SUMMABILITY OF ORDINARY DIRICHLET SERIES BY PERRON-TYPE MATRICES

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1. INTRODUCTION

Let $f(z)$ be a function which is analytic for $|z| < R$ $(R > 1)$, and suppose that $f(1) = 1$. Then given a series Σu_n, we formally rearrange the series $\Sigma u_n [f(z)]^n$ into powers of z so that it becomes $\Sigma U_n z^n$. Explicitly, if we let

$$[f(z)]^n = \sum_k f_{kn} z^k \qquad (n = 1, 2, \cdots),$$

$$f_{00} = 1, \quad f_{0k} = 0 \qquad (k = 1, 2, \cdots),$$

then

$$U_n = \sum_k f_{nk} u_k \qquad (n = 0, 1, 2, \cdots).$$

The matrix $F = (f_{nk})$ is said to be generated by $f(z)$, and it will be called a Perron-type matrix. The series ΣU_n is called the F-transform of Σu_n; and if ΣU_n converges, then Σu_n is said to be F-summable. If $\Sigma |U_n|$ converges, then Σu_n is said to be $|F|$-summable. Such transformations have been studied by Perron [6], Knopp [2], and Macphail [3, 4].

Given an ordinary Dirichlet series $\Sigma u_n(n + 1)^{-s}$ and a suitable Perron-type matrix F, we will here be concerned with several problems, which together with their solutions comprise the following theorem.

THEOREM. *Let* $w = f(z)$ *be a function that is analytic and univalent in* $|z| < R$ $(R > 1)$ *and let* $z = g(w)$ *be its inverse function. Assume that*

(1) $g(w)$ *generates the matrix* (g_{nk}),

(2) $\Re\left[\dfrac{e^{i\theta} f'(e^{i\theta})}{f(e^{i\theta})}\right] > 0 \qquad (0 \leq \theta < 2\pi)$,

(3) $f(1) = 1$, $|f(z)| < 1$ *if* $|z| < 1$,

(4) $\sum_j \sum_k |f_{nk} g_{kj}|$ *converges* $\qquad (n = 0, 1, 2, \cdots)$.

Assume that $u_n = O(r^n)$, *where* $|rf(0)| < 1$. *Then*

(A) *if* $\Sigma u_n(n + 1)^{-s_0}$ *is* F-*summable, then* $\Sigma u_n(n + 1)^{-s}$ *is* F-*summable for* $\Re s > \Re s_0$,

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(B) *if* $\Sigma u_n(n+1)^{-s_0}$ *is* $|F|$-*summable, then* $\Sigma u_n(n+1)^{-s}$ *is* $|F|$-*summable for* $\Re s > \Re s_0$, *and*

(C) *if* $\Sigma u_n(n+1)^{-s_0}$ *is* F-*summable, then* $\Sigma u_n(n+1)^{-s}$ *is* $|F|$-*summable for* $\Re s > 1 + \Re s_0$.

These results generalize results of Obrechkoff [5] concerning Euler methods and results of Cowling and Piranian [1] concerning Taylor methods.

Before we proceed with the proof, there are several remarks to be made concerning the hypotheses on f(z) and g(w). A function f(z) satisfying (2) maps the circle $|z| < 1$ onto a "star-shaped" domain, that is, one whose boundary is cut at only one point by each ray from the origin. It is well-known that (3) implies that $f'(1) > 0$. The assumption (4) is somewhat restrictive, requiring in the case of the Taylor T_p matrix, for example, that $0 \leq p < 1/3$. If $f(0) = 0$, then (f_{nk}) and (g_{nk}) are triangular matrices, in which case (4) is satisfied.

2. THREE LEMMAS

Our proof of the theorem depends on three lemmas.

LEMMA 1. *Assume that* $w = f(z)$ *and* $z = g(w)$ *are as described in the hypotheses of the theorem. Suppose that* $u_n = O(r^n)$, *where* $|rf(0)| < 1$, *and that* ΣU_n, *the* F-*transform of* Σu_n, *converges. Then*

$$u_n = \sum_k g_{nk} U_k \qquad (n = 0, 1, 2, \cdots).$$

Proof. By definition,

$$U_n = \sum_k f_{nk} u_k \qquad (n = 0, 1, 2, \cdots).$$

Then, since ΣU_n converges,

$$\sum U_n z^n = \sum_n z^n \sum_k f_{nk} u_k \qquad (|z| < 1).$$

But since $|rf(0)| < 1$, there exists a number θ $(0 < \theta < 1)$ and a neighborhood of $z = 0$ where $|rf(z)| < \theta$. Thus, since $u_n = O(r^n)$, the series $\Sigma u_k[f(z)]^k$ converges uniformly in this neighborhood, and so we may apply Weierstrass' theorem on double series and write

$$\sum U_n z^n = \sum_k u_k [f(z)]^k$$

for sufficiently small z. Since f(z) maps $|z| < 1$ onto a star-shaped domain, $|g(0)| < 1$, so we may reapply Weierstrass' theorem, obtaining the result

$$\sum u_n w^n = \sum_n w^n \sum_k g_{nk} U_k,$$

for sufficiently small w. Equating coefficients of w^n completes the proof of Lemma 1.

LEMMA 2. *Let* $w = f(z)$ *and* $z = g(w)$ *be as before. Then*

(a) *there exists an* $R_0 > 1$ *such that for all* $t \geq 0$ *the function* $g(f(z)e^{-t})$ *is an analytic function of* z *for* $|z| \leq R_0$,

(b) *there exist numbers* $T_1 > 0$ *and* R_1 $(0 < R_1 < 1)$ *such that*

$$g(f(z)e^{-t}) = z \exp\{t\phi(z, t)\}, \quad \phi(z, t) = -[f(z)/zf'(z)] + tp(z, t)$$

and $p(z, t), \dfrac{\partial p(z, t)}{\partial z}$ *are bounded functions of* z *and* t *for* $0 \leq t \leq T_1$ *and* $R_1 \leq |z| \leq R_0$,

(c) *there exist positive numbers* $\rho_2 > 1$, T_2, *and* ψ *such that* $\Re\phi(z, t) < -\psi$ *on* $\rho_2^{-1} \leq |z| \leq \rho_2$ *for* $0 \leq t \leq T_2$, *and*

(d) *if* $T = \min(T_1, T_2)$ *then there exist numbers* θ $(0 < \theta < 1)$ *and* $\rho_1 > 1$ *such that*

$$|g(f(z)e^{-t})| \leq \theta < 1$$

if $|z| \leq \rho_1$ *and* $t \geq T$.

Proof. (a). This conclusion follows because $f(z)$ is analytic for $|z| < R$ $(R > 1)$, because $g(w)$ is the inverse function of $f(z)$, and from (2) and the fact that $0 < e^{-t} \leq 1$, if $t \geq 0$.

(b) Let $H(z, t) = \log g(f(z)e^{-t})$. Restrict t and z so that $g(f(z)e^{-t})$ is analytic and nonzero for $0 \leq t \leq T_1$ and $R_1 \leq |z| \leq R_0$. Then since

$$\frac{\partial H(z, 0)}{\partial t} = -\frac{f(z)}{z f'(z)},$$

we may write

$$H(z, t) = \log z - \frac{f(z)}{z f'(z)} t + t^2 p(z, t)$$

for $0 \leq t \leq T_1$ and $R_1 \leq |z| \leq R_0$.

(c) From (2) it follows that there exist numbers ρ_2 $(R > \rho_2 > 1)$ and $\psi_2 > 0$ such that

$$\Re\left[\frac{z f'(z)}{f(z)}\right] > \psi_2 \qquad (\rho_2^{-1} \leq |z| \leq \rho_2).$$

Hence there exists a number $\psi_1 > 0$ such that

$$\Re\left[\frac{f(z)}{z f'(z)}\right] = \left|\frac{f(z)}{z f'(z)}\right|^2 \Re\left[\frac{z f'(z)}{f(z)}\right] > \psi_1 \quad (\rho_2^{-1} \leq |z| \leq \rho_2).$$

Since

$$\phi(z, t) = -\frac{f(z)}{z f'(z)} + O(t),$$

the proof of (c) follows immediately.

(d) The image of $|z| = 1$ under the mapping given by $\zeta = g(f(z)e^{-T})$ is a curve C, lying inside $|\zeta| = 1$. This follows from the hypotheses made on f(z) and the fact that $e^{-T} < 1$. Hence, there exists a circle of radius $\rho_1 > 1$ whose image in the ζ-plane lies between C and $|\zeta| = 1$, and hence along which

$$|g(f(z)e^{-T})| \leq \theta < 1.$$

Thus, since $e^{-t} \leq e^{-T}$ for $t \geq T$,

$$\max_{|z|=\rho_1} |g(f(z)e^{-t})| \leq \theta < 1 \qquad (t \geq T).$$

This completes the proof of Lemma 2.

In all that follows let $\rho_0 = \min(\rho_1, \rho_2, R_0, R_1^{-1})$.

LEMMA 3. *If* f(z) *and* g(w) *are as above, then there exists a constant* M *such that*

$$|g(w) - 1| \leq M(1 - |g(w)|) \qquad (0 \leq w \leq 1).$$

Proof. We need to show that there exists a sector of the half-plane $\Re z < 1$ inside of which the image of $0 \leq w \leq 1$ lies. But g(w) is analytic at $w = 1$; and since f(z) maps $|z| < 1$ onto a star-shaped domain and $f(1) = 1$, g(w) is analytic in an open connected set about $[0, 1]$, and $|g(w)| < 1$ for $0 \leq w \leq 1$. Furthermore, (3) implies that $f'(1) > 0$ so that $g'(1) > 0$ also. Thus the image of $0 \leq w \leq 1$ under the mapping by g(w) is perpendicular to $|z| = 1$ at $z = 1$. Hence it is possible to find a circle of sufficiently small radius about $z = 1$ that the image of $0 \leq w \leq 1$ will intersect at some point interior to $|z| < 1$. The points where this small circle intersects $|z| = 1$ together with $z = 1$ determine the desired sector. This completes the proof of Lemma 3.

3. PROOF OF THE THEOREM

Next note that since $u_n = O(r^n)$, where $|rf(0)| < 1$, there exists a number r' such that $|r' f(0)| < 1$ and such that, for sufficiently large n, $u_n(n + 1)^{-s_0} = O(r'^n)$. Thus we may assume without loss of generality that $s_0 = 0$.

We now proceed with the proof of the main theorem.

Proof of (A). Set

$$U_n = U_n(0) = \sum_k f_{nk} u_k \qquad (n = 0, 1, 2, \cdots).$$

Then by Lemma 1,

$$u_n = \sum_k g_{nk} U_k \qquad (n = 0, 1, 2, \cdots).$$

Thus,

$$U_n(s) = \sum_k f_{nk} u_k (k + 1)^{-s} = \sum_k f_{nk}(k + 1)^{-s} \sum_j g_{kj} U_j \qquad (n = 0, 1, 2, \cdots).$$

Then using (4) and that $\Re s > 0$ and $U_j \to 0$, we may reverse the order of summation, obtaining the result

$$U_n(s) = \sum_j A_{nj} U_j \qquad (n = 0, 1, 2, \cdots),$$

where

$$A_{nj} = \sum_k f_{nk} g_{kj} (k + 1)^{-s} \qquad (n, j = 0, 1, 2, \cdots).$$

Further, since $\Re s > 0$,

$$(k + 1)^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-(k+1)t} dt \qquad (k = 0, 1, 2, \cdots).$$

But (4) implies that $\sum_k |f_{nk} g_{kj}|$ converges for all n, j. Thus we may write

$$A_{nj} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-t} \sum_k f_{nk} g_{kj} e^{-kt} dt.$$

Also by (4),

$$\sum_n z^n \sum_k f_{nk} g_{kj} e^{-kt} = \sum_k g_{kj} e^{-kt} [f(z)]^k = [g(f(z)e^{-t})]^j.$$

Thus,

$$A_{nj} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-t} \left[\frac{1}{2\pi i} \int_C \frac{[g(f(z)e^{-t})]^j}{z^{n+1}} dz \right] dt,$$

where C is a circle of sufficiently small radius about the origin. So, by partial summation,

$$\sum_{n=0}^m A_{nj} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-t} \left[\frac{1}{2\pi i} \int_C [g(f(z)e^{-t})]^j \left(\frac{1 - z^{-m-1}}{z - 1} \right) dz \right] dt.$$

Hence, recalling part (a) of Lemma 2, we see that

$$\sum_{n=0}^\infty A_{nj} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-t} [g(e^{-t})]^j dt.$$

Thus in order for the convergence of ΣU_n to imply the convergence of $\Sigma U_n(s)$, it suffices to show that

$$\sup_m \sum_j \left| \sum_{n=0}^{m} (A_{nj} - A_{n,j+1}) \right| < \infty .$$

But

$$\sum_{n=0}^{m} (A_{nj} - A_{n,j+1}) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-t} \beta_{mj}(t)\, dt ,$$

where

$$\beta_{mj}(t) = \frac{1}{2\pi i} \int_C \frac{g(f(z)e^{-t}) - 1}{z - 1} \frac{[g(f(z)e^{-t})]^j}{z^{m+1}} dz .$$

Evaluating the residue at $z = 1$, we find that

$$\beta_{mj}(t) = [g(e^{-t}) - 1][g(e^{-t})]^j + \frac{1}{2\pi i} \int_{|z|=\rho_0} \frac{(g(f(z)e^{-t}) - 1)}{(z - 1)} \frac{[g(f(z)e^{-t})]^j}{z^{m+1}} dz \tag{5}$$

(Recall that ρ_0 is defined to be

$$\rho_0 = \min(\rho_1, \rho_2, R_0, R_1^{-1}) > 1.)$$

If $t \geq T$, then by part (d) of Lemma 2,

$$\int_{|z|=\rho_0} \frac{g(f(z)e^{-t}) - 1}{z - 1} \frac{[g(f(z)e^{-t})]^j}{z^{m+1}} dz = O\left(\frac{\theta^j}{\rho_0^m} \right) .$$

If $0 \leq t \leq T$ and $0 \leq j \leq m$, then, using Lemma 2 (b) and 2 (c), we conclude that

$$\int_{|z|=\rho_0} \frac{g(f(z)e^{-t}) - 1}{z - 1} \frac{[g(f(z)e^{-t})]^j}{z^{m+1}} dz = O(\rho_0^{j-m}) .$$

If $0 \leq t \leq T$ and $m + 1 \leq j$, then, using parts (b) and (c) of Lemma 2, we conclude that

$$\beta_{mj}(t) = \frac{1}{2\pi i} \int_{|z|=\rho_0^{-1}} \frac{(g(f(z)e^{-t}) - 1)}{z - 1} \frac{[g(f(z)e^{-t})]^j}{z^{m+1}} dz = O(\rho_0^{m-j}) .$$

Therefore by (5) and Lemma 3,

$$\sum_{j=0}^{m} \left| \sum_{n=0}^{m} (A_{nj} - A_{n,j+1}) \right|$$

$$\leq \sum_{j=0}^{m} \frac{1}{|\Gamma(s)|} \int_T^\infty t^{\Re s - 1} e^{-t} \{ |g(e^{-t}) - 1| \cdot |g(e^{-t})|^j + O(\theta^j/\rho_0^m) \} dt$$

$$+ \sum_{j=0}^{m} \frac{1}{\Gamma(s)} \int_0^T t^{\Re s-1} e^{-t} \{ |g(e^{-t}) - 1| \cdot |g(e^{-t})|^j + O(\rho_0^{j-m}) \} \, dt$$

$$\leq \frac{1}{|\Gamma(s)|} \int_0^\infty t^{\Re s-1} e^{-t} M(1 - |g(e^{-t})|^{m+1}) \, dt$$

$$+ \frac{1}{|\Gamma(s)|} \int_T^\infty t^{\Re s-1} e^{-t} O\left(\frac{1 - \theta^{m+1}}{\rho_0^m (1 - \theta)} \right) dt$$

$$+ \frac{1}{|\Gamma(s)|} \int_0^T t^{\Re s-1} e^{-t} O\left(\frac{\rho_0(1 - \rho_0^{-m-1})}{\rho_0 - 1} \right) dt$$

$$\leq \frac{1}{|\Gamma(s)|} \int_0^\infty t^{\Re s-1} e^{-t} O(1) \, dt = O(1).$$

And

$$\sum_{j=m+1}^{\infty} \left| \sum_{n=0}^{m} (A_{nj} - A_{n,j+1}) \right| \leq \sum_{j=m+1}^{\infty} \frac{1}{|\Gamma(s)|} \cdot \int_0^T t^{\Re s-1} e^{-t} O(\rho_0^{m-j}) \, dt$$

$$+ \sum_{j=m+1}^{\infty} \frac{1}{|\Gamma(s)|} \int_T^\infty t^{\Re s-1} e^{-t} \{ |g(e^{-t}) - 1| \cdot |g(e^{-t})|^j + O(\theta^j / \rho_0^m) \} \, dt$$

$$\leq \frac{1}{|\Gamma(s)|} \int_0^T t^{\Re s-1} e^{-t} O\left(\frac{1}{\rho_0 - 1} \right) dt$$

$$+ \frac{1}{|\Gamma(s)|} \int_T^\infty t^{\Re s-1} e^{-t} \left\{ 2M \frac{\lambda^{m+1}}{1 - \lambda} + O\left(\frac{\theta^{m+1}}{\rho_0^m (1 - \theta)} \right) \right\} dt,$$

where $\lambda = \max_{t \geq T} |g(e^{-t})| < 1$. Combining these results, we obtain the estimate

$$\sum_{j=0}^{\infty} \left| \sum_{n=0}^{m} (A_{nj} - A_{n,j+1}) \right| = O(1),$$

where the constant implicit in $O(1)$ is independent of m. This completes the proof of (A).

Proof of (B). Here it suffices to show that

$$\sup_j \sum_n |A_{nj}| < \infty.$$

Let

$$\alpha_{nj} = \frac{1}{2\pi i}\int_C \frac{[g(f(z)e^{-t})]^j}{z^{n+1}}\,dz,$$

where C is a circle of radius ρ ($\rho_0^{-1} \leq \rho \leq \rho_0$). Then if $t \geq T$, let $\rho = \rho_0$ so that by part (d) of Lemma 2,

$$\alpha_{nj}(t) = O(\theta^j/\rho_0^n) \qquad (n, j = 0, 1, 2, \cdots).$$

If $0 \leq t \leq T$, then by part (b) of Lemma 2 and by letting $\rho = 1$,

$$\alpha_{nj}(t) = \frac{1}{2\pi i}\int_{|z|=1} z^{j-n-1}\exp[jt\phi(z, t)]\,dz$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{i(j-n)\theta}\exp[jt\phi(e^{i\theta}, t)]\,d\theta.$$

If $j = n$, then by part (c) of Lemma 2,

$$\alpha_{jj}(t) = \frac{1}{2\pi}\int_0^{2\pi}\exp[jt\phi(e^{i\theta}, t)]\,d\theta = O(e^{-j\Psi t}).$$

If $j \neq n$, then integrating twice by parts, we find that

$$\alpha_{nj}(t) = \frac{1}{2\pi(j-n)^2}\int_0^{2\pi}\exp[i(j-n)\theta + jt\phi]\left\{jt\frac{\partial^2\phi}{\partial\theta^2} + (jt)^2\left(\frac{\partial\phi}{\partial\theta}\right)^2\right\}d\theta$$

$$= O\left(\frac{e^{-jt\Psi}}{(j-n)^2}(jt + (jt)^2)\right) \qquad (0 \leq t \leq T).$$

Since

$$A_{nj} = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-t}\alpha_{nj}(t)\,dt,$$

we therefore obtain the result

$$\sum_n |A_{nj}| \leq \sum_{n\neq j}\frac{1}{|\Gamma(s)|}\int_0^T t^{\Re s-1}e^{-t}O\left[e^{-jt\Psi}\frac{(jt+(jt)^2)}{(j-n)^2}\right]dt$$

$$+ \frac{1}{|\Gamma(s)|}\int_0^T t^{\Re s-1}e^{-t}O(e^{-jt\Psi})\,dt$$

$$+ \sum_n \frac{1}{|\Gamma(s)|}\int_T^\infty t^{\Re s-1}e^{-t}O(\theta^j/\rho_0^n)\,dt.$$

This implies that

$$\sum_n |A_{nj}| \leq \sum_{n \neq j} O\left(\frac{1}{(j - n)^2 (j + 1)^{\Re s}}\right) + O\left(\frac{1}{(j + 1)^{\Re s}}\right) + O(\theta^j). \tag{6}$$

Finally, we see that for $\Re s > 0$,

$$\sup_j \sum_n |A_{nj}| < \infty .$$

This completes the proof of (B).

Proof of (C). Here it suffices to show that

$$\sum_n \sum_j |A_{nj}| < \infty .$$

But from (6) it follows that this is true if $\Re s > 1$.

REFERENCES

1. V. F. Cowling and G. Piranian, *On the summability of ordinary Dirichlet series by Taylor methods,* Michigan Math. J. 1 (1952), 73-78.

2. K. Knopp, *Über Polynomentwicklungen im Mittag-Lefflerschen Stern durch Anwendung der Eulerschen Reihentransformation,* Acta Math. 47 (1926), 313-335.

3. M. S. Macphail, *On Perron's extension of the Euler-Knopp summation method,* Trans. Roy. Soc. Canada. Sect. III. (3) 42 (1948), 43-49.

4. ———, *The extended Euler-Knopp transformation,* Trans. Roy. Soc. Canada. Sect. III. (3) 46 (1952), 39-43.

5. N. Obrechkoff, *Sommation par le Transformation d'Euler. Les séries de Dirichlet, les séries de facultés et la série de Newton,* Ann. Univ. Sofia II. Fac. Phys. Math., Livre 1, 35 (1939), 1-156.

6. O. Perron, *Über eine Verallgemeinerung der Eulerschen Reihentransformationen,* Math. Z. 18 (1923), 157-172.

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