin interpolating Blaschke products in Theorem 3.7. This will be shown by the following remark.

REMARK. *For every $n \in \mathbf{N}$ there exists an interpolating set $E_n \subset G^*$ such that at least n thin interpolating Blaschke products $B_1, \dots, B_n$ are needed to get $Z(B_1 \cdots B_n) \supset E$.*

Proof. Let B be a thin interpolating Blaschke product with the zero sequence (z_n). By a well-known result (see, e.g., Garnett [4, p. 404, Lemma 1.4]) there exist numbers r, λ satisfying $0 < r < 1$ and $0 < \lambda < 1$ which have the following property: If $|w| < r$ then

$$B_w(z) = \frac{B(z) - w}{1 - \bar{w}B(z)}$$

is an interpolating Blaschke product. The zero sequence of (a_n) of B_w fulfills $\rho(a_n, z_n) < \lambda$. By Lemma 2.1 of Hedenmalm [6, p. 491], B_w is thin. Choose n different products $B_{w_1}, \dots, B_{w_n}$ with $|w_n| < r$. Define $E_n = Z(B_{w_1} \cdots B_{w_n})$. Clearly, $E_n \subset G^*$. We claim that E_n is an interpolating set. Let $x, y \in \mathbf{D}$ be zeros of B_{w_j} and B_{w_k}, respectively $(j \neq k)$. Note that $B(x) = w_j$ and $B(y) = w_k$. Applying Schwarz–Pick's lemma we obtain $\rho(x, y) \geq \rho(B(x), B(y)) = \rho(w_j, w_k)$. Note that the ρ-distance of the finitely many points $w_1, \dots, w_n$ is positive. Hence it follows by Lemma 2 in [9, p. 338] that $B_{w_1} \cdots B_{w_n}$ is interpolating. Consequently, E_n is an interpolating set. Let $m \in Z(B) \backslash \mathbf{D}$. Since B is thin by Proposition 2.3 in [6, p. 491], there exist n points $v_1, \dots, v_n \in P(m)$ such that $B(v_k) = w_k$. In other words, $v_k \in E_n$ for $k = 1, \dots, n$. Since every thin interpolating Blaschke product has at most one zero in $P(m)$ (see, e.g., [6, p. 491, Lemma 2.1]), we obtain the assertion. □

Analogously to Corollary 3.6, we have the following result.

COROLLARY 3.8. *Let E be an interpolating set that contains at least two points. Then the interpolation problem*

$$f|_E \equiv g \tag{14}$$

has for every $g \in C(E)$ a solution of the form $f = cb_1 \cdots b_N$, where $c \in \mathbf{C}$ and $b_1, \dots, b_N$ are thin interpolating Blaschke products, if and only if $E \subset G^*$.

Proof. Let $E \not\subset G^*$. Then there exists a point $x \in E \backslash G^*$. Let $g \in C(E)$ with $g(x) = 0$ and $g \not\equiv 0$. Since no thin interpolating Blaschke product vanishes on x, there is no solution of (14) of the desired form.

Now let conversely $E \subset G^*$ and $g \in C(E)$. By Theorem 3.7 there exists an interpolating Blaschke product B with $Z(B) \supset E$ and $B = B_1 \cdots B_N$, where

$B_1, B_2, \ldots, B_N$ are thin. By the Tietze extension theorem there exists a continuous extension $\tilde{g}$ of g on $Z(B)$. Let (z_n) denote the zero sequence of the interpolating Blaschke product B and let $\delta = \delta(B)$. Define

$$r = \frac{\delta}{1+\sqrt{1-\delta^2}}.$$

By [3], the pseudohyperbolic discs $K_r(z_n) = \{z \in \mathbf{D} : \rho(z, z_n) < r\}$ are pairwise disjoint, and there exists an interpolating Blaschke product b with a zero sequence (ζ_n) and a constant $c \in \mathbf{C}$ such that

$$cb(z_n) = \tilde{g}(z_n) \quad \text{and} \quad \zeta_n \in K_r(z_n)$$

for all $n \in \mathbf{N}$. Let $(z_{n,k})$ denote the zero sequence of B_k $(k = 1, \ldots, N)$. Let $\zeta_{n,k}$ be the unique zero of b in $K_r(z_{n,k})$. By [6], the sequence $(\zeta_{n,k})$ is thin for $k = 1, \ldots, N$. Therefore we have $b = b_1 \cdots b_N$ with thin interpolating Blaschke products $b_1, \ldots, b_N$. Since $\overline{\{z_n : n \in \mathbf{N}\}} \supset E$, it follows that $b|_E \equiv g$. $\square$

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