ZEROS OF LIPSCHITZ FUNCTIONS ANALYTIC IN THE UNIT DISC

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1. INTRODUCTION

Let D denote the open unit disc in the complex plane, and let $\bar{D}$ denote its closure. Let Lip α be the class of functions f analytic in D and satisfying a Lipschitz condition of order α,

$$(1.1) \qquad |f(z) - f(z')| \leq C\,|z - z'|^{\alpha}.$$

L. Carleson [1] gave a necessary and sufficient condition for a closed set $E \subset \partial D = \bar{D} \setminus D$ to be the zero set of a function $f \in \text{Lip}\,\alpha$. If $\rho(z, E)$ denotes the Euclidean distance from z to E, then evidently (1.1) implies that

$$\log|f(z)| \leq \alpha \log \rho(z, E) + \log C,$$

and consequently

$$(1.2) \qquad \int_{-\pi}^{\pi} \log \rho(e^{i\theta}, E)\, d\theta > -\infty,$$

by a well-known theorem of F. Riesz. Conversely, Carleson showed that if (1.2) holds, then there exists an outer function f such that $f(e^{i\theta}) = 0$ if and only if $e^{i\theta} \in E$, and that for each integer $m > 0$ the function f can be constructed so that it belongs to the class A^m of functions that are analytic in D and whose first m derivatives are continuous in $\bar{D}$. W. P. Novinger [3] and we extended this result independently by showing that f can be constructed so that it belongs to the class $A^\infty = \bigcap_{m=1}^{\infty} A^m$. Also, a result has recently been proved by Carleson and S. Jacobs that implies the following: if $f \in A = A^0$, if f is an outer function, and if $|f(e^{i\theta})|$ has 2m continuous derivatives as a function of θ, then $f \in A^m$. This theorem yields an easy proof of the extension of Carleson's theorem discussed above.

In this paper, we solve the analogous problem for zero sets in $\bar{D}$. In the following, Z denotes a closed subset of $\bar{D}$ such that $Z \cap D$ is countable. To each element of $Z \cap D$ we assign a multiplicity, and we let $\{z_j\}_{j=1}^{\infty}$ be an enumeration of $Z \cap D$ with each element of $Z \cap D$ appearing in the sequence a number of times equal to its multiplicity. Also, $\rho(z)$ $(z \in \bar{D})$ denotes the Euclidean distance from z to Z.

THEOREM. *In order that for some* α $(0 < \alpha \leq 1)$ *there exist a function* $f \in \text{Lip}\,\alpha$ *whose zero set is* Z *(counting multiplicities), it is necessary that*

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$$\sum_{j=1}^{\infty} (1 - |z_j|) < \infty \tag{1.3}$$

and

$$\int_{-\pi}^{\pi} \log \rho(e^{i\theta})\, d\theta > -\infty . \tag{1.4}$$

On the other hand, if (1.3) *and* (1.4) *hold, then there exists an* $f \in A^{\infty}$ *whose zero set is* Z (*counting multiplicities*).

Carleson [2] has established the regularity condition

$$\int_{-\pi}^{\pi} \log\left(\sum_{j=1}^{\infty} \frac{1 - |z_j|^2}{|e^{i\theta} - z_j|^2} \right) d\theta < +\infty$$

under the weaker assumption that

$$\int_0^1 \int_{-\pi}^{\pi} |f'(re^{i\theta})|^2\, r\, dr\, d\theta < +\infty .$$

We wish to thank Professor Carleson for some helpful comments on Lemma 3.2.

2. PROOF OF THE THEOREM

The necessity of (1.3) is well-known, and the necessity of (1.4) follows from F. Riesz's theorem by the argument outlined in the introduction.

We shall now state several technical lemmas and use them to establish the second part of the theorem. In Section 3, we shall prove the lemmas.

We suppose that the set Z and the distance function ρ satisfy conditions (1.3) and (1.4). Without loss of generality, we may assume that $|z_j| \geq 1/2$ $(j = 1, 2, \cdots)$ and $\rho(z) \leq 1/2$ for all $z \in \overline{D}$. Let B denote the Blaschke product

$$B(z) = \prod_{j=1}^{\infty} \frac{\bar{z}_j}{|z_j|} \left(\frac{z_j - z}{1 - \bar{z}_j z} \right) .$$

LEMMA 2.1. *If* $E = Z \cap \partial D$ *and* $U = \partial D \setminus E$, *then some infinitely differentiable function* $h\colon U \to R$ *satisfies the conditions*

(i) $h(e^{i\theta}) \geq 2 \quad (e^{i\theta} \in U)$,

(ii) $[\rho(e^{i\theta})]^{-1} - 2 \leq h(e^{i\theta}) \leq [\rho(e^{i\theta})]^{-1} + 2 \quad (e^{i\theta} \in U)$,

(iii) *for* $n = 1, 2, \cdots$, *there exist positive constants* C_n *and* p_n *such that*

$$|h^{(n)}(e^{i\theta})| \leq C_n [\rho(e^{i\theta})]^{-p_n} \qquad (e^{i\theta} \in U) .$$

LEMMA 2.2. *If* g *is a nonnegative, Lebesgue-integrable function on* $[-\pi, \pi]$, *then some positive, infinitely differentiable function* $\omega(x)$ *defined for* $x \geq 0$ *satisfies the conditions*

(i) $x^{-1}\omega(x) \to +\infty$ *as* $x \to \infty$,

(ii) $\int_{-\pi}^{\pi} \omega \circ g(\theta)\, d\theta < +\infty$,

(iii) *for* $n = 0, 1, \cdots$, *there exist constants* C_n *such that*

$$|\omega^{(n)}(x)| \le C_n(1 + |x|^2) .$$

The function to be constructed for the proof of our theorem is the Blaschke product B multiplied by the outer function F associated with the function

$$\phi(e^{i\theta}) = \omega(\log h(e^{i\theta})) , \tag{2.1}$$

where h is the function of Lemma 2.1 and ω is the function of Lemma 2.2 when $g(\theta) = \log h(e^{i\theta})$. That is, we define $f = BF$, where

$$G(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \phi(e^{i\theta})\, d\theta \tag{2.2}$$

and

$$F = e^{-G} . \tag{2.3}$$

We shall need estimates on the derivatives of G and of B.

LEMMA 2.3. *If* G *is defined as above, then there exist positive constants* C_n *and* p_n $(n = 0, 1, \cdots)$ *such that*

$$|G^{(n)}(z)| \le C_n [\rho(z)]^{-p_n} \qquad (z \in \overline{D}) .$$

LEMMA 2.4. *There exist positive constants* C_n *such that*

$$|B^{(n)}(z)| \le C_n [\rho(z)]^{-2n} \qquad (z \in D;\ n = 0, 1, \cdots) .$$

Our final lemma concerns the distance function ρ.

LEMMA 2.5. *If* $\sum_{j=1}^{\infty} (1 - |z_j|) < \infty$ *and*

$$I = \int_{-\pi}^{\pi} \log \rho(e^{i\theta})\, d\theta > -\infty ,$$

then

$$\lim_{r \to 1^-} \int_{-\pi}^{\pi} \log \rho(re^{i\theta})\, d\theta = I .$$

Proof of the sufficiency. Let B, G, and F be defined as above, and set $f = BF$. By the Leibnitz formula, every derivative of f is a sum of terms of the form FH, where H is a product of B and powers of its derivatives and powers of derivatives of G. To show that $f \in A^{\infty}$, it suffices to show that each FH is bounded in D.

According to Lemmas 2.3 and 2.4, there exist constants $C > 1$ and $p > 0$ such that $|H(z)| \leq C[\rho(z)]^{-p}$ for $z = re^{i\theta} \in D$. Thus

$$\log |H(re^{i\theta})| \leq -p \log \rho(re^{i\theta}) + \log C ;$$

and therefore, by our assumption that $\rho(re^{i\theta}) \leq 1/2$,

$$\log^+ |H(re^{i\theta})| \leq -p \log \rho(re^{i\theta}) + \log C .$$

By Lemma 2.5, H has bounded characteristic; that is, it belongs to the class N. Moreover, Lemma 2.5 and a well-known generalization of Lebesgue's dominated-convergence theorem (see [5, p. 89], for example) imply that

$$\lim_{r \to 1^-} \int_{-\pi}^{\pi} \log^+ |H(re^{i\theta})| \, d\theta = \int_{-\pi}^{\pi} \log^+ |H(e^{i\theta})| \, d\theta .$$

Consequently, H has the factorization $B_1 S_1 H_1$, where B_1 is a Blaschke product, S_1 is a singular inner function, and H_1 is an outer function for the class N (see [4, p. 82]). Thus $FH = B_1 S_1 FH_1$, and FH is bounded in D if and only if $|F(e^{i\theta}) H_1(e^{i\theta})|$ is essentially bounded. Now

$$\begin{aligned} |F(e^{i\theta}) H_1(e^{i\theta})| &= |F(e^{i\theta}) H(e^{i\theta})| \leq C\,|F(e^{i\theta})|\,[\rho(e^{i\theta})]^{-p} \\ &= C[\rho(e^{i\theta})]^{-p} \exp[-\omega(\log h(e^{i\theta}))] \qquad \text{a.e.} \end{aligned}$$

If we write $\omega(x) = x\varepsilon(x)$, where $\varepsilon(x) \to +\infty$ as $x \to +\infty$, then

$$(2.4) \qquad |F(e^{i\theta}) H_1(e^{i\theta})| \leq C[h(e^{i\theta})]^{-\varepsilon(\log h(e^{i\theta}))} [\rho(e^{i\theta})]^{-p} \qquad \text{a.e.}$$

It follows from (ii) of Lemma 2.1 that $|F(e^{i\theta}) H_1(e^{i\theta})|$ is essentially bounded.

We have demonstrated that $f \in A^\infty$. Since B is the Blachke factor of f, we conclude that $\{z_j\}$ is the zero set of f in D. On ∂D it is clear from the definition of F that $f(z) \neq 0$ for $z \in \partial D \setminus Z$ and that f vanishes on $Z \cap \partial D$. In fact, (2.4) implies that all the derivatives of f vanish on $Z \cap \partial D$.

3. PROOFS OF THE LEMMAS

We shall prove the lemmas of Section 2 in the order in which they are stated. For Lemma 2.1, we first note that in $\overline{D}$ the distance function ρ satisfies the Lipschitz condition

$$|\rho(z + \Delta z) - \rho(z)| \leq |\Delta z| .$$

Consequently, the function $f(\theta) = \rho[(e^{i\theta})]^{-1}$ satisfies in each complementary interval of $\partial D \setminus Z$ the weak Lipschitz condition

$$(3.1) \qquad |f(\theta + \Delta\theta) - f(\theta)| \leq 2\,|\Delta\theta|\,[f(\theta)]^2 ,$$

whenever $|\Delta\theta| \leq (1/2)[f(\theta)]^{-1}$. Therefore, Lemma 2.1 is an immediate consequence of the following lemma.

LEMMA 3.1. *Let* f *be a continuous, real-valued function defined on an open interval* (a, b). *Assume that* f *satisfies the weak Lipschitz condition* (3.1) *on* (a, b) *and that*

(i) $f(\theta) \geq 2 \quad (\theta \in (a, b))$,

(ii) $f(\theta) \to +\infty$ *as* $\theta \to a^+$ *and as* $\theta \to b^-$.

Then some real-valued, infinitely differentiable function h *defined on* (a, b) *satisfies the conditions*

(i') $h(\theta) \geq 2 \quad (\theta \in (a, b))$,

(ii') $f(\theta) - 2 \leq h(\theta) \leq f(\theta) + 2 \quad (\theta \in (a, b))$,

(iii') *for* $n = 1, 2, \cdots$, *there exist positive constants* C_n, *independent of* f, a, *and* b, *such that* $|h^{(n)}(\theta)| \leq C_n[f(\theta)]^{3n}$.

Proof. Let $\theta_0 = (a + b)/2$. Define sequences $\{u_j\}$ and $\{v_j\}$ inductively by $u_0 = v_0 = \theta_0$ and

$$u_{j+1} = u_j - [f(u_j)]^{-3}, \tag{3.2}$$

$$v_{j+1} = v_j + [f(v_j)]^{-3}, \tag{3.3}$$

The weak Lipschitz condition (3.1) together with (ii) implies that $u_j \in (a, b)$ and $v_j \in (a, b)$ $(j = 1, 2, \cdots)$. Clearly, $\{u_j\}$ is decreasing and $\{v_j\}$ is increasing. We also note that $u_j \to a$ as $j \to \infty$; for if $u_j \to u > a$, then, by taking limits in (3.2), we deduce that $u = u - [f(u)]^{-3}$, which is a contradiction to the continuity of f on (a, b). Similarly, $v_j \to b$ as $j \to \infty$.

Construct h as follows. Choose a real-valued, nondecreasing, infinitely differentiable function χ defined on R with $0 \leq \chi(x) \leq 1$, $\chi(x) = 0$ for $x \leq 0$, and $\chi(x) = 1$ for $x \geq 1$. For $v_j \leq \theta < v_{j+1}$, define

$$h(\theta) = f(v_j) + [f(v_{j+1}) - f(v_j)]\chi\left(\frac{\theta - v_j}{v_{j+1} - v_j}\right)$$

and for $u_{j+1} < \theta \leq u_j$, define

$$h(\theta) = f(u_j) + [f(u_{j+1}) - f(u_j)]\chi\left(\frac{u_j - \theta}{u_j - u_{j+1}}\right).$$

Since all the derivatives of χ vanish at $x = 0$ and $x = 1$, the function h is infinitely differentiable on (a, b). That (i') is satisfied is clear from (i) and the monotonicity of χ. To see that (ii') holds, note first that for $u_{j+1} \leq \theta \leq u_j$,

$$|\theta - u_j| \leq |u_{j+1} - u_j| = [f(u_j)]^{-3},$$

so that, by (3.1) and (i),

$$|f(\theta) - f(u_j)| \leq 2[f(u_j)]^{-3}[f(u_j)]^2 \leq 1.$$

Similar inequalities hold with v_j replacing u_j. Since $0 \leq \chi(x) \leq 1$, the inequality (ii') follows. The proof of (iii') is now easy. For example, when $u_{j+1} < \theta \leq u_j$, then

$$|h^{(n)}(\theta)| = |f(u_{j+1}) - f(u_j)| \left|\chi^{(n)}\left(\frac{u_j - \theta}{u_j - u_{j+1}}\right)\right| |u_j - u_{j+1}|^{-n}$$

$$\leq C_n |f(u_j)|^{3n} = C_n[f(u_j)/f(\theta)]^{3n}[f(\theta)]^{3n} \leq C_n 2^{3n}[f(\theta)]^{3n},$$

where $C_n = \max\{\chi^{(n)}(x): -\infty < x < +\infty\}$. This completes the proof of Lemma 3.1.

Lemma 2.2 is a statement about integrable functions; we omit the proof.

Proof of Lemma 2.3. The proof is essentially the same as for a corresponding estimate given by Carleson in [1]. If $z \in D$ and $\rho(z) \leq 8(1 - |z|)$, then the estimate for

$$|G^{(n)}(z)| = \left|\frac{n!}{\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta}}{(e^{i\theta} - z)^{n+1}} \phi(e^{i\theta}) d\theta\right|$$

follows easily. If $z \in D$ and $\rho(z) > 8(1 - |z|)$, consider the arc

$$\{e^{i\theta}: \alpha \leq \theta \leq \beta\} = \left\{e^{i\theta}: |e^{i\theta} - z| \leq \tfrac{1}{2}\rho(z)\right\} \subset \partial D \setminus Z.$$

Note that because of the Lipschitz condition on ρ,

$$\rho(e^{i\theta}) \geq \frac{1}{2}\rho(z) \qquad (\alpha \leq \theta \leq \beta). \tag{3.4}$$

The lemma will be proved if we obtain the desired estimate for

$$J_n(z) = \frac{n!}{\pi} \int_{\alpha}^{\beta} \frac{e^{i\theta}}{(e^{i\theta} - z)^{n+1}} \phi(e^{i\theta}) d\theta,$$

since the estimate for $G^{(n)}(z) - J_n(z)$ is clear.

To estimate $J_n(z)$, integrate by parts n times, obtaining the formula

$$\begin{aligned} J_n(z) = &- \sum_{k=0}^{n-1} \frac{(n-k-1)!}{(i)^{k+1}\pi}\left[\frac{\phi^k(e^{i\beta})}{(e^{i\beta} - z)^{n-k}} - \frac{\phi^k(e^{i\alpha})}{(e^{i\alpha} - z)^{n-k}}\right] \\ &+ \frac{1}{(i)^n \pi} \int_{\alpha}^{\beta} \frac{e^{i\theta}}{e^{i\theta} - z} \phi^n(e^{i\theta}) d\theta, \end{aligned} \tag{3.5}$$

where $\phi^0 = \phi$ and

$$\phi^k(e^{i\theta}) = e^{-i\theta} \frac{d}{d\theta} \phi^{k-1}(e^{i\theta}) \qquad (k = 1, 2, \cdots, n).$$

We shall estimate $J_n(z)$ by the sum of the absolute values of the terms of (3.5). First, it is easy to verify from the bounds given in Lemmas 2.1 and 2.2 that

$$|\phi^{(n)}(e^{i\theta})| \leq D_n[\rho(e^{i\theta})]^{-m_n} \qquad (e^{i\theta} \in U), \tag{3.6}$$

for some positive constants D_n and m_n $(n = 0, 1, 2, \cdots)$. Now, since

$\rho(e^{i\beta}) \geq \frac{1}{2}\rho(z)$, $\rho(e^{i\alpha}) \geq \frac{1}{2}\rho(z)$, and

$$|e^{i\alpha} - z| = |e^{i\beta} - z| = \frac{1}{2}\rho(z),$$

it is clear that each term of the sum on the right side of (3.5) satisfies an inequality of the desired form. Therefore, it remains only to estimate the integral in (3.5).

In this integral, let $z = re^{it}$, and write

$$\phi^n(e^{i\theta}) = \phi^n(e^{it}) + \varepsilon(\theta).$$

By the mean-value theorem, $|\varepsilon(\theta)| \leq M_n|\theta - t|$, where

$$M_n = \max\left\{\left|\frac{d}{d\theta}\phi^n(e^{i\theta})\right| : \alpha \leq \theta \leq \beta\right\}.$$

Thus, the integral does not exceed

$$|\phi^{(n)}(e^{it})| \left|\int_\alpha^\beta \frac{e^{i\theta}}{e^{i\theta} - z}\, d\theta\right| + M_n \int_\alpha^\beta \frac{|\theta - t|}{|e^{i\theta} - z|}\, dt \leq \text{const.}\,(|\phi^{(n)}(e^{it})| + |\beta - \alpha| M_n),$$

and therefore it satisfies the desired inequality, by virtue of (3.4) and (3.6).

Proof of Lemma 2.4. These estimates are very easy. A direct computation shows that

$$B'(z) = -B(z)\sum_{j=1}^{\infty} \frac{1 - |z_j|^2}{(1 - \bar{z}_j z)(z_j - z)}.$$

Since $|B(z)| \leq 1$, $\sum_{j=1}^{\infty}(1 - |z_j|) < \infty$, and $|1 - \bar{z}_j z| \geq |z_j - z|$ for $z \in D$, the estimate for B' is clear. The higher derivatives are handled similarly.

It remains to prove Lemma 2.5. Actually, we shall prove a slightly stronger result, namely Lemma 3.2 below. Let $n(r)$ denote the number of elements of $\{z_j\}$ in $\{z: |z| \leq r\}$. We claim that $n(r) = O((1 - r)^{-1})$. To see this, recall that the convergence of $\sum_{j=1}^{\infty}(1 - |z_j|)$ is equivalent to the convergence of $\sum_{j=1}^{\infty} \log|z_j|^{-1}$. Write this last sum as the integral $M = \int_0^1 \log t^{-1}\, dn(t)$. Since $n(0) = 0$, we have for every $r < 1$ the estimate

$$M \geq \int_0^r \log t^{-1}\, dn(t) = n(r)\log r^{-1} + \int_0^r t^{-1} n(t)\, dt.$$

Thus $n(r) \leq \frac{M}{\log r^{-1}} \sim \frac{M}{1 - r}$, as $r \to 1^-$.

LEMMA 3.2. *Let* Z, $\{z_j\}$, ρ, *and* $n(r)$ *be defined as above. If*

(i) $\log n(r) = O\left(\log \frac{1}{1-r}\right)$ as $r \to 1^-$

and

(ii) $I = \int_{-\pi}^{\pi} \log \rho(e^{i\theta})\, d\theta > -\infty$,

then $\lim_{r \to 1^-} \int_{-\pi}^{\pi} \log \rho(re^{i\theta})\, d\theta = I.$

Proof. It will suffice to consider the interval $3/4 \le r < 1$. Let

$$S_r = \left\{ \theta \in (-\pi, \pi] \colon \rho(re^{i\theta}) < \tfrac{1}{4}\rho(e^{i\theta}) \right\},$$

and let $S_r' = (-\pi, \pi] \setminus S_r$. If $\chi_r(\theta)$ is the characteristic function of S_r', then

$$\chi_r(\theta) \log \rho(re^{i\theta}) \ge \log \rho(e^{i\theta}) - \log 4 \qquad (\theta \in (-\pi, \pi]).$$

Further,

$$\lim_{r \to 1^-} \chi_r(\theta) \log \rho(re^{i\theta}) = \log \rho(e^{i\theta}) \qquad (\theta \in (-\pi, \pi]).$$

Consequently, by (ii) and the dominated-convergence theorem,

$$\lim_{r \to 1^-} \int_{S_r'} \log \rho(re^{i\theta})\, d\theta = I.$$

To prove the lemma, it thus suffices to show that

$$\lim_{r \to 1^-} \int_{S_r} \log \rho(re^{i\theta})\, d\theta = 0.$$

In fact, since $\log \rho(re^{i\theta}) < 0$ for $\theta \in S_r$, it suffices to prove that

$$\liminf_{r \to 1^-} \int_{S_r} \log \rho(re^{i\theta})\, d\theta \ge 0. \tag{3.7}$$

To prove (3.7), we first note that $\rho(e^{i\theta}) < 2(1 - r)$ for all $\theta \in S_r$. For if this is not the case, then

$$\rho(re^{i\theta}) \ge \rho(e^{i\theta}) - (1 - r) \ge \tfrac{1}{2}\rho(e^{i\theta}),$$

which contradicts the assumption that $\theta \in S_r$. In particular,

$$\rho(re^{i\theta}) < \tfrac{1}{4}\rho(e^{i\theta}) < \tfrac{1}{2}(1 - r) \qquad (\theta \in S_r). \tag{3.8}$$

Hence, for each $\theta \in S_r$, the disc

$$D_r(\theta) = \left\{ w \in D \colon |w - re^{i\theta}| < \tfrac{1}{2}(1 - r) \right\}$$

contains at least one point of Z. Let

$$Z_r = \bigcup \{Z \cap D_r(\theta) \colon \theta \in S_r\},$$

and let Z_r^* be the set of points obtained by the radial projection of Z_r onto the circle $\{z\colon |z| = r\}$. Let $\rho_r^*(re^{i\theta})$ denote the distance measured along the circle of radius r from $re^{i\theta}$ to Z_r^*. Then it is easy to see that

$$\rho_r^*(re^{i\theta}) \leq \sqrt{2}\,\rho(re^{i\theta}) \qquad (\theta \in S_r)\,. \tag{3.9}$$

Now let $N(r)$ denote the number of points in Z_r^*. Since

$$Z_r^* \subset \left\{z\colon |z| \leq \frac{1}{2}(1+r)\right\},$$

it follows from (i) that

$$\log N(r) = O\left(\log \frac{1}{1-r}\right) \qquad (r \to 1^-)\,. \tag{3.10}$$

Next we introduce the set

$$S_r^* = \{\theta \in (-\pi, \pi]\colon \rho_r^*(re^{i\theta}) < (\sqrt{2}/4)\,\rho(e^{i\theta}) < (1/\sqrt{2})(1-r)\}\,.$$

By (3.9), $S_r \subset S_r^*$, and therefore

$$\int_{S_r^*} \log \rho_r^*(re^{i\theta})\,d\theta \leq \int_{S_r} \log \rho_r^*(re^{i\theta})\,d\theta \leq m(S_r)\log\sqrt{2} + \int_{S_r} \log \rho(re^{i\theta})\,d\theta\,,$$

where m denotes linear Lebesgue measure. Since

$$S_r \subset \{\theta \in (-\pi, \pi]\colon \rho(e^{i\theta}) < 2(1-r)\}\,,$$

it is clear from (ii) that $m(S_r) \to 0$ as $r \to 1^-$. Thus, to prove (3.7) it suffices to prove that

$$\liminf_{r \to 1^-} \int_{S_r^*} \log \rho_r^*(re^{i\theta})\,d\theta \geq 0\,. \tag{3.11}$$

For later reference, we show that

$$m(S_r^*) = o(1)\left(\log \frac{1}{1-r}\right)^{-1} \qquad (r \to 1^-)\,. \tag{3.12}$$

To this end, we observe that

$$S_r^* \subset \{\theta \in (-\pi, \pi]\colon \rho(e^{i\theta}) < 2(1-r)\}\,;$$

from (ii), it follows that $m(S_r^*) \to 0$ as $r \to 1^-$. Therefore

$$o(1) = \int_{S_r^*} \log \rho(e^{i\theta})\,d\theta \leq m(S_r^*)\log 2(1-r)\,,$$

which implies (3.12).

We shall now prove that

$$\int_{S_r^*} \log \rho_r^*(re^{i\theta})\,d\theta \geq m(S_r^*) \log \frac{m(S_r^*)}{2N(r)} - (1 + \log 2)\, m(S_r^*)\ . \tag{3.13}$$

This will suffice to prove (3.11), and hence the lemma, since by (3.10) and (3.12),

$$\lim_{r \to 1^-} [m(S_r^*) \log m(S_r^*) + m(S_r^*) \log (2N(r))^{-1} - (1 + \log 2)\, m(S_r^*)] = 0\,.$$

To prove (3.13), it is necessary to consider the geometry of S_r^* and Z_r^*. Let $r \exp(i\theta_1), \cdots, r \exp(i\theta_{N(r)})$ be the points of Z_r^*, ordered so that

$$-\pi < \theta_1 < \cdots < \theta_{N(r)} \leq \pi\,.$$

The open set S_r^* is clearly a finite union of open intervals, say (a_k, b_k) $(k = 1, \cdots, m_r)$. To estimate the integral in (3.13), it is necessary to obtain a bound on m_r. We shall show that $m_r \leq 2N(r)$ by showing that each interval $[a_k, b_k]$ must contain at least one θ_i. Suppose this is not the case, and choose i so that $\theta_{i-1} < a_k < b_k < \theta_i$ (trivial modifications may be necessary near $\pm\pi$). Since $a_k \in \partial S_r^*$, either $\rho(e^{ia_k}) = 2(1 - r)$ or $\rho_r^*(re^{ia_k}) = (\sqrt{2}/4)\rho(e^{ia_k})$. In the first case, it is easy to see that $\rho_r^*(re^{ia_k}) \geq 1 - r$. But $\rho_r^*(re^{i\theta}) < (1/\sqrt{2})(1 - r)$ for all $\theta \in S_r^*$, and therefore the continuity of ρ_r^* implies that if θ is sufficiently near a_k, then no point $re^{i\theta}$ belongs to S_r^*. This contradicts the assumption that $a_k \in \partial S_r^*$. Thus $\rho(e^{ia_k}) < 2(1 - r)$. Similarly, $\rho(e^{ib_k}) < 2(1 - r)$. Consequently,

$$\rho_r^*(re^{ia_k}) = (\sqrt{2}/4)\rho(e^{ia_k}) \quad \text{and} \quad \rho_r^*(re^{ib_k}) = (\sqrt{2}/4)\rho(e^{ib_k})\,.$$

Choose the one of a_k, b_k that lies nearer to an endpoint of $[\theta_{i-1}, \theta_i]$. To be specific, assume this point is a_k, so that

$$\rho_r^*(re^{i\theta}) = r(\theta - \theta_i)$$

when θ is near a_k. On the other hand, $\sqrt{2}\rho(e^{i\theta})/4$ is a Lipschitz function with constant at most $\sqrt{2}/4$. Thus, since $r \geq 3/4 > \sqrt{2}/4$, we have the inequality

$$\rho_r^*(re^{i\theta}) > \sqrt{2}\rho(re^{i\theta})/4$$

when θ is in (a_k, b_k) and near a_k. This contradicts the assumption that $(a_k, b_k) \subset S_r^*$.

Now, for each interval (a_k, b_k) of S_r^*, let p and q denote the integers for which

$$\theta_p \leq a_k < \theta_{p+1} < \cdots < \theta_{q-1} < b_k \leq \theta_q\,.$$

Let $u(x) = x \log x$, for $x > 0$. A direct computation shows that

$$\int_{a_k}^{b_k} \log \rho_r^*(re^{i\theta})\, d\theta$$

$$\geq u(\theta_{p+1} - a_k) + \sum_{i=p+2}^{q-1} u(\theta_i - \theta_{i-1}) + u(b_k - \theta_{q-1}) - (1 + \log 2)(b_k - a_k) .$$

Therefore, a lower bound for

$$\int_{S_r^*} \log \rho_r^*(re^{i\theta})\, d\theta + (1 + \log 2)\, m(S_r^*)$$

is given by a sum of the form $\sum_i u(\varepsilon_i)$, where the ε_i are positive numbers such that $\sum_i \varepsilon_i = m(S_r^*)$; moreover, there are at most $2N(r)$ terms in the sum. It is well known that under these constraints the minimum value for such a sum occurs when all the ε_i are equal. Thus

$$\int_{S_r^*} \log \rho_r^*(re^{i\theta})\, d\theta + (1 + \log 2)\, m(S_r^*) \geq 2N(r)\; u(m(S_r^*)/2N(r)) ,$$

which is precisely the desired inequality (3.13).

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