COMPARISON OF SINGULAR AND ČECH HOMOLOGY IN LOCALLY CONNECTED SPACES

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The main result of this paper is Theorem 1 (see Section 2). It shows that a natural homomorphism ν_* (ν^*) of the singular homology (Čech cohomology) theory into the Čech homology (singular cohomology) theory becomes an isomorphism in dimensions $0 \leq q \leq p + 1$, provided that we restrict ourselves to the category of paracompact Hausdorff spaces which are lc_S^p and semi-$(p + 1)$-lc_S (for these notations see Section 2). The proof is carried out in Sections 1 to 5.

In the case of triangulable spaces, the equivalence of singular and Čech theory is a well-known fact (a consequence of the uniqueness theorem of S. Eilenberg and N. Steenrod [4]). The same fact has been established more recently for ANR-s (see J. Dugundji [3], Y. Kodama [9] and S. Mardešić [11]). For metrizable compacta which are homotopy locally connected, the equivalence has been established by S. Lefschetz ([10], p. 107). A proposition, closely related to the part of Theorem 1 which is concerned with cohomology, can be derived from Cartan's uniqueness theorem for cohomology with coefficients in sheaves [1]; this approach is not applicable to homology. Finally, Theorem 1 generalizes a result obtained by H. B. Griffiths in [6]. (In an unpublished paper, Griffiths has recently developed a general theory of "locally trivial homology," and Theorem 1 is there derived in the framework of that theory.)

Section 7 contains a proof of an analogue of Theorem 1, for Čech homology with compact carriers. In Section 8, we show that, for locally paracompact spaces, $lc_S^p \Rightarrow lc_C^p$ (for notation see Section 2). In Section 9, an application of Theorem 1 gives a criterion for unicoherence of locally arcwise connected semi-1-lc_S paracompact Hausdorff spaces in terms of the first singular homology group.

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1. NATURAL HOMOMORPHISMS ν_* AND ν^*

Let (X, A) be a pair of topological spaces, $A \subset X$ (A need not be closed). $H_q(X, A; G; S)$ and $H^q(X, A; G; S)$ will denote the q-th singular homology group and the q-th singular cohomology group, both taken with coefficients in a (discrete) group G. The corresponding Čech groups will be distinguished by a letter C replacing S. For purposes of this paper, we adopt a definition of Čech homology and cohomology which naturally generalizes the classical notions of the Vietoris theory, and which was introduced by E. Spanier in [13] and by C. H. Dowker in [2]. The definition gives groups which are naturally isomorphic to usual Čech groups for arbitrary pairs (X, A) (for a proof see [2]). Here is the Spanier-Dowker definition.

Let $\alpha = (\alpha_1, \alpha_2)$ be an open covering of (X, A). This means that α_1 is an open covering of X, that $\alpha_2 \subset \alpha_1$ and that the union of all the members of α_2 is a subset of X which contains A. Let $K_{\alpha_1}(X)$ ($K_{\alpha_2}(A)$) be the simplicial complex whose vertices are all the points of X (of A); a finite set of vertices forms a simplex of

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$K_{\alpha_1}(X)$ (of $K_{\alpha_2}(A)$) if and only if it is contained in a $U \in \alpha_1$ ($U \in \alpha_2$). Clearly, $K_\alpha(X, A) = K_{\alpha_1}(X)/K_{\alpha_2}(A)$ is a chain complex, K_{α_2} being a subcomplex of K_{α_1}. ($K_{\alpha_1}(X)$ and $K_{\alpha_2}(A)$ denote also the corresponding (ordered) chain complexes.) $\beta = (\beta_1, \beta_2)$ is said to refine $\alpha = (\alpha_1, \alpha_2)$ (in symbols: $\alpha < \beta$) if one can associate with each $V \in \beta_1$ some $U(V) \in \alpha_1$ in such a way that $U(V) \supset V$ and that $V \in \beta_2$ implies $U(V) \in \alpha_2$. If $\alpha < \beta$, let $\pi_{\beta\alpha}: K_\beta \to K_\alpha$ be the chain mapping induced by the identity map of vertices. $\pi_{*\beta\alpha}$ and $\pi^*{}_{\beta\alpha}$ will denote the induced homomorphisms of the corresponding homology and cohomology groups. The set Ω of all open coverings of (X, A), ordered by $<$, is directed. Homology and cohomology groups of K_α ($\alpha \in \Omega$) form thus an inverse and a direct system of groups; the corresponding limits are, by definition, the Čech groups of (X, A). It is clear how to define the homomorphisms f_* and f^*, induced by a mapping $f: (X, A) \to (Y, B)$, and the boundary homomorphism ∂_* and δ^*.

Let $S(X)$ denote the chain complex of singular chains of X. $S(A)$ is a subcomplex of $S(X)$, and $S(X, A) = S(X)/S(A)$ is the singular chain complex of the pair (X, A). If s is a singular simplex, we denote by $||s||$ its carrier, that is, the image of the unit simplex under the mapping s. Let $S_{\alpha_1}(X)$ denote the subcomplex of $S(X)$ generated by singular simplices s which belong to the cover α_1, that is, for which one can find a $U \in \alpha_1$ containing $||s||$. Let $\eta_{\alpha_1}: S_{\alpha_1}(X) \to S(X)$ be the natural injection. Similarly, we have $\eta_{\alpha_2}: S_{\alpha_2}(A) \to S(A)$. If $S_\alpha(X, A) = S_{\alpha_1}(X)/S_{\alpha_2}(A)$, we define $\eta_\alpha: S_\alpha(X, A) \to S(X, A)$ in the natural way. According to a result in [4] (Theorem 8.2, p. 197), η_α is a natural equivalence, that is, there exists a chain mapping $\varepsilon_\alpha: S(X, A) \to S_\alpha(X, A)$ such that

$$\varepsilon_\alpha \eta_\alpha \simeq \eta_\alpha \varepsilon_\alpha \simeq 1, \tag{1}$$

$\simeq$ denoting chain homotopy and 1 denoting the identity mapping. (For the case of absolute homology, an explicit ε_α is given in [11].) If $\alpha < \beta$, let

$$\eta_{\beta\alpha}: S_\beta(X, A) \to S_\alpha(X, A)$$

denote the chain mapping induced by inclusions $S_{\beta_1}(X) \subset S_{\alpha_1}(X)$, $S_{\beta_2}(A) \subset S_{\alpha_2}(A)$. Clearly, $\eta_\alpha \eta_{\beta\alpha} = \eta_\beta$ and thus

$$\varepsilon_\alpha \simeq \eta_{\beta\alpha} \varepsilon_\beta. \tag{2}$$

We define now a chain mapping $\mu_{\alpha_1}: S_{\alpha_1}(X) \to K_{\alpha_1}(X)$ by assigning to every (ordered) singular simplex s of $S_{\alpha_1}(X)$ the array of its vertices. Since μ_{α_1} maps $S_{\alpha_2}(A)$ into $K_{\alpha_2}(A)$, it induces a chain mapping $\mu_\alpha: S_\alpha(X, A) \to K_\alpha(X, A)$. If $\alpha < \beta$, then obviously $\mu_\alpha \eta_{\beta\alpha} = \pi_{\beta\alpha} \mu_\beta$.

Consider now the composite chain mapping

$$\nu_\alpha = \mu_\alpha \varepsilon_\alpha, \tag{3}$$

$\nu_\alpha: S(X, A) \to K_\alpha(X, A)$. According to (2), we have

$$\nu_\alpha \simeq \pi_{\beta\alpha} \nu_\beta. \tag{4}$$

This shows that mappings ν_α give rise (by a standard procedure) to a homomorphism

$$\nu_*\colon H_q(X, A; G; S) \to H_q(X, A; G; C) = \varprojlim H_q(K_\alpha(X, A); G)$$

and a homomorphism

$$\nu^*\colon H^q(X, A; G; C) = \varinjlim H^q(K_\alpha(X, A); G) \to H^q(X, A; G; S).$$

It is easily verified that ν_* and ν^* commute with f_*, ∂_* and f^*, δ^* respectively. ν_* and ν^* are therefore natural homomorphisms of the corresponding homology theories over the category of arbitrary pairs of spaces.

2. BASIC DEFINITIONS CONCERNING COVERINGS

Let (M, N) be a pair of spaces, $N \subset M$. We denote by $H_q(N|M; G; S)$ the image of $H_q(N; G; S)$ in $H_q(M; G; S)$ under the homomorphism induced by the inclusion $N \subset M$. The analogous meaning is given to $H_q(N|M; G; C)$. Let Z denote the integers, as usual. A space X is said to be $q\text{-lc}_s$ at a point $x \in X$ if, for every open $U \subset X$ containing x, there is an open V $(x \in V \subset U)$ such that $H_q(V|U; Z; S) = 0$. X is said to be lc^p_s at $x \in X$ if it is $q\text{-lc}_s$ at x, for $0 \le q \le p$. X is semi-q-lc_s at x if there is an open V $(x \in V)$ such that $H_q(V|X; Z; S) = 0$. X is said to be lc^p_s or semi-p-lc_s if it has these properties at every $x \in X$. If $q = 0$, we have to use augmented homology. Replacing singular homology by Čech homology we obtain the notions lc^p_c, and so forth.

We can now state the main result.

THEOREM 1. *Let* (X, A) *be a pair of paracompact Hausdorff spaces. If both* X *and* A *are* lc^p_s *and semi-*$(p+1)$-lc_s, *then the homomorphisms*

$$\nu_*\colon H_q(X, A; G; S) \to H_q(X, A; G; C) \quad \textit{and} \quad \nu^*\colon H^q(X, A; G; C) \to H^q(X, A; G; S)$$

are isomorphisms (onto), *for all* $q \le p + 1$.

(In the first draft of this paper, Theorem 1 was proved only under the hypothesis that X and A are lc^{p+1}_s. The author is indebted to Professor R. L. Wilder for suggesting the weaker conditions which appear in the present form of the theorem.)

The proof is given in the following sections. We first introduce some definitions concerning coverings.

If α_1 is a covering of X and $V \in \alpha_1$, we shall denote by $\text{St}_{\alpha_1}(V)$ the star of V with respect to α_1, that is, the union of all $V_1 \in \alpha_1$ with $V \cap V_1 \neq \emptyset$. If $\alpha = (\alpha_1, \alpha_2)$ is a covering of (X, A), we shall denote by $\alpha_2 \cap A$ the covering of A consisting of sets $V_2 \cap A$ $(V_2 \in \alpha_2)$.

DEFINITION 1. A covering $\beta = (\beta_1, \beta_2)$ of (X, A) is said to be a p-*refinement of the covering* $\alpha = (\alpha_1, \alpha_2)$ (in symbols: $\alpha <_p \beta$) if the following conditions are satisfied:

(i) For each $V_1 \in \beta_1$ there is a $U_1 \in \alpha_1$ such that

$$\text{St}_{\beta_1}(V_1) \subset U_1 \quad \text{and} \quad H_q((\text{St}_{\beta_1}(V_1))|U_1; Z; S) = 0,$$

for all $q \le p$.

(ii) For each $V_2 \in \beta_2$ there is a $U_2 \in \alpha_2$ such that

$$\mathrm{St}_{\beta_2 \cap A}(V_2 \cap A) \subset U_2 \cap A \quad \text{and} \quad H_q((\mathrm{St}_{\beta_2 \cap A}(V_2 \cap A))|(U_2 \cap A); Z; S) = 0,$$

for all $q \leq p$.

LEMMA 1. *If* (X, A) *is a pair of paracompact Hausdorff spaces which are both* lc_s^p, *then every open covering* α *of* (X, A) *admits a* p-*refinement* β.

(A covering β_1 is said to star-refine α_1 (in symbols: $\alpha_1 <_* \beta_1$) if for each $V_1 \in \beta_1$ there is a $U_1 \in \alpha_1$ such that $\mathrm{St}_{\beta_1}(V_1) \subset U_1$. E. E. Floyd has introduced in [5] a notion of p-refinement (for closed coverings and Čech homology) which differs from our concept. In fact, denoting by $\alpha_1 <^p \beta_1$ the statement that β_1 p-refines α_1 in the sense of Floyd, we can say that $\alpha_1 <_p \beta_1$ means essentially the existence of a γ_1 such that $\alpha_1 <^p \gamma_1 <_* \beta_1$. Note that Floyd's notion of strong p-refinement corresponds essentially to $\alpha_1 <_* \gamma_1 <^p \beta_1$ (see (2.2) [5]). Compare also our Lemma 1 with Floyd's (3.2).)

Proof. For each $x \in A$, choose a $U_2(x) \in \alpha_2$ containing x, and choose an open set $V_2(x)$ of X such that

$$x \in V_2(x) \subset U_2(x) \quad \text{and} \quad H_q((V_2(x) \cap A)| (U_2(x) \cap A); Z; S) = 0,$$

for all $q \leq p$. This choice is possible, because A is lc_s^p. Let γ_2' be an open star-refinement of the covering $\{V_2(x) \cap A|\ x \in A\}$ of A (A is a paracompact Hausdorff space). For each element $W_2' \in \gamma_2'$ choose an open set $W_2 = W_2(W_2')$ of X in such a way that $W_2 \cap A = W_2'$. Let $\gamma_2 = \{W_2(W_2')|\ W_2' \in \gamma_2'\}$.

On the other hand, consider for each $x \in X$ a $U_1(x) \in \alpha_1$ containing x, and choose an open set $V_1(x)$ such that $x \in V_1(x) \subset U_1(x)$ and $H_q(V_1(x)| U_1(x); Z; S) = 0$, for all $q \leq p$. Let $\gamma_1 = \{W_1\}$ be an open star-refinement of the covering $\{V_1(x)|\ x \in X\}$.

We now define $\beta = (\beta_1, \beta_2)$ by

$$\beta_2 = \{W_1 \cap W_2|\ W_1 \in \gamma_1,\ W_2 \in \gamma_2\}, \tag{5}$$

$$\beta_1 = \gamma_1 \cup \beta_2. \tag{6}$$

It is easily verified that $\alpha <_p \beta$ (note that β_2 refines both γ_1 and γ_2).

DEFINITION 2. An open covering $\alpha = (\alpha_1, \alpha_2)$ of (X, A) is said to be a p-*covering* if, for each $V_1 \in \alpha_1$, $H_p(V_1|X; Z; S) = 0$ and if, for each $V_2 \in \alpha_2$,

$$H_p((V_2 \cap A)|A; Z; S) = 0.$$

LEMMA 2. *If both* X *and* A *are semi*-p-lc_s, *then a* p-*covering of* (X, A) *exists.*

Proof. Choose an open covering β_1 of X such that $V_1 \in \beta_1$ implies

$$H_p(V_1|X; Z; S) = 0,$$

and choose an open covering β_2' of A such that $V_2' \in \beta_2'$ implies $H_p(V_2'|A; Z; S) = 0$. Furthermore, choose for each $V_2' \in \beta_2'$ an open set $V_2(V_2')$ of X such that $V_2 \cap A = V_2'$. Let $\beta_2 = \{V_2(V_2')|\ V_2' \in \beta_2'\}$. Now define $\alpha = (\alpha_1, \alpha_2)$ by

$$\alpha_2 = \{V_1 \cap V_2|\ V_1 \in \beta_1,\ V_2 \in \beta_2\}, \tag{7}$$

(8) $$\alpha_1 = \beta_1 \cup \alpha_2 .$$

It is easily verified that α is a p-covering of (X, A).

DEFINITION 3. A sequence $(\alpha^i) = \alpha^0, \alpha^1, \cdots, \alpha^{p+1}$ $(p \geq 0)$, consisting of $p + 2$ open coverings $\alpha^i = (\alpha^i_1, \alpha^i_2)$ of (X, A), is said to be a p-*sequence* if α^{i+1} is a p-refinement of α^i, for $0 \leq i \leq p$. The coverings $\alpha = \alpha^0$ and $\beta = \alpha^{p+1}$ will be referred to as *end-terms of the sequence*. A p-sequence (α'^i) is said to *refine* the p-sequence (α^i) (in symbols: $(\alpha^i) < (\alpha'^i)$) if $\alpha^i < \alpha'^i$ for all $0 \leq i \leq p$.

LEMMA 3. *Let* (X, A) *be a pair of paracompact Hausdorff spaces which are both* lc^p_s. *Then the set of all* p-*sequences, ordered by* $(\alpha^i) < (\alpha'^i)$, *is a nonempty directed set.*

Proof. Choose an open covering α''^0 which refines all the coverings $\alpha^0, \cdots, \alpha^{p+1}, \alpha'^0, \cdots, \alpha'^{p+1}$. Apply Lemma 1 $(p + 1)$ times to obtain a p-sequence (α''^i) with the first term α''^0. Clearly, (α''^i) refines both (α^i) and (α'^i).

DEFINITION 4. Let (α^i) be a p-sequence of coverings for (X, A), $\alpha = \alpha^0$, $\beta = \alpha^{p+1}$. A chain mapping $\lambda_{\beta\alpha}: K^{p+1}_\beta(X, A) \to S_\alpha(X, A)$ (K^n denotes the nth skeleton of the complex K) is said to be a *projection belonging to the* p-*sequence* (α^i) if there is a chain mapping $\lambda_{\beta_1\alpha_1} = \lambda: K^{p+1}_{\beta_1}(X) \to S_{\alpha_1}(X)$ such that the following four conditions are satisfied.

(i) For each 0-simplex v of $K^{p+1}_{\beta_1}(X)$ we have $\lambda(v) = v$.

(ii) For each q-simplex v of $K^{p+1}_{\beta_1}(X)$ $(0 \leq q \leq p + 1)$ there is a $U_1 \in \alpha^{p+1-q}_1$ such that $(U_1 \supset \|v\| \cup \|\lambda v\|$. (A simplex v of $K_{\beta_1}(X)$ is a finite set of points of X; when referring to that set, we use the notation $\|v\|$, to distinguish it from the case where v is considered as a chain. If $x = \Sigma a_i s_i$ is a singular chain, then $\|x\|$ denotes the union of all $\|s_i\|$ with a coefficient $a_i \neq 0$.)

(iii) For each q-simplex v of $K^{p+1}_{\beta_2}(A)$ $(0 \leq q \leq p + 1)$ there is a $U_2 \in \alpha^{p+1-q}_2$ such that $(U_2 \cap A) \supset \|v\| \cup \|\lambda v\|$.

(iv) Since $\lambda(K^{p+1}_{\beta_2}(A)) \subset S_{\beta_2}(A)$ (by (iii)), λ induces a chain mapping of $K^{p+1}_\beta(X, A)$ into $S_\alpha(X, A)$. This mapping coincides with $\lambda_{\beta\alpha}$.

3. PROPERTIES OF PROJECTIONS $\lambda_{\beta\alpha}$

Let L be a subcomplex of $K^{p+1}_{\beta_1}(X)$, and let λ be a chain mapping of L into $S_{\alpha_1}(X)$. When we say that λ satisfies (i), (ii) and (iii), we mean the corresponding conditions with v being a simplex of L and of $L \cap K^{p+1}_{\beta_2}(A)$, respectively.

LEMMA 4. *Let* (α^i) *be a* p-*sequence with end-terms* α *and* β; *let* L *be a subcomplex of* $K^{p+1}_{\beta_1}(X)$, *and* λ *a chain mapping,* $\lambda: L \to S_{\alpha_1}(X)$. *If* λ *satisfies* (i), (ii) *and* (iii), *then it can be extended to a chain mapping* $\lambda: K^{p+1}_{\beta_1}(X) \to S_{\alpha_1}(X)$ *satisfying* (i), (ii) *and* (iii).

We shall prove inductively that λ can be extended to a map of $L \cup K^q_{\beta_1}(X)$ into

$S_{\alpha_1}(X)$ which satisfies (i), (ii) and (iii) $(q = 0, 1, \cdots, p+1)$. For $q = 0$, the extension is achieved by setting $\lambda(v) = v$, for all 0-simplexes of $K^0_{\beta_1}(X)$. Case $q = 1$: Let v be a 1-simplex of $K^1_{\beta_1}(X) \setminus (L \cup K^1_{\beta_2}(A))$. Choose a $V \in \beta_1$ containing $||v||$ and a $U \in \alpha^p_1$ such that $V \subset U$ and $H_1(V|U; Z; S) = 0$ (recall that $\alpha^p_1 <_p \beta_1$). Since $\lambda\partial v = \partial v$ is a singular 0-cycle (with respect to augmented homology), there exists a singular 1-chain λv in U, such that $\partial(\lambda v) = \partial v$. If v is a 1-simplex of $K^{p+1}_{\beta_2}(A) \setminus L$, then we choose V from β_2, U from α^p_2 and λv from $A \cap U$.

Assume now that λ has already been extended to $L \cup K^q_{\beta_1}(X)$ so as to satisfy all our requirements, and that $1 \leq q < p+1$. Let v be a $(q+1)$-simplex of $K^{q+1}_{\beta_1}(X) \setminus (L \cup K^{q+1}_{\beta_2}(A))$, and let $\partial_i v$ be the q-faces of v, so that $\partial v = \Sigma(-1)^i \partial_i v$. Since $\partial_i v$ belongs to $K^q_{\beta_1}(X)$, there exists a $U_i \in \alpha^{p+1-q}_1$ with $U_i \supset ||\partial_i v|| \cup ||\lambda\partial_i v||$ $(0 \leq i \leq q+1)$ (see (ii)). Since $q + 1 \geq 2$, it follows that, for each $i > 0$,

$$U_0 \cap U_i \supset ||\partial_0 v|| \cap ||\partial_i v|| = ||\partial_0 \partial_i v|| \neq \emptyset .$$

This implies that all U_i $(i \geq 0)$ belong to $\mathrm{St}_{\alpha_1^{(p+1-q)}} U_0$. Hence, there exists a $U \in \alpha^{p-q}_1$ such that

$$U \supset \mathrm{St}_{\alpha_1^{(p+1-q)}} U_0 \supset U_i \supset ||\partial_i v|| \quad (0 \leq i \leq q+1)$$

and such that

$$H_q((\mathrm{St}_{\alpha_1^{(p+1-q)}} U_0)|U; Z; S) = 0 .$$

Since $\lambda\partial v$ is a singular cycle of $\mathrm{St}_{\alpha_1^{(p+1-q)}} U_0$, we can choose a $(q+1)$-chain λv such that $\partial\lambda v = \lambda\partial v$ and $||\lambda v|| \subset U$. Clearly, $||v|| = \bigcup ||\partial_i v||$; hence, $||\partial_i v|| \subset U$ and $||\lambda v|| \subset U$ imply (ii). In the case when v is a simplex of $K^{q+1}_{\beta_2}(A) \setminus L$, one can select U in such a way as to obtain (iii).

Taking for L the 0-skeleton $K^0_{\beta_1}(X)$ and defining $\lambda: L \to S_{\alpha_1}(X)$ by $\lambda(v) = v$ (v any 0-simplex), we obtain from Lemma 4

COROLLARY 1. *The set of projections belonging to a* p-*sequence is not empty.*

LEMMA 5. *Let* (α^i) *and* (α'^i) *be two* p-*sequences with end-terms* α, β *and* α', β', *respectively, and let* $(\alpha^i) < (\alpha'^i)$. *For each projection* $\lambda_{\beta'\alpha'}$ *belonging to* (α'^i), *there exists a projection* $\lambda_{\beta\alpha}$ *belonging to* (α^i) *and satisfying*

$$\eta_{\alpha'\alpha}\lambda_{\beta'\alpha'} = \lambda_{\beta\alpha}\pi_{\beta'\beta} . \tag{9}$$

Proof. Choose a chain mapping $\lambda': K^{p+1}_{\beta'_1}(X) \to S_{\alpha'_1}(X)$ in accordance with Definition 4. It suffices to find a mapping $\lambda: K^{p+1}_{\beta_1}(X) \to S_{\alpha_1}(X)$ satisfying conditions (i),

(ii) and (iii) and, in addition,

$$\eta_{\alpha_1'\alpha_1}\lambda' = \lambda\pi_{\beta_1'\beta_1}. \tag{10}$$

$L = K^{p+1}_{\beta_1'}(X)$ is a subcomplex of $K^{p+1}_{\beta_1}(X)$, because $\beta_1 < \beta_1'$. It is readily verified that $\eta_{\alpha_1'\alpha_1}\lambda'$ is a chain mapping of L into $S_{\alpha_1}(X)$ which satisfies (i), (ii) and (iii) (recall $\alpha_1^{p+1-q} < \alpha_1'^{p+1-q}$ and $\alpha_2^{p+1-q} < \alpha_2'^{p+1-q}$). Applying Lemma 4, we obtain an extension λ of $\eta_{\alpha_1'\alpha_1}\lambda'$ to $K^{p+1}_{\beta_1}(X)$. It is clear that λ satisfies (10) (recall that $\pi_{\beta_1'\beta_1}$ is an injection).

LEMMA 6. *Let* $\gamma = (\gamma_1, \gamma_2)$ *be an open* $(p + 1)$-*covering of* (X, A) (see Def. 2), *let* α *be a star-refinement of* γ, *and* $(\alpha^i) = \alpha, \alpha^1, \cdots, \alpha^p, \beta$ *a* p-*sequence of coverings. Then, for any two projections* $\lambda_{\beta\alpha}$ *and* $\lambda'_{\beta\alpha}$ *belonging to* (α^i), *we have a homotopy in* $S(X, A)$:

$$\eta_\alpha\lambda_{\beta\alpha} \simeq \eta_\alpha\lambda'_{\beta\alpha}. \tag{11}$$

Proof. The first step consists in defining a chain homotopy D of the p-skeleton $K^p_{\beta_1}(X)$ into $S_{\alpha_1}(X)$ in such a way that

$$\partial D + D\partial = \lambda - \lambda' \tag{12}$$

($\lambda\colon K^{p+1}_{\beta_1}(X) \to S_{\alpha_1}(X)$ and λ' are as in Def. 4). Furthermore, for every q-simplex v of $K^p_{\beta_1}(X)$ there has to be a $U_1 \in \alpha_1^{p-q}$ such that

$$U_1 \supset \|v\| \cup \|Dv\|, \tag{13}$$

and for every q-simplex v of $K^p_{\beta_2}(A)$ there has to be a $U_2 \in \alpha_2^{p-q}$ such that

$$(A \cap U_2) \supset \|v\| \cup \|Dv\|. \tag{14}$$

For 0-simplices v we set $D(v) = 0$. Let now v be a 1-simplex of $K^1_{\beta_1}(X) \setminus K^1_{\beta_2}(A)$. Choose V and V' from α_1^p in accordance with (ii), so that $(V \cap V') \supset \|v\| \neq \emptyset$ and thus $(\mathrm{St}_{\alpha_1^p} V) \supset (V \cup V')$. Now choose $U \in \alpha_1^{p-1}$ so that

$$\mathrm{St}_{\alpha_1^p} V \subset U \qquad \text{and} \qquad H_1((\mathrm{St}_{\alpha_1^p} V) \,|\, U; Z; S) = 0.$$

Since $\lambda v - \lambda' v$ is a singular 1-cycle of $\mathrm{St}_{\alpha_1^p} V$, there exists a singular 2-chain Dv in U, such that $\partial Dv = \lambda v - \lambda' v$. If v belongs to $K^1_{\beta_2}(A)$, we can choose V and V' from α_2^p, U from α_2^{p-1} and Dv from $A \cap U$.

Assume now that D has already been defined on $K^q_{\beta_1}(X)$ in accordance with our requirements $(1 \leq q < p)$. Let v be a $(q + 1)$-simplex of $K^{p+1}_{\beta_1}(X) \setminus K^{p+1}_{\beta_2}(A)$. Choose

V and V' from α_1^{p-q} (by (ii)) such that $V \supset ||v|| \cup ||\lambda v||$ and $V' \supset ||v|| \cup ||\lambda' v||$. Furthermore, choose $U_i \in \alpha_1^{p-q}$ such that $U_i \supset ||\partial_i v|| \cup ||D\partial_i v||$ (use the induction hypothesis). Since $q + 1 \geq 2$, we have $(U_i \cap U_0) \supset ||\partial_0 \partial_i v|| \neq \emptyset$ for $i > 0$. Also,

$$(V \cap U_0) \supset ||\partial_0 v|| \neq \emptyset \quad \text{and} \quad (V' \cap U_0) \supset ||\partial_0 v|| \neq \emptyset .$$

We conclude thus that all U_i and $V \cup V'$ lie in $\mathrm{St}_{\alpha_1^{p-q}} U_0$. Notice that this set contains also the singular $(q + 1)$-chain $\lambda v - \lambda' v - D\partial v$; this is actually a cycle, according to (12). Now choose a $U \in \alpha_1^{p-q-1}$ such that

$$\mathrm{St}_{\alpha_1^{p-q}} U_0 \subset U \quad \text{and} \quad H_{q+1}((\mathrm{St}_{\alpha_1^{p-q}} U_0) \,|U; Z; S) = 0 .$$

Clearly, we can now choose a $(q + 2)$-chain Dv from U such that $\partial Dv = \lambda v - \lambda' v - D\partial v$. If v belongs to $K_{\beta_2}^{p+1}(A)$, we can make our choices in such a way as to satisfy (14).

To complete the proof we extend D to $K_{\beta_1}^{p+1}(X)$ by an analogous procedure. In the case $p = 0$ and for v from $K_{\beta_1}^1(X) \setminus K_{\beta_2}^1(A)$, we choose $U \in \gamma_1$ such that $U \supset \mathrm{St}_{\alpha_1}(V)$ (note that α_1 is a star-refinement of γ_1). $\eta_{\alpha_1 \gamma_1} \lambda v - \eta_{\alpha_1 \gamma_1} \lambda' v$ is a cycle of U. Using the fact that γ is a 1-covering, we can find a singular chain Dv in X such that $\partial Dv = \eta_{\alpha_1} \lambda v - \eta_{\alpha_1} \lambda' v$. If v belongs to $K_{\beta_2}^1(A)$, we apply obvious modifications. Now suppose that $p > 0$, and let v be a $(p + 1)$-simplex of $K_{\beta_1}^{p+1}(X) \setminus K_{\beta_2}^{p+1}(A)$. The same argument as in the case $\dim v \leq p$ shows that $\lambda v - \lambda' v - D\partial v$ is a cycle of $\mathrm{St}_{\alpha_1}(U_0)$. Choose now $U \in \gamma_1$ such that $U \supset \mathrm{St}_{\alpha_1}(U_0)$. Since γ is a $(p + 1)$-covering of (X, A), one can find a singular chain Dv in X such that $\partial Dv = \eta_{\alpha_1} \lambda v - \eta_{\alpha_1} \lambda' v - D\partial v$. One proceeds in a similar way for $(p + 1)$-simplexes v of $K_{\beta_2}^{p+1}(A)$, obtaining

$$D(K_{\beta_2}^{p+1}(A)) \subset S(A) .$$

Hence, D induces a homotopy of $K_\beta^{p+1}(X, A)$ into $S(X, A)$ connecting $\eta_\alpha \lambda_{\beta\alpha}$ and $\eta_\alpha \lambda'_{\beta\alpha}$.

LEMMA 7. *Let* (X, A) *be a pair of paracompact Hausdorff spaces which are both* lc_s^p, *and let* γ *be a* $(p + 1)$-*covering* $(p \geq 0)$. *Furthermore, let* (α^i) *and* (α'^i) *be any two* p-*sequences with end-terms* α, β *and* α', β', *respectively, and let* α *and* α' *be star-refinements of* γ. *Then there exists a common refinement* τ *of* β *and* β' *such that*

$$\eta_\alpha \lambda_{\beta\alpha} \pi_{\tau\beta} \simeq \eta_{\alpha'} \lambda_{\beta'\alpha'} \pi_{\tau\beta'} , \tag{15}$$

for any pair of projections $\lambda_{\beta\alpha}$ *and* $\lambda_{\beta'\alpha'}$ *belonging to* (α^i) *and* (α'^i), *respectively.*

Proof. Let (α''^i) be a p-sequence refining both (α^i) and (α'^i) (Lemma 3), and let its end-terms be σ and τ. Choose an arbitrary projection $\lambda_{\tau\sigma}$ belonging to (α''^i) (Corollary 1). According to Lemma 5, there exist projections $\lambda_{\beta\alpha}^0$ and $\lambda_{\beta'\alpha'}^0$

belonging to (α^i) and (α'^i), respectively, and satisfying

$$\eta_{\sigma\alpha}\lambda_{\tau\sigma} = \lambda^0_{\beta\alpha}\pi_{\tau\beta} \tag{16}$$

and

$$\eta_{\sigma\alpha'}\lambda_{\tau\sigma} = \lambda^0_{\beta'\alpha'}\pi_{\tau\beta'}. \tag{17}$$

On the other hand, by Lemma 6,

$$\eta_\alpha\lambda_{\beta\alpha} \simeq \eta_\alpha\lambda^0_{\beta\alpha} \tag{18}$$

and

$$\eta_{\alpha'}\lambda_{\beta'\alpha'} \simeq \eta_{\alpha'}\lambda^0_{\beta'\alpha'}. \tag{19}$$

From these relations we readily infer that both sides in (15) are homotopic to $\eta_\sigma\lambda_{\tau\sigma}$.

4. THE COMPOSITE CHAIN MAPPINGS $\lambda_{\beta\alpha}\mu_\beta$ AND $\mu_\alpha\lambda_{\beta\alpha}$

LEMMA 8. *Let γ be a $(p+1)$-covering of (X, A), let (α^i) be a p-sequence of coverings with end-terms α and β, and let α be a star-refinement of γ. If μ_β and η_β are restricted to $S^{p+1}_\beta(X, A)$, then the homotopy*

$$\eta_\alpha\lambda_{\beta\alpha}\mu_\beta \simeq \eta_\beta \tag{20}$$

holds in $S(X, A)$, for every projection $\lambda_{\beta\alpha}$ belonging to (α^i).

The proof parallels closely the proof of Lemma 6, and details are omitted. The first step consists in defining (by induction) a chain homotopy D of $S^p_{\beta_1}(X)$ into $S_{\alpha_1}(X)$ in such a way that

$$\partial D + D\partial = \lambda\mu_{\beta_1} - \eta_{\beta_1\alpha_1} \tag{21}$$

(λ being as in Def. 4). Furthermore, for each q-simplex s of $S^p_{\beta_1}(X)$, there must exist a $U_1 \in \alpha_1^{p-q}$ such that

$$U_1 \supset ||s|| \cup ||Ds||, \tag{22}$$

and for each 2-simplex s of $S^p_{\beta_2}(A)$ there must exist a $U_2 \in \alpha_2^{p-q}$ such that

$$(A \cap U_2) \supset ||s|| \cup ||Ds||. \tag{23}$$

The second step consists in an extension of D to $S^{p+1}_{\beta_1}(X)$ with values in S(X); D has to send $S^{p+1}_{\beta_2}(A)$ into S(A), so that a passage to the quotients is possible. Observe that, for each singular simplex s, $||\mu_{\beta_1}s|| \subset ||s||$.

LEMMA 9. *Let* (α^i) *be a* p-*sequence with end-terms* α *and* β, *and let* $\lambda_{\beta\alpha}$ *be any projection belonging to* (α^i). *Then*

$$\mu_\alpha \lambda_{\beta\alpha} \simeq \pi_{\beta\alpha}\,. \tag{24}$$

Proof. It suffices to define a homotopy $D\colon K_{\beta_1}^{p+1}(X) \to K_{\alpha_1}^{p+1}(X)$, connecting $\mu_{\alpha_1}\lambda$ and $\pi_{\beta_1\alpha_1}$ and satisfying

$$D(K_{\beta_2}^{p+1}(A)) \subset K_{\alpha_2}^{p+1}(A)\,. \tag{25}$$

For each simplex v of $K_{\beta_1}^{p+1}(X) \setminus K_{\beta_2}^{p+1}(A)$ $(\dim v \le p + 1)$, choose a $U(v) \in \alpha_1$ such that

$$U(v) \supset \|v\| \cup \|\lambda v\|, \tag{26}$$

and for each simplex v of $K_{\beta_2}^{p+1}(A)$ $(\dim v \le p + 1)$, choose a $U(v) \in \alpha_2$ such that

$$A \cap U(v) \supset \|v\| \cup \|\lambda v\|. \tag{27}$$

Furthermore, choose a point x(v) from U(v) and from $A \cap U(v)$, respectively. Define D by

$$Dv = (x(v))(\mu_{\alpha_1}\lambda v - \pi_{\beta_1\alpha_1} v)\,, \tag{28}$$

where the expression on the right side denotes the join of $(\mu_{\alpha_1}\lambda v - \pi_{\beta_1\alpha_1} v)$ and x(v). (If $v = (a_0, \cdots, a_q)$ and $x \in X$, then the join xv is the simplex $(x, a_0, \cdots, a_q)$. If $x \in U$ and $\|v\| \subset U$, then obviously $\|xv\| \subset U$.) We conclude from (26) that

$$\|Dv\| \subset U(v)\,, \tag{29}$$

and from (27) that

$$\|Dv\| \subset U(v) \cap A\,, \tag{30}$$

for $v \in K_{\beta_2}^{p+1}(A)$, so that D satisfies (25). It follows from (28) that D is indeed a homotopy connecting $\mu_{\alpha_1}\lambda$ and $\pi_{\beta_1\alpha_1}$.

5. PROOF OF THEOREM 1

Choose a $(p + 1)$-covering γ of (X, A) (Lemma 2), a star-refinement α of γ ((X, A) is a pair of paracompact Hausdorff spaces) and a p-sequence (α^i) with end-terms α and β (Lemma 1). Let $h_* \in H_q(X, A; G; S)$ $(q \le p + 1)$ be such that $\nu_* h_* = 0$. In other words, for each open covering ω of (X, A),

$$\nu_{*\omega} h_* = \mu_{*\omega}\,\varepsilon_{*\omega}\, h_* = 0\,; \tag{31}$$

in particular $\nu_{*\beta} h_* = 0$. Applying (1), (3) and Lemma 8, we obtain

(32) $$h_* = \eta_{*\beta}\, \varepsilon_{*\beta} h_* = \eta_{*\alpha}\, \lambda_{*\beta\alpha}\, \mu_{*\beta}\, \varepsilon_{*\beta}\, h_* = \eta_{*\alpha}\, \lambda_{*\beta\alpha}\, \nu_{*\beta} h_* = 0,$$

proving that ν_* is a monomorphism in dimensions up to $p + 1$. Using the same facts, we deduce that for every $h^* \in H^q(X, A; G; S)$ $(q \leq p + 1)$,

(33) $$h^* = \varepsilon^*{}_\beta \eta^*{}_\beta h^* = \varepsilon^*{}_\beta \mu^*{}_\beta \lambda^*{}_{\beta\alpha} \eta^*{}_\alpha h^* = \nu^*{}_\beta(\lambda^*{}_{\beta\alpha} \eta^*{}_\alpha h^*),$$

proving that ν^* is an epimorphism.

To prove that ν_* is an epimorphism, take a $(p + 1)$-covering of (X, A), and consider the set P of all p-sequences (α^i) with end-terms α and β which have the property that α star-refines γ. Let Π be the set of all α, where $(\alpha^i) \in P$. The set Π is cofinal in the set Ω of all open coverings of (X, A) (Lemma 1). Let $h_* = \{h_\omega\}$ $(\omega \in \Omega)$ be an arbitrary element of $H_q(X, A; G; C)$ $(q \leq p + 1)$. For every $(\alpha^i) \in P$ define an $h((\alpha^i)) \in H_q(H, A; G; S)$ by

(34) $$h((\alpha^i)) = \eta_{*\alpha}\, \lambda_{*\beta\alpha}\, h_\beta,$$

where $\lambda_{\beta\alpha}$ is any projection belonging to (α^i). According to Lemma 6, $h((\alpha^i))$ is well-defined by (34). If $(\alpha^i) \in P$ and $(\alpha'^i) \in P$, choose a common refinement τ of β and β' in accordance with Lemma 7. Since $h_\beta = \pi_{*\tau\beta} h_\tau$ and $h_{\beta'} = \pi_{*\tau\beta'} h_\tau$ (definition of h as a "thread" in an inverse system), we obtain (Lemma 7)

(35) $$h((\alpha^i)) = \eta_{*\alpha}\, \lambda_{*\beta\alpha}\, \pi_{*\tau\beta} h_\tau = \eta_{*\alpha'}\, \lambda_{*\beta'\alpha'}\, \pi_{*\tau\beta'} h_\tau = h((\alpha'^i)).$$

In other words,

(36) $$h = h((\alpha^i)) \qquad ((\alpha^i) \in P),$$

is a uniquely determined element of $H_q(X, A; G; S)$ (independent of the choice of $(\alpha^i) \in P$). To prove that $\nu_*(h) = h_*$, apply (1) and Lemma 9. Then, for all $\alpha \in \Pi$,

(37) $$\nu_{*\alpha} h = \mu_{*\alpha}\, \varepsilon_{*\alpha}\, \eta_{*\alpha}\, \lambda_{*\beta\alpha}\, h_\beta = \mu_{*\alpha}\, \lambda_{*\beta\alpha}\, h_\beta = \pi_{*\beta\alpha}\, h_\beta = h_\alpha.$$

The conclusion follows now from the fact that Π is cofinal in Ω.

Proof that ν^* is a monomorphism. Let $h^* \in H^q(X, A; G; C)$ $q \leq p + 1$, and $\nu^* h^* = 0$. Let h^α be a representative of h^* in $H^q(K_\alpha(X, A); G)$. By supposition, $\nu^*{}_\alpha h^\alpha = \varepsilon^*{}_\alpha \mu^*{}_\alpha h^\alpha = 0$, so that (1) yields

(38) $$\mu^*{}_\alpha h^\alpha = 0.$$

Let (α^i) be a p-sequence starting with α and ending with β (Lemma 1), and let $\lambda_{\beta\alpha}$ be a projection belonging to (α^i) (Corollary 1). By Lemma 9 and (38), we obtain

(39) $$\pi^*{}_{\beta\alpha} h^\alpha = \lambda^*{}_{\beta\alpha}\, \mu^*{}_\alpha h^\alpha = 0.$$

Since $\pi^*{}_{\beta\alpha} h^\alpha$ is merely another representative of h^*, we conclude that $h^* = 0$. This completes the proof of Theorem 1.

Examining the proof of the monomorphism of ν^*, we find that the supposition that X and A are semi-$(p + 1)$-lc_s is not needed. Therefore, we have

COROLLARY 2. *If* (X, A) *is a pair of paracompact Hausdorff spaces which are both* lc_s^p, *then* $\nu^*\colon H^{p+1}(X, A; G; C) \to H^{p+1}(X, A; G; S)$ *is a monomorphism.*

It is natural to ask the corresponding question for ν_*, which reads as follows.

Let (X, A) be a pair of paracompact Hausdorff spaces which are both lc_s^p. Is $\nu_*\colon H_{p+1}(X, A; G; S) \to H_{p+1}(X, A; G; C)$ necessarily an epimorphism? (This question was raised by E. R. Fadell during a discussion with the author.) In the next section we answer the question negatively by producing a counter-example.

Note that in the case of cohomology, the proof of Theorem 1 is considerably simpler than in the case of homology (most of Section 3 is not needed). Moreover, singular and Čech cohomology are both exact, and therefore the theorem for the case of absolute cohomology would imply the result in the relative case.

6. AN lc_s^0-COMPACT SPACE FOR WHICH ν_* IS NOT AN EPIMORPHISM IN DIMENSION 1

Let ω_1 be the first uncountable ordinal, and let Ω be the set of all ordinals $\alpha \leq \omega_1$. For each $\alpha \in \Omega \setminus \{\omega_1\}$, choose a copy I_α of the open interval (0, 1) of reals, and order the set $L = \Omega \cup (\bigcup_{\alpha<\omega_1} I_\alpha)$ as follows. In Ω and I_α, preserve the natural orderings; consider α as preceding all points of I_α, and consider I_α as preceding $\alpha + 1$. Ω and L are compact Hausdorff spaces under the order topology (see [8], L, p. 164). Let $X = L/\Omega$, the topology being the quotient topology (see [8], p. 94). We shall prove the following proposition:

(i) X *is a compact Hausdorff space which is* lc_s^0. *The homomorphism* $\nu_*\colon H_1(X; Z; S) \to H_1(X; Z; C)$ *is not an epimorphism.*

That X is a compact Hausdorff space is immediate. Let $f\colon L \to X$ denote the natural quotient mapping, and let $\xi = f(\Omega)$.

Given any $\beta \in \Omega$, let L_β denote the set $\{x \mid x \in L,\ x \leq \beta\}$, and let L^β denote the set $\{x \mid x \in L,\ x \geq \beta\}$. It is easy to see that L_β is an arc, for all $\beta < \omega_1$ (apply for instance Theorem 2.8, p. 168 of [7]). With the help of this fact, it is readily seen that X is locally arcwise connected, that is, lc_s^0. The space X can be considered as a "transfinite bouquet" of circles attached at a common base point ξ.

Let Y_β denote the space $L/(\Omega \cup L_\beta)$, and let $f_\beta\colon X \to Y_\beta$ be the corresponding natural projection. We shall prove first the following proposition:

(ii) *Given any singular homology class* $h_* \in H_1(X; Z; S)$, *there is a* $\beta < \omega_1$ *such that* $(f_\beta)_* h_* = 0$.

Proof of (ii). Consider any path $\phi\colon I \to X$. Let $U = \{x \mid x \in I,\ \phi(x) \neq \xi\}$. Then U consists of at most countably many open intervals V. Clearly, for each such V there exists a unique $\alpha < \omega_1$ such that $\phi(V) \subset f(I_\alpha)$. The least upper bound of these α is a $\beta < \omega_1$ such that $\phi(I) \subset f(L_\beta)$. Now take any singular 1-cycle z representing the class h_*. The preceding argument proves the existence of a $\beta < \omega_1$ such that z lies in $f(L_\beta)$. Passing thus to Y_β, we see that z is mapped into a point of Y_β.

An immediate consequence of (ii) and of the naturality of ν_* is the following proposition:

(iii) *For every Čech homology class* $h \in \nu_*(H_1(X; Z; S)) \subset H_1(X; Z; C)$, *there is a* $\beta < \omega_1$ *such that* $(f_\beta)_* h = 0$.

In order to prove (i) it now suffices to show the following:

(iv) *There exists a Čech homology class* $h \in H_1(X; Z; C)$ *such that* $(f_\beta)_* h \neq 0$, *for all* $\beta < \omega_1$.

Let X_α denote the space $L/(\Omega \cup L^\alpha)$ $(\alpha \in \Omega)$, and let $g^\alpha : L \to X_\alpha$ be the corresponding natural projection. We define maps $\pi_{\alpha'\alpha} : X_{\alpha'} \to X_\alpha$ $(\alpha < \alpha')$ as follows. If $\xi_\alpha = g^\alpha(\Omega \cup L^\alpha)$, then $\pi_{\alpha'\alpha}(\xi_{\alpha'}) = \xi_\alpha$. For $x = g^{\alpha'}(y)$ and $y \in L \setminus (\Omega \cup L^\alpha)$, we distinguish two cases: if $y < \alpha$, then $\pi_{\alpha'\alpha}(g^{\alpha'}(y)) = g^\alpha(y)$; if $\alpha < y < \alpha'$, then $\pi_{\alpha'\alpha}(g^{\alpha'}(y)) = \xi_\alpha$. It is readily verified that $\{X_\alpha; \pi_{\alpha'\alpha}\}$ $(\alpha < \beta)$ is an inverse system of compact Hausdorff spaces for all $\beta \in \Omega$ which have no immediate predecessor, and that X_β is the inverse limit of that system with mappings $\pi_{\beta\alpha} : X_\beta \to X_\alpha$ as corresponding natural projections. This is true in particular for $\beta = \omega_1$ and $X_{\omega_1} = X = \lim X_\alpha$ $(\alpha < \omega_1)$.

We shall now define (by transfinite induction) a Čech homology class

$$h_\beta \in H_1(X_\beta; Z; C),$$

for every $\beta \in \Omega$. Since X_0 is a point, h_0 has to be the trivial class 0. Suppose now that h_α has been defined for all $\alpha < \beta$ in such a way that $(\pi_{\alpha'\alpha})_* h_{\alpha'} = h_\alpha$, for all $\alpha < \alpha' < \beta$. In defining h_β, we distinguish two cases.

First case: β has an immediate predecessor β', that is, $\beta = \beta' + 1$. Here, X_β consists obviously of $X_{\beta'}$ and a 1-sphere (image of $\bar{I}_{\beta'}$) having only the point $\xi_{\beta'}$ in common with $X_{\beta'}$. Therefore, $H_1(X_\beta; Z; C)$ is the direct sum of $H_1(X_{\beta'}; Z; C)$ and a free cyclic group. We define h_β as the direct sum of $h_{\beta'}$ and a nontrivial element of this free cyclic group. Observe the following property of h_β: if

$$Z_{\beta',\beta'+1} = L/(\Omega \cup L^{\beta'+1} \cup L_{\beta'})$$

($Z_{\beta',\beta'+1}$ is homeomorphic to a 1-sphere) and we pass from $X_{\beta'+1}$ to $Z_{\beta',\beta'+1}$ by the natural projection, $h_{\beta'+1}$ is mapped into a nonzero element.

Second case: β has no immediate predecessor. Here $X_\beta = \lim X_\alpha$ $(\alpha < \beta)$. The h_α $(\alpha < \beta)$ form by supposition an element of $\lim_{\alpha<\beta} H_1(X_\alpha; Z; C)$, and they determine an element $h_\beta \in H_1(X_\beta; Z; C)$ such that $(\pi_{\beta\alpha})_* h = h_\alpha$ (apply the continuity theorem for Čech homology, p. 261 of [4]).

Now let $h = h_{\omega_1} \in H_1(X; Z; C)$, and denote $\pi_{\omega_1\beta}$ by π_β. Then $(\pi_\beta)_* h = h_\beta$, for all $\beta < \omega_1$. Note further that, for all $\beta < \omega_1$, the following diagram of mappings is commutative:

$$\begin{tikzcd} X \arrow[r, "f_\beta"] \arrow[d, "\pi_{\beta+1}"'] & Y_\beta \arrow[d] \\ X_{\beta+1} \arrow[r] & Z_{\beta,\beta+1} \end{tikzcd}$$

All the mappings in the diagram are natural projections. Since $(\pi_{\beta+1})_* h = h_{\beta+1}$ and $h_{\beta+1}$ is mapped into a nonzero element after being projected in $Z_{\beta,\beta+1}$, it follows that $(f_\beta)_* h$ can not be zero.

7. COMPARISON OF SINGULAR HOMOLOGY AND ČECH HOMOLOGY WITH COMPACT CARRIERS

Let (X, A) be a pair of Hausdorff spaces $(A \subset X)$. We consider the family Φ of all pairs (F_1, F_2) $(F_1 \subset X,\ F_2 \subset F_1 \cap A)$ such that both F_1 and F_2 are compact. The family Φ, ordered by inclusion, is obviously a directed set. The Čech homology group of (X, A) with compact carriers is defined as the direct limit of

$$H_q(F_1, F_2; G; C),$$

and it will be denoted by $H_q^{\Phi}(X, A; G; C)$. There is a natural homomorphism ν of $H_q(X, A; G; S)$ into $H_q^{\Phi}(X, A; G; C)$, defined as follows. Let $h \in H_q(X, A; G; S)$, and let z be a singular cycle of X mod A belonging to the class h. Then $\|z\|$ is a compact subset of X, while $\|\partial z\|$ is a compact subset of A. Consider the pair $(F_1, F_2) = (\|z\|, \|\partial z\|) \in \Phi$. The cycle z determines a singular homology class $h' \in H_q(F_1, F_2; G; S)$. Now take $\nu_* h' \in H_q(F_1, F_2; G; C)$, where ν_* is the homomorphism defined in Section 1. Finally, take for νh the element of $H_q^{\Phi}(X, A; G; C)$ determined by $\nu_* h'$. It is not difficult to see that νh is independent of the choices involved in this description, and that ν is actually a natural homomorphism.

THEOREM 2. *Let* (X, A) *be a pair of locally compact Hausdorff spaces which are both* lc_s^p. *Then* $\nu\colon H_q(X, A; G; S) \to H_q^{\Phi}(X, A; G; C)$ *is an isomorphism (onto) for* $q \leq p$.

(In this form, the theorem is actually due to H. B. Griffiths (unpublished); the author proved it originally under the additional supposition that both X and A are paracompact. Only later did he find the present proof, which reduces the theorem to an easy consequence of Theorem 1.)

The proof is based on the following

LEMMA 10. *Let* X *be a paracompact Hausdorff space. Let* C *be a closed and* U *an open subset of* X $(C \subset U \subset X)$. *Then there exists an open subset* V *of* X *such that* $C \subset V \subset U$ *and* V *is paracompact.*

Proof. The space X is necessarily normal. Therefore, one can construct successively a sequence of sets $V_1, V_2, \cdots$, open in X, and such that

$$C \subset V_1 \subset \bar{V}_1 \subset V_2 \subset \bar{V}_2 \subset \cdots \subset U. \tag{40}$$

Let

$$V = \bigcup_{n=1}^{\infty} V_n. \tag{41}$$

V is open in X, and it clearly satisfies $C \subset V \subset U$. On the other hand, $V = \bigcup_{n=1}^{\infty} \bar{V}_n$ and is thus an F_σ. A theorem of E. Michael ([12], p. 835) asserts that every F_σ of a paracompact Hausdorff space is itself paracompact. This proves the lemma. Notice that open sets of paracompact Hausdorff spaces are not necessarily paracompact. However, according to Lemma 10 there exists a basis of open sets consisting only of paracompact members.

Proof of Theorem 2. Given any compact pair $(F_1, F_2) \in \Phi$, one can find an open set U_1 of X such that $\bar{U}_1$ is compact and $F_1 \subset U_1$ (X is locally compact). Moreover, one can find a set $U_2 \subset A$, open in A and such that the closure $\overline{U}_2$ (taken with respect to A) is compact and such that $F_2 \subset U_2 \subset U_1$. Clearly, $\overline{U}_1$ and $\overline{U}_2$ are paracompact spaces, and we can apply Lemma 10 to obtain paracompact spaces V_1 and V_2 which satisfy $F_1 \subset V_1 \subset U_1$, $F_2 \subset V_2 \subset U_2$, and are open in $\overline{U}_1$ and $\overline{U}_2$, respectively. Consequently, V_1 and V_2 are open in X and A, respectively, and are thus lc_s^p.

Let us now prove that ν is an epimorphism. Let $h \in H_q(F_1, F_2; G; C)$ be a representative of an element h_* of $H_q^{\Phi}(X, A; G; C)$ $(q \leq p)$. Let h' be its image under the homomorphism induced by inclusion $(F_1, F_2) \subset (V_1, V_2)$. Applying Theorem 1 to (V_1, V_2), we find that h' is the ν_*-image of a class $h'' \in H_q(V_1, V_2; G; S)$. Passing from (V_1, V_2) to the compact pair $(\overline{U}_1, \overline{U}_2)$, we get the same element h_* of $H_q^{\Phi}(X, A; G; C)$, which is now obtained as the ν-image of the singular class h''.

Proof that ν is a monomorphism. Let z be a cycle of X mod A representing a singular class $h \in H_q(X, A; G; S)$ $(q \leq p)$, and let $h' \in H_q(\|z\|, \|\partial z\|; G; S)$ be the class determined by z. Then $\nu h = 0$ means that there is a compact pair (F_1, F_2) such that $(F_1, F_2) \supset (\|z\|, \|\partial z\|)$ and such that $\nu_* h'$ goes into 0 under the homomorphism induced by this inclusion. Imbedding further (F_1, F_2) into (V_1, V_2) and applying Theorem 1, we conclude that h' goes into 0 when mapped into

$$H_q(V_1, V_2; G; S).$$

This ends the proof of Theorem 2.

8. THE IMPLICATION $\mathrm{lc}_s^p \Rightarrow \mathrm{lc}_c^p$

A Hausdorff space X is said to be *locally paracompact* if for each $x \in X$ there is an open set U $(x \in U)$ such that $\overline{U}$ is paracompact. Clearly, locally compact spaces as well as paracompact spaces are special cases of locally paracompact spaces. An equivalent definition is the following. A Hausdorff space is locally paracompact if for each $x \in X$ there is an open set V $(x \in V)$ which is paracompact. The equivalence is an immediate consequence of Lemma 10, and of the fact that a closed subset of a paracompact Hausdorff space is itself paracompact.

THEOREM 3. *If* X *is a locally paracompact Hausdorff space and is* lc_s^p, *then* X *is also* lc_c^p.

Proof. Let $x \in X$, and let U be an open set of X $(x \in U)$. Choose an open paracompact set U' around x (by local paracompactness) and choose an open set $V' \subset U' \cap U$ $(x \in V')$ such that $H_q(V' \mid U; Z; S) = 0$ $(q \leq p)$. Finally, choose a paracompact open set $V \subset V'$ $(x \in V)$ (apply Lemma 10 to U'). Clearly,

$$H_q(V \mid U; Z; S) = 0 \quad (q \leq p). \tag{42}$$

V is a paracompact lc_s^p Hausdorff space, and Theorem 1 is applicable. Using the naturality of ν_*, we conclude from (42) that

$$H_q(V \mid U; Z; C) = 0 \quad (q \leq p). \tag{43}$$

9. A HOMOLOGY CRITERION FOR UNICOHERENCE

In a previous paper [11], the author has derived a criterion for unicoherence of ANR-s, using the fact that singular and Čech cohomology coincide for ANR-s. Using Theorem 1, one now obtains the following improved criterion.

THEOREM 4. *Let* X *be a paracompact Hausdorff space which is connected, locally arcwise connected and semi-*1-lc_S. *Then* X *is unicoherent if and only if* $\mathrm{Hom}(H_1(X; Z; S), Z) = 0$, *that is, if and only if* $H_1(X; Z; S)$ *does not admit* Z *as a direct summand.*

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