THE CONSTRUCTION OF THE KERVAIRE SPHERE BY MEANS OF AN INVOLUTION

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One of the most significant results in differential topology is Milnor's discovery of exotic smooth structures on the standard *PL* sphere [5]. Much work has been done to understand precisely how these exotic structures differ from the standard structure. Every homotopy sphere, Σ^n, for $n \neq 3, 4$, may be obtained from two n-disks by identifying their boundaries via a diffeomorphism of S^{n-1} [7], [9]. For the standard sphere the diffeomorphism may be chosen to be quite simple, namely, the antipodal map on the equator S^{n-1}. In this paper we show how other homotopy spheres may also be formed by quite simple diffeomorphisms, namely smooth involutions.

Kervaire and Milnor, [4] considered homotopy spheres Σ^{2n-1} which bound parallelizable manifolds and showed that these homotopy spheres form a finite cyclic group, called bP_{2n}, under connected sum. Hirsch and Milnor [3], have shown in dimension 7 that the Milnor exotic sphere M_3^7, the generator of bP_8, admits a fixed point free involution that has an invariant codimension one standard S^6. This free involution is defined as the antipodal map in the fibers where M_3^7 is exhibited as a 3-sphere bundle over the 4-sphere. Hence, as Hirsch and Milnor noted, M_3^7, like S^7, may be realized by identifying two copies of D^7 by a fixed point free involution on their boundary.

Our main result shows that the Kervaire sphere, K^{4k+1}, the generator of bP_{4k+2}, may also be obtained by gluing two disks via a smooth involution. Unlike the Hirsch-Milnor example we must allow the involution to have a set of fixed points.

THEOREM 1. *There exists a smooth involution $T : \partial D^{4k+1} \to \partial D^{4k+1}$ with fixed point set,* Fix $T = S^{2k}$ *such that the Kervaire sphere K^{4k+1} is diffeomorphic to the disjoint union of two copies of D^{4k+1} with their boundaries identified by T.*

That is

$$K^{4k+1} \cong \{D^{4k+1} \amalg D^{4k+1} \mid x \sim Tx \quad \text{for} \quad x \in \partial D^{4k+1}\} \tag{1}$$

and

$$\text{Fix } T = S^{2k}.$$

Note that the involution $\hat{T}$ obtained by switching the two disks extends T to an involution of all of K^{4k+1}.

Our proof of Theorem 1 is direct and constructive. First, following Milnor [6],

Received October 17, 1978. Revision received October 29, 1979.
This work was partially supported by NSF.

Michigan Math. J. 27 (1980).

we construct an explicit Morse function g for K^{4k+1} which has just two critical points (both non-degenerate with critical values $+1$ and -1). Next we define a differentiable involution $\hat{T} : K^{4k+1} \to K^{4k+1}$ with the property $g(\hat{T}z) = -g(z)$. The proof of Theorem 1 is then completed by the following argument.

By Morse theory [5], $g^{-1}([0,\infty))$ and $g^{-1}((-\infty,0])$ are two $(4k + 1)$-disks glued along their common boundary, the standard sphere $g^{-1}(0)$. Also, $\hat{T}$ produces an explicit identification between the "upper" disk $D^{4k+1} \xrightarrow{\ell} g^{-1}([0,\infty))$ and the "lower" disk $D^{4k+1} \xrightarrow{\ell} g^{-1}([0,\infty)) \xrightarrow{\hat{T}} g^{-1}((-\infty,0])$. Using these identifications, K^{4k+1} is given by

$$K^{4k+1} \cong \{D^{4k+1} \amalg D^{4k+1} / x \sim \ell^{-1} \circ \hat{T} \circ \ell(x) \quad \text{for} \quad x \in \partial D^{4k+1}\}. \tag{2}$$

where the map $T = (\ell^{-1} \circ \hat{T} \circ \ell) : \partial D^{4k+1} \to \partial D^{4k+1}$ is the desired involution.

In Section 2, we recall the definition of the Kervaire sphere K^{4k+1} in terms of plumbing together two copies of the unit disk bundle $D(TS^{2k+1})$ of the tangent bundle TS^{2k+1} of the $(2k + 1)$-sphere. Next we show that K^{4k+1} is an example of the more general class of the differentiable manifolds $M(f_1, f_2)$ considered by Milnor in his paper "Differentiable Structures on Spheres" [6]. (See Section 3).

Our basic observation (Section 3) is that we may first, (following Milnor [6]), construct a Morse function $g : M(f_1, f_2) \to \mathbf{R}$ with just two critical points for $M(f_1, f_2)$ and then construct an involution $\hat{T} : M(f_1, f_2) \to M(f_1, f_2)$ with $g(\hat{T}z) = -g(z)$ if minor restrictions are placed on the differentiable maps of spheres into rotation groups

$$f_1 : S^m \to SO(r + 1), \qquad f_2 : S^r \to SO(m + 1) \tag{3}$$

used to define $M(f_1, f_2)$.

The following theorem explains precisely what restrictions must be placed on the maps f_1 and f_2 to insure $M(f_1, f_2)$ is a homotopy sphere which admits the desired involution.

THEOREM 2. *If the differentiable maps*

$$f_1 : S^m \to SO(n + 1), \qquad f_2 : S^n \to SO(m + 1)$$

have the properties

1) *f_1 factors through $SO(n)$,*

2) *$f_2(-y) = f_2(y)$ for $y \in S^n$,*

then there is a Morse function $g : M(f_1, f_2) \to \mathbf{R}$ for $M(f_1, f_2)$ with just two critical points and an involution $\hat{T} : M(f_1, f_2) \to M(f_1, f_2)$ with fixed points an m-sphere S^m such that $g(\hat{T}z) = -g(z)$. In particular, $M(f_1, f_2)$ is a homotopy sphere which is obtained from two $(m + n + 1)$-disks by gluing their boundaries together via an involution T with fixed points an m-sphere S^m.

We use theorem 2 to prove theorem 1.

SECTION 2

The standard model for the Kervaire sphere K^{4k+1} is given by the boundary of the plumbing of two copies of the unit disk bundle $D(TS^{2k+1})$ of the tangent bundle of the $(2k+1)$-sphere S^{2k+1} [4].

In terms of the characteristic map $\phi : S^{2k} \to SO(2k+1)$ of the tangent bundle TS^{2k+1}, the unit disk bundle $D(TS^{2k+1})$ is

$$\begin{array}{rcl} D(TS^{2k+1}) & = & \left\{ \begin{array}{l} (D^{2k+1} \times D^{2k+1})_A \amalg (D^{2k+1} \times D^{2k+1})_B \\ \text{with } (x,y)_A \sim (x,\phi(x)y)_B \\ \text{for } (x,y) \in \partial D^{2n+1} \times D^{2n+1} \end{array} \right\} \\ \Big\downarrow \pi & & \\ S^{2k+1} & = & \{(D^{2k+1})_A \amalg (D^{2k+1})_B / x_A \sim x_B \quad \text{for} \quad x \in \partial D^{2k+1}\} \end{array} \tag{4}$$

where $\pi((x,y)) = x$.

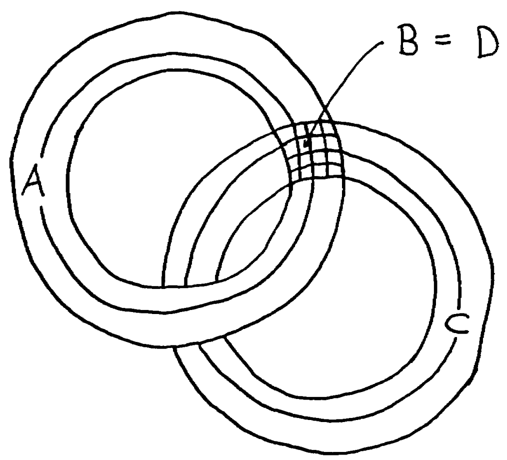

Figure 1.

We may plumb two copies of the unit disk bundle $D(TS^{2k+1})_1$, $D(TS^{2k+1})_2$.

$$\begin{aligned} D(TS^{2k+1})_1 &= \{(D^{2k+1} \times D^{2k+1})_A \amalg (D^{2k+1} \times D^{2k+1})_B / \sim\} \\ D(TS^{2k+1})_2 &= \{(D^{2k+1} \times D^{2k+1})_C \amalg (D^{2k+1} \times D^{2k+1})_D / \sim\} \end{aligned} \tag{5}$$

by matching $(D^{2k+1} \times D^{2k+1})_B$ with $(D^{2k+1} \times D^{2k+1})_D$ by $(x,y)_B \sim (y,x)_D$ ("switching base and fibers"). It will be convenient to use two possibly different (but of course homotopic) characteristic maps ϕ_1, ϕ_2 in these two disk bundles $D(TS^{2k+1})_1$, $D(TS^{2k+1})_2$.

The Kervaire sphere K^{4k+1} is the boundary of the resulting manifold W^{4k+2}. Note that for $(x,y) \in S^{2k} \times S^{2k}$, the points $(x,y)_A$, $(x,\phi_1(x)y)_B$, $(\phi_1(x)y,x)_D$, $(\phi_1(x)y, \phi_2(\phi_1(x)y)^{-1}x)_C$ are all identified in W^{4k+2}. Hence, the Kervaire sphere K^{4k+1} has the representation

$$K^{4k+1} \cong \left\{\begin{array}{l}(D^{2k+1} \times S^{2k})_A \amalg (D^{2k+1} \times S^{2k})_C \\ \text{with } (x,y)_A \sim (\phi_1(x)y, \phi_2(\phi_1(x)y)^{-1}x)_C \\ \text{for } (x,y) \in S^{2k} \times S^{2k}\end{array}\right\} \tag{6}$$

It is also convenient (see next section) to write the last factor as $S^{2k} \times D^{2k+1}$ instead of $D^{2k+1} \times S^{2k}$; that is,

$$K^{4k+1} \cong \left\{\begin{array}{l}(D^{2k+1} \times S^{2k})_A \amalg (S^{2k} \times D^{2k+1})_E \\ \text{with } (x,y)_A \sim (\phi_2(\phi_1(x)y)^{-1}x, \phi_1(x)y)_E \\ \text{for } (x,y) \in S^{2k} \times S^{2k}\end{array}\right\} \tag{7}$$

where $\phi_1, \phi_2 : S^{2k} \to SO(2k+1)$ are any two characteristic maps for the unit disk bundle $D(TS^{2k+1})$.

LEMMA 1. *We may choose* ϕ_1, ϕ_2 *such that*

1) ϕ_1 *factors through* $SO(2k)$, $S^{2k} \to SO(2k) \to SO(2k+1)$

2) ϕ_2 *satisfies* $\phi_2(-y) = \phi_2(y)$.

Proof. (1) The $(2k+1)$-sphere has a nonvanishing vector field ($\chi(S^{2k+1}) = 0$). That is, the structure group $SO(2k+1)$ of TS^{2k+1} may be reduced to $SO(2k) \subset SO(2k+1)$. Equivalently, the characteristic map of TS^{2k+1} factors through $SO(2k)$.

(2) The standard form for the characteristic map of the tangent bundle TS^{2k+1} [8, page 120] is $\phi_2(x) = \alpha(x)$ for $x \in S^{2k+1}$ where $\alpha(x)$ is the reflection through the hyperplane perpendicular to the line spanned by x in $\mathbf{R}^{2k+1}$. For this choice of characteristic map, clearly $\phi_2(-x) = \phi_2(x)$.

SECTION 3

Given smooth maps $f_1 : S^m \to SO(n+1)$, $f_2 : S^n \to SO(m+1)$ Milnor [6] defines a smooth manifold $M(f_1, f_2)$ by matching the boundaries of $D^{m+1} \times S^n$ and $S^m \times D^{n+1}$ via the identification

$$\begin{gathered} f : \partial D^{m+1} \times S^n \to S^m \times \partial D^{n+1} \\ f(x,y) = (x',y') \qquad \text{for} \qquad x \in S^m, y \in S^n \end{gathered} \tag{8}$$

where

$$y' = f_1(x) \cdot y, \quad x' = f_2(y')^{-1}x = f_2(f_1(x) \cdot y)^{-1} \cdot x \tag{9}$$

This identification f has the inverse

$$x = f_2(y') \cdot x', \quad y = f_1(x)^{-1} \cdot y' = f_1(f_2(y') \cdot x')^{-1} \cdot y'. \tag{10}$$

Thus

$$M(f_1,f_2) = \left\{\begin{array}{l}(D^{m+1}\times S^n)_1 \amalg (S^m \times D^{n+1})_2 \quad \text{with} \\ (x,y)_1 \sim (x',y')_2 = (f_2(f_1(x)\cdot y)^{-1}x, f_1(x)y)_2 \\ \text{for} \quad (x,y)\in S^m\times S^n \end{array}\right\} \tag{11}$$

If $f_1 : S^m \to SO(n+1)$ factors as $S^m \to SO(n) \to SO(n+1)$, then Milnor defines a Morse function $g : M(f_1,f_2) \to [-1,+1]$ with just two critical points (both nondegenerate) by

$$\begin{cases} (tx,y)_1 \to h(y)/(1+t^2)^{1/2} \text{ (in first coordinate system)} \\ (x',t'y')_2 \to t'h(y')/(1+(t')^2)^{1/2} \text{ (in the second)} \end{cases} \tag{12}$$

where $h(\tilde{y}) = y_n$ if $\tilde{y} \in S^n$ has coordinates $(y_0,y_1,\ldots,y_n)$. (Here $(x,y)\in S^{2n}\times S^{2n}$ and $0 \le t \le 1$, $(x',y') \in S^{2n}\times S^{2n}$ and $0 \le t' \le 1$).

If $f_2 : S^n \to SO(m+1)$ satisfies $f_2(-y) = f_2(y)$ for $y \in S^n$, then we may define a differentiable involution

$$\hat{T} : M(f_1,f_2) \to M(f_1,f_2)$$

by

$$\begin{array}{ll} \hat{T}[(tx,y)_1] = (tx,-y)_1 & \text{(in the first coordinate system)} \\ \hat{T}[(x',t'y')_2] = (x',-t'y')_2 & \text{(in the second).} \end{array} \tag{13}$$

(For $(x,y)\in S^m\times S^n$ and $0\le t\le 1$ and $(x',y')\in S^m\times S^n$ and $0 \le t \le 1$). $\hat{T}$ is compatible with the identification of $(\partial D^{m+1}\times S^n)_1$ and $(S^m\times \partial D^{n+1})_2$ since for $(x,y)\in S^m\times S^n$

$$\begin{array}{ccc} (x,y)_1 & \to & (x',y')_2 = (f_2(f_1(x)\cdot y)^{-1}x, f_1(x)\cdot y)_2 \\ & & \downarrow \hat{T} \\ \Big\downarrow \hat{T} & & (f_2(f_1(x)\cdot y)^{-1}x, -f_1(x)y)_2 \\ & & \| \\ (x,-y)_1 & \to & (f_2(f_1(x)\cdot[-y])^{-1}x, f_1(x)[-y])_2 \end{array} \tag{14}$$

The asserted equality arises since

$$f_1(x)[-y] = -f_1(x)\cdot y$$

(as $f_1(x)$ is linear) and

$$f_2(f_1(x)[-y])^{-1}\cdot x = f_2(-f_1(x)\cdot y)^{-1}\cdot x = f_2(f_1(x)\cdot y)^{-1}x$$

(since $f_2(-\alpha) = f_2(\alpha)$ by assumption).

Since $h((y_0,\ldots,y_n)) = y_n$, it is clear that

$$g(\hat{T}[(tx,y)_1]) = g[(tx,-y)_1] = \frac{-h(y)}{(1+t)^2)^{1/2}} = g[(tx,y)_1] \tag{15}$$

(in the first coordinate system)

$$g(\hat{T}[(x',t'y')_2]) = g[(x',-t'y')_2] = \frac{-t'h(y')}{(1+(t')^2)^{1/2}} = -g[(x',t'y')_2] \tag{15}$$

(in the second coordinate system).

Hence, if f_1 factors $S^m \to SO(n) \subset SO(n+1)$ and if $f_2 : S^n \to SO(m+1)$ satisfies $f_2(-y) = f_2(y)$, then

1) g is a Morse function on $M(f_1,f_2)$ with just two critical points (both nondegenerate) and with critical values ± 1),

2) $\hat{T}$ is a differentiable involution with fixed point set $S^m = \{(x',0)_2 | x' \in S^m\}$

3) $g(\hat{T}z) = -g(z)$ for z in $M(f_1,f_2)$.

This proves Theorem 2.

Proof of Theorem 1. Theorem 1 follows from Theorem 2 as Lemma 1 implies the Kervaire manifold K^{4k+1} may be exhibited as $K^{4k+1} \cong M(\phi_1,\phi_2)$ where $\phi_1 : S^{2k} \to SO(2k+1)$ factors $S^{2k} \to SO(2k) \to SO(2k+1)$ and $\phi_2 : S^{2k} \to SO(2k+1)$ satisfies $\phi_2(-y) = \phi_2(y)$ for $y \in S^{2k}$.

If we modify the second condition in theorem 2 slightly we obtain a free involution on $M(f_1,f_2)$.

THEOREM 3. *If the differentiable maps*

$$f_1 : S^m \to SO(n+1), \qquad f_2 : S^n \to SO(m+1)$$

have the properties

1) f_1 factors through $SO(n)$,

2) $f_2(-y) = -f_2(y)$ *for* $y \in S^n$,

then there is a Morse function $g : M(f_1,f_2) \to \mathbf{R}$ *for* $M(f_1,f_2)$ *with just two critical points and a free involution* $\hat{T} : M(f_1,f_2) \to M(f_1,f_2)$ *such that* $g(\hat{T}z) = -g(z)$. *In particular,* $M(f_1,f_2)$ *is a homotopy sphere which is obtained from two* $(m+n+1)$*-disks by gluing their boundaries together via a free involution* T.

Theorem 3 is proved in precisely the same way as is theorem 2. We define a Morse function $g : M(f_1,f_2) \to \mathbf{R}$ by equation (12). We define a *free* involution $\hat{T} : M(f_1,f_2) \to M(f_1,f_2)$ by

$$\begin{aligned} &\hat{T}[(tx,y)_1] = (tx,-y)_1 && \text{(in the first coordinate system)} \\ &\hat{T}[(x',t'y')_2] = (-x',-t'y')_2 && \text{(in the second).} \end{aligned} \tag{16}$$

The compatibility of this involution $\hat{T}$ with the identification of $(\partial D^{m+1} \times S^n)_1$

with $(S^m \times \partial D^{n+1})_2$ follows as in equation (14) except $f_2(-y) = -f_2(y)$ by assumption. The equality $g(\hat{T}z) = -g(z)$ is easily checked as in equation (15).

Theorem 3 may be used to recover results of Hirsch and Milnor [3] in dimensions 7 and 15. Let M^7 and M^{15} be generators for bP_8 and bP_{16} respectively.

THEOREM 1′ [3]. *There are smooth fixed point free involutions*

$$T_7 : \partial D^7 \to \partial D^7$$

and

$$T_{15} : \partial D^{15} \to \partial D^{15}$$

such that M^7 is diffeomorphic to the disjoint union of two copies of D^7 with their boundaries identified by T_7 and M^{15} is diffeomorphic to the disjoint union of two copies of D^{15} with their boundaries identified by T_{15}.

The result for M^7 actually appears in [3] while the result for M^{15} is implicit there.

To recover theorem 1′ from theorem 3 one uses the techniques of [2] and [6] to show that in dimension 7, respectively 15, one may choose f_1 and f_2 to be characteristic maps of 4 plane bundles over S^4, respectively of 8 plane bundles over S^8, so that $M(f_1, f_2) = M^7$, respectively $M(f_1, f_2) = M^{15}$, where f_1 and f_2 satisfy the hypothesis of theorem 3.

The following observation of Hirsch and Milnor allows the construction of many other homotopy spheres by involutions.

THEOREM 4 [3]. *Let M^{2n+1} and M_0^{2n+1} be homotopy spheres $(n > 1)$ where M^{2n+1} can be decomposed as the union of two disks identified along their boundaries via a smooth fixed point free involution. Then the connected sum, $M^{2n+1} \# 2(M_0^{2n+1})$, can also be so decomposed.*

Theorems 1′ and 4 imply that all 28 elements of bP_8 and all 8128 elements of bP_{16} may be decomposed as in theorem 1′.

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