PERIODIC HOMEOMORPHISMS OF THE 3-SPHERE AND RELATED SPACES

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1. INTRODUCTION

All objects in this paper are in the PL category. Let h be a periodic homeomorphism of a space M. The cyclic group generated by h shall be denoted by $\langle h \rangle$. Two actions of $\langle h \rangle$ and $\langle h' \rangle$ on M are said to be *conjugate* if there exists a homeomorphism t of M such that $\langle tht^{-1} \rangle = \langle h' \rangle$. In this case, h and h' are called *weakly equivalent.* If $tht^{-1} = h'$, then h and h' are said to be *equivalent.*

E. E. Moise [11] and F. Waldhausen [17] have shown that up to weak equivalences, the 3-sphere S^3 admits exactly one orientation-preserving homeomorphism of even period with nonempty fixed-point set (see P. A. Smith [15] and Kim [4] for alternative proofs). In the present paper, we show that up to weak equivalences S^3 admits exactly one orientation-reversing homeomorphism of period 4k. It follows that there are exactly four Z_4-actions on S^3, up to conjugation (see P. M. Rice [13] for free actions and Kim [4] for semi-free actions). Therefore, all Z_{2n}-actions $(n \leq 2)$ on S^3 are classified (for Z_2-actions, see [8], [9], and [17]). We show further that no lens space L(p, q) $(p > 2)$ admits an orientation-reversing homeomorphism of period n for all $n \neq 4$. We also discuss some free involutions on a lens space L(p, q).

Let h be a homeomorphism of period n on L = L(p, q). Then there exists a homeomorphism $\bar{h}$ of $L/\langle h^k \rangle$, uniquely determined by h, such that $\bar{h}g = gh$, where $g: L \to L/\langle h^k \rangle$ is the orbit map generated by $\langle h^k \rangle$. We call $\bar{h}$ the homeomorphism on $L/\langle h^k \rangle$ induced by h. We say that h is *sense-preserving* if $h_\#$ induces the identity on $H_1(L)$. We shall denote the fixed-point set of h by Fix(h). Note that if h is orientation-reversing, then n must be even, and Fix(h) $\neq \emptyset$ by the Lefschetz fixed-point theorem.

2. ACTIONS ON S^3

Consider S^3 as a subset of C^2, defined by $\{(z_1, z_2) \in C^2 \mid z_1 \bar{z}_1 + z_2 \bar{z}_2 = 1\}$. Define an orientation-reversing homeomorphism T of S^3 by $T(z_1, z_2) = (\omega z_1, \bar{z}_2)$, where $\omega = e^{2\pi i/n}$ and n is even. We call T the standard homeomorphism (of period n). Remark 2.1 may be helpful in elucidating the meaning of Theorem 2.2.

Remark 2.1. Let ϕ be an orientation-preserving homeomorphism of period n on S^3 and with Fix$(\phi) \neq \emptyset$. It is known [11] that Fix(ϕ) is a simple closed curve. By Waldhausen [17], Fix(ϕ) is unknotted for n = 2, and it is unknotted for n = 2k for all k. A well-known conjecture, due to P. A. Smith, asserts that Fix(ϕ) is unknotted for all n (see S. Eilenberg [1]). It can be seen that the fixed-point set of each orientation-reversing periodic homeomorphism on S^3 consists of two points. In

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Theorem 2.2, h^2 is obviously an orientation-preserving homeomorphism whose fixed-point set is a simple closed curve. All orientation-reversing involutions on S^3 (the case $n = 2$) are known [9].

THEOREM 2.2. *Let* h *be an orientation-reversing homeomorphism of period* $n > 2$ *on* S^3. *If* $\mathrm{Fix}(h^2)$ *is unknotted, then* h *is weakly equivalent to the standard homeomorphism.*

Proof. Since n must be even, $n = 2k$ for some k. Let h_1 and h_2 be two orientation-reversing homeomorphisms of period n. Let $F_i = \mathrm{Fix}(h_i^2)$ and $M_i = S^3/\langle h_i^2 \rangle$, and let $g_i\colon S^3 \to M_i$ be the orbit map ($i = 1, 2$). It is known that M_i is homeomorphic to S^3 and $\pi_1(M_i - g_i(F_i)) = Z$ (see [4]). Let $\bar{h}_i$ be the homeomorphism on M_i induced by h_i. Then $\bar{h}_i$ is an orientation-reversing involution. Therefore, $\mathrm{Fix}(h_i)$ must consist of two points, say x_{ij} ($j = 1, 2$). Let $g(x_{ij}) = y_{ij}$ and $g_i(F_i) = J_i$. Then $\bar{h}_i$ interchanges the two open arcs $J_i - \{y_{i1}, y_{i2}\}$ and $\bar{h}_i(y_{ij}) = y_{ij}$. Take invariant balls B_{ij} in M_i containing y_{ij} such that $B_{i1} \cap B_{i2} = \emptyset$. Let $C_i = \mathrm{cl}(J_i - B_{i1} - B_{i2})$. Let K_i be an invariant regular neighborhood of C_i in $\mathrm{cl}(M - B_{i1} - B_{i2})$ such that $K_i \cup B_{i1} \cup B_{i2}$ is a regular neighborhood of J_i. Then K_i has two components, say N_i and N_i' (see Figure 1), and $\bar{h}_i$ interchanges N_i and N_i'. Let $U_i = B_{i1} \cup B_{i2} \cup N_i \cup N_i'$. Since $\pi_1(M_i - U_i) = Z$, one can show by a result of J. Stallings [16] that $\mathrm{cl}(M_i - U_i)$ is a solid torus. Therefore we may reparametrize $\mathrm{cl}(M_i - U_i)$ in terms of $A_i \times I$ so that

$$\partial A_i \times I \approx \partial K_i \cap \partial U_i, \qquad A_i \times 0 \approx \mathrm{cl}(\partial B_{i1} - N_i - N_i'), \qquad A_i \times 1 \approx \mathrm{cl}(\partial B_{i2} - N_i - N_i')$$

(see Figure 2), where each A_i is an annulus. It is known [5] that there exists an involution α on A_i such that the product structure on $A_i \times I$ can be defined so that $\bar{h}_i(x, t) = (\alpha(x), t)$ for $x \in A_i$ and $0 \le t \le 1$. Furthermore, by the argument in [5], we may choose an equivalence t' between the old and the new $\bar{h}_i$ such that $t(\mathrm{cl}(\partial B_{ij} - N_i - N_i')) = A_i \times 0$ if $j = 1$ and $A_i \times 1$ if $j = 2$. Therefore, since $h_i \mid B_i$ is essentially the cone over $h_i \mid \partial B_i$ (see [9]), there exists an equivalence t between $\bar{h}_1$ and $\bar{h}_2$ such that $tg_1(F_1) = g_2(F_2)$. Since $\pi_1(M_i - g_i(F_i)) = Z$, one may conclude by the lifting theorem that h_1 and h_2 are weakly equivalent in the usual way.

By Remark 2.1, we have the following corollary.

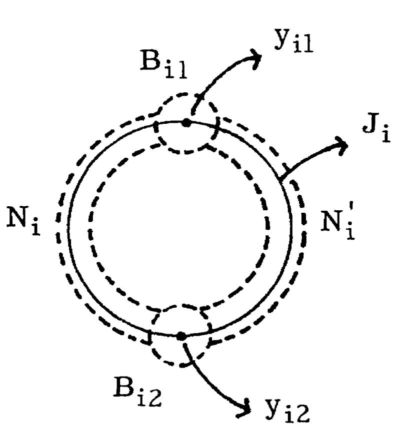

Figure 1.

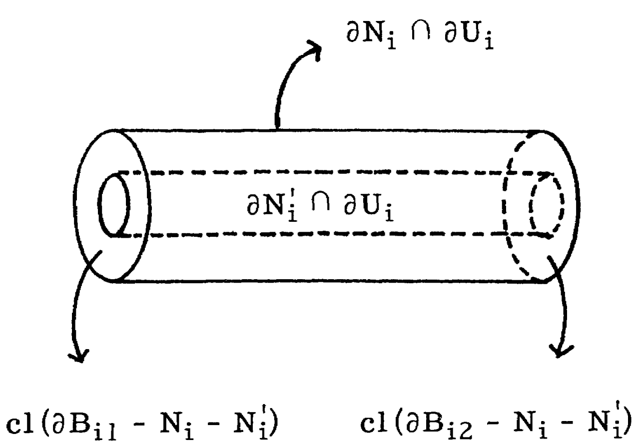

Figure 2.

COROLLARY 2.3. *Up to weak equivalences, there is exactly one orientation-reversing homeomorphism of period* 4k *on* S^3.

It is known that up to conjugation, there exist only one free action [13] and only one semifree action [4] on S^3. Therefore, we have the following result.

THEOREM 2.4. *Up to conjugation,* S^3 *admits exactly four* Z_4*-actions.*

3. ACTIONS ON L(p, q)

Define a homeomorphism T of period p on S^3 by $T(z_1, z_2) = (\omega z_1, \omega^q z_2)$, where $\omega = e^{2\pi i/p}$ and p, q are relatively prime. We consider the lens space L(p, q) as the orbit space $S^3/\langle T \rangle$. K. W. Kwun [6] showed that no lens space L(p, q) $(p > 2)$ admits an orientation-reversing involution. Motivated by this, we shall show the following.

THEOREM 3.1. *No lens space* L(p, q) $(p > 2)$ *admits an orientation-reversing homeomorphism of period* n *for all* $n \neq 4$.

LEMMA 3.2. *Every homeomorphism* h *on* L = L(4s, q) *is orientation-preserving.*

Proof. Let $h_\#(a) = ka$ for $a \in \pi_1(L)$. Then k must be odd. Therefore, $k^2 \equiv 1 \pmod 4$. By a result of P. Olum (see [12, p. 467]), $k^2 \equiv \deg h \pmod{4s}$. Hence, $\deg h \equiv 1 \pmod 4$, and h is orientation-preserving.

Proof of Theorem 3.1. Let h be an orientation-reversing homeomorphism of period n on L = L(p, q) $(p > 2)$. By the Lefschetz fixed-point theorem, $\mathrm{Fix}(h) \neq \emptyset$. Obviously, n is even, say $n = 2^k m$ for some odd m. If k = 1, there exists an orientation-reversing involution on L, which is a contradiction. Therefore, $k \geq 2$ (if k = 2, then $m > 1$, since $n \neq 4$). Let $h_\#(a) = ra$ for $a \in \pi_1(L)$. Then $r^2 \equiv \deg h \pmod p$ [12]. Therefore, h^4 is sense-preserving. Let $g: S^3 \to L$ be the natural projection, and let F_0 be a component of $\mathrm{Fix}(h^2)$. Then $F_0 \subset \mathrm{Fix}(h^4)$. Since h^4 is sense-preserving, $g^{-1}(F_0)$ is connected (use an argument similar to that used in [7]). Let y be a point of S^3 such that $g(y) \in F_0$. Then there exists a lifting homeomorphism $\tilde{h}$ of period $2^{k-1}m$ on S^3 such that $g\tilde{h} = h^2 g$ and $\tilde{h}(y) = y$. Since $g^{-1}(F_0)$ is connected, $\mathrm{Fix}(\tilde{h}) = g^{-1}(F_0)$. Since $\tilde{h}$ is of even period, an easy application of Waldhausen's result [17] shows that $g^{-1}(F_0)$ is unknotted. Therefore, $\pi_1(S^3 - g^{-1}(F_0)) = Z$, and it can be seen that $\pi_1(L - F_0) = Z$ (for a proof, see [7]). Let $T = h^m$. Then T is an orientation-reversing homeomorphism of period 2^k. Since $\mathrm{Fix}(h^2) \subset \mathrm{Fix}(T^2)$, we see that $F_0 \subset \mathrm{Fix}(T^2)$.

Consider first the case where p is odd. Since p and the period of T^2 are relatively prime and $\pi_1(L - F_0) = Z$, the orbit space $L/\langle T^2 \rangle$ is homeomorphic to L(p, q') for some q' [4] (in fact, $F_0 = \mathrm{Fix}(T^2)$ in this case). Let $\overline{T}$ be the homeomorphism on $L/\langle T^2 \rangle$ induced by T. Then $\overline{T}$ is an orientation-reversing involution on L(p, q') $(p > 2)$, which is a contradiction.

Now let p be even; then, by Lemma 3.2, p = 2p' for some odd $p' \geq 3$ (note that $p > 2$). Since $F_0 \subset \mathrm{Fix}(T^2)$, we see that $F_0 \subset \mathrm{Fix}(T^{2^i})$ $(1 \leq i < k)$. Since $T^{2^{k-1}}$ is an orientation-preserving involution on L(2p', q) and $\pi_1(L - F_0) = Z$, the orbit space $M = L/\langle T^{2^{k-1}} \rangle$ is homeomorphic to L(p', q') for some q', and $\pi_1(M - \bar{g}(F_0)) = Z$, where $\bar{g}: L \to M$ is the orbit map [3]. Since $k - 1 \geq 1$, the induced homeomorphism

$\overline{T}$ on $L/\langle T^{2^{k-1}}\rangle$ by T is orientation-reversing and of period 2^{ℓ} for some $\ell \geq 1$. Since $\bar{g}(F_0) \subset \mathrm{Fix}(\overline{T}^2)$, we can now return to the case where p is odd. This completes the proof.

Remark 3.3. Let h be a homeomorphism of period 4 on $L = L(p, q)$, where p is a prime of the form $4\ell + 3$. Since $h_{\#}^4 = 1$ and the automorphism group on Z_p is Z_{p-1}, the order of $h_{\#}$ is 1 or 2. Therefore, deg h = 1 (see [12, p. 461]) and h must be orientation-preserving. Hence, a lens space $L(p, q)$ does not admit an orientation-reversing homeomorphism of period 4 if either $p \equiv 0 \pmod 4$ or p has a prime factor of the form $4\ell + 3$.

Now we discuss some properties of sense-preserving free involutions on $L(p, q)$. All sense-preserving involutions on $L(p, q)$ with nonempty fixed-point sets are known ([3], [6], [7]).

PROPOSITION 3.4. *A free involution* h *on* $L = L(p, q)$ *is sense-preserving if and only if* $\pi_1(L/\langle h\rangle)$ *is abelian.*

Proof. Let α be any path class in L based at x_0. Consider a path class $\omega \cdot h_{\#}\alpha \cdot \omega^{-1} \cdot \alpha^{-1}$, where ω is a path joining x_0 to $h(x_0)$. Suppose that h is sense-preserving. Since $\pi_1(L)$ is abelian, $\omega \cdot h_{\#}\alpha \cdot \omega^{-1} \cdot \alpha^{-1} = 1 \in Z_p = \pi_1(L)$. On the other hand,

$$g_{\#}(\omega \cdot h_{\#}\alpha \cdot \omega^{-1} \cdot \alpha^{-1}) = g_{\#}[\omega] \cdot g_{\#}\alpha \cdot (g_{\#}[\omega])^{-1} \cdot (g_{\#}\alpha)^{-1},$$

where g is the orbit map induced by h. Hence, letting $a = g_{\#}[\omega]$ and $b = g_{\#}\alpha$, we see that $aba^{-1}b^{-1} = 1$. Since $b \notin g_{\#}\pi_1(L)$, the group $\pi_1(L/\langle h\rangle)$ must be abelian. Conversely, if $\pi_1(L/\langle h\rangle)$ is abelian, one can reverse the argument to complete the proof.

COROLLARY 3.5. *If* p *is odd, then every free involution* h *on* $L = L(p, q)$ *is sense-preserving.*

Proof. Let $G = \pi_1(L/\langle h\rangle)$. Then we have the obvious short exact sequence $0 \to Z_p \xrightarrow{f} G \xrightarrow{g} Z_2 \to 0$. Since the order of G is 2p, the group G has an element β of order 2. Since G acts freely on S^3, it follows from a theorem of J. Milnor [10] that β is in the center of G. Since p is odd, $g(\beta) \neq 0$. Therefore, G must be abelian. Now the result follows from Proposition 3.4.

THEOREM 3.6. *Let* h *be a sense-preserving free involution on* $L = L(p, q)$, *where* $p = 4k$ *for some* k. *Then the orbit space* $L/\langle h\rangle$ *is a lens space* $L(2p, q')$, *where* $q'q \equiv \pm 1$ *or* $q' \equiv \pm q \pmod p$. *All such* q' *can occur. Accordingly, up to equivalences, those free involutions* h *on* L *are completely determined by the set of nonhomeomorphic lens spaces* $L(2p, q')$, *where* $q'q \equiv \pm 1$ *or* $q' \equiv \pm q \pmod p$.

Proof. Let h be a free involution on $L = L(p, q)$, where $p \equiv 0 \pmod 4$, and let $M = L/\langle h\rangle$. Suppose that $\pi_1(M)$ is abelian. Then, since the order of $\pi_1(M)$ is 2p, we see by a result of D. B. A. Epstein [2] that $\pi_1(M) = Z_{2p}$. We may assume that $\pi_1(M)$ acts freely on S^3 and admits $\pi_1(L)$ as a subgroup. Let t be a generator of $\pi_1(M)$. Then t^2 is a generator of $\pi_1(L)$. Hence, t^2 is equivalent to an orthogonal transformation. Recently, G. X. Ritter [14] showed that if $\langle t\rangle$ acts freely on S^3 and t^2 is equivalent to an orthogonal transformation, then t is also equivalent to an orthogonal transformation. Therefore M is a lens space $L(2p, q')$, for some integer q'. Define a homeomorphism T of S^3 by $T(z_1, z_2) = (e^{\pi i/p}z_1, e^{\pi q' i/p}z_2)$. Then the orbit space $S^3/\langle T^2 \rangle$ is L(p, q'), and L(p, q') ≈ L(p, q). Recall that L(p, q') and L(p, q) are homeomorphic if and only if $q'q \equiv \pm 1$ or $q' \equiv \pm q$ (mod p). Hence $q'q \equiv \pm 1$ or $q' \equiv \pm q$ (mod p).

Conversely, consider a lens space L(2p, q'), where $q'q \equiv \pm 1$ or $q' \equiv \pm q$ (mod p). Since $q'q \equiv \pm 1$ or $q' \equiv \pm q$ (mod p), there exists a homeomorphism k of L(p, q) onto L(p, q'). Notice that some free involution h on L(p, q') is a covering transformation of L(2p, q'). Define a free involution $\tilde{h}$ on L(p, q) by $\tilde{h} = k^{-1}hk$. Then the orbit space $L(p, q)/\langle \tilde{h} \rangle$ is homeomorphic to L(2p, q'). Notice that two free involutions on L(p, q) are equivalent if and only if their orbit spaces are homeomorphic. Therefore, the result follows from Proposition 3.4.

Finally, we remark that for each pair p, q there exists a sense-preserving free involution on L(p, q).

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