A JACOBIAN CONDITION FOR INTERIORITY

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Let D be an open set in Euclidean n-space, E^n, and let f: D → E^n be a continuous function from D into E^n. There are a number of topological conditions on f so that f is interior on D; that is, so that the image of every open subset of D is open in E^n. However, these conditions are not formulated in such a way as to yield simple proofs of interiority for functions as they occur naturally in analysis. For example, a function analytic in a plane domain is interior, but the writers know of no proof of interiority that does not employ such tools as Taylor series or the integration theory of analytic functions. (For a discussion of this case, see Whyburn's Memoir [4].) In this note, we provide simple sets of conditions for interiority and for quasi-interiority.

Definitions. A mapping f: A → B is light if for each point b of B, $f^{-1}(b)$ is totally disconnected. It is monotone if for each point b of B, $f^{-1}(b)$ is connected. It is quasi-interior (Whyburn) if for each point b is B and for each open set U in A that contains a compact component of $f^{-1}(b)$, b is in Int f(U), the interior of f(U). Clearly a light quasi-interior transformation is interior.

Theorem 1. *Let D be an open set in E^n. Let f: D → E^n be of class C^1, and let the Jacobian J(f(p)) be zero only on a compact subset of D of dimension less than n − 1. Then f is quasi-interior on D.*

Proof. If Z denotes the subset of D on which

J(f) vanishes, then the n-dimensional measure of f(Z) is given by $\int_Z J(f(p))dp = 0$. Hence f(Z) has no interior points. Let p be in f(D). If C is a component of $f^{-1}(p)$, and C intersects D - Z, then C is a point, For at each point of $C \cap (D - Z)$, the fact that $J(f) \neq 0$ there implies that f is locally one-to-one there. If C is in Z, then C is compact. In the first case, sufficiently small open sets containing C are mapped homeomorphically into open sets containing p. In the second case, given an open set U containing C, there is an open set V containing C such that $\overline{V}$, the closure of V, is the union of the simplexes of a finite simplicial complex, K, and such that $\overline{V} - V$ does not intersect $f^{-1}(p)$. Under these conditions we can define the local degree of p with respect to f and $\overline{V}$, $d(p, f, \overline{V})$. Concerning $d(p, f, \overline{V})$, the following properties are well known.

(1) There is an open subset W of $E^n - f(\overline{V} - V)$ that contains p and is such that for each point q in W, $d(p, f, \overline{V}) = d(q, f, \overline{V})$. ([1], p. 473-4).

(2) If $d(q, f, \overline{V}) \neq 0$, q in E^n, then q is in $f(\overline{V})$. ([1], p. 468).

Since f(Z) contains no interior point, there is a point q in W - f(Z). Then at each point x in $f^{-1}(q)$, $J(f(x)) \neq 0$. Hence f is one-to-one in a neighborhood U_x of such a point. It follows that the compact set $\overline{V}$ can contain only a finite number of such points, $x_1, x_2, \ldots, x_k$, all lying in V. Since dim $Z < n - 1$, D - Z is connected ([2], p. 98), so the continuous function J(f(x)) is of the same sign throughout D - Z. If J(f) is positive in D - Z, each set $f(U_{x_i})$ has its orientation preserved, and contributes a +1 to $d(q, f, \overline{V})$; if J(f) is negative in D - Z, each such set has its orientation reversed, and contributes a -1 to $d(q, f, \overline{V})$. (This is the essential content of [1], p. 477.) Hence $d(q, f, \overline{V}) = \pm k \neq 0$. It follows from (1) that this is true for all other points of W, and from (2) that W is in f(V). So p is in Int f(V), which completes the proof.

We will discuss the implications of this result after the next theorem.

Theorem 2. Let D be an open set in E^n, and let $f: D \rightarrow E^n$ be light, of class C^1, and have a non-negative (non-positive) Jacobian in D. Then f is interior.

Proof. Let X be a compact subset of D that is the closure of an open subset of D. Since f is light, a theorem of Hurewicz ([2], p. 91) shows that dim f(X) = n, so that by ([2], p. 44), f(X) contains an open subset of E^n. This remark, with continuity, shows that Int f(D) is dense in f(D). If Z again denotes the set on which J(f) vanishes, the n-dimensional measure of f(Z) is zero, so that Int f(D) - f(Z) is dense in f(D). That is, arbitrarily close to each point of f(D) there are points q such that $J(f(x)) \neq 0$ for any x in $f^{-1}(q)$. This is all that is needed to carry through the argument of Theorem 1. So f is quasi-interior and light, and hence interior.

We can now see that in Theorem 1 the condition that dim Z < n - 1 could be replaced by the conditions that J(f) have the same sign on D - Z and that dim $Z < n$.

A few remarks on Theorem 2 may be helpful to the non-topologist. For each point p in f(D), the points in $f^{-1}(p)$ not in Z are isolated, so form a countable set having no limit point in D - Z. But that the same is true for $f^{-1}(p) \cap Z$ is not at all obvious, if indeed it is true. For $n = 2$, the work of Stoïlow and of Whyburn [3] gives a complete analysis of the situation, but very little is known about interior light transformations of open sets in E^n, $n > 2$. For example, if it were known that each point inverse for such a transformation were countable, Hilbert's fifth problem would be solved.

Part of the significance of Theorem 1 is the following. Any continuous map f: $A \to B$ on a compact metric space A can be factored into a monotone map m: $A \to M$ followed by a light interior map l: $M \to B$ so that $f = lm$. (Eilenberg-Whyburn, [3], p. 141.) We can prove the following theorem.

Theorem 3. *Let D be an open set in E^n with a compact closure, and let f: $\overline{D} \to E^n$ be continuous and of class C^1 in D. Let the set Z where $J(f) = 0$ be compact in D and of dimension less than $n - 1$. Then f can be factored into a monotone map m: $\overline{D} \to M$ such that m(D) is an open subset of M, followed by a light map l: $M \to E^n$ such that l is interior on M(D).*

Proof. The "middle space" M is the space whose points are the components of sets $f^{-1}(p)$, p in $f(\overline{D})$, with a topology such that a subset of M is open if and only if the union of its elements is open in D. The map m is the natural map assigning to each point x in

$\overline{D}$ the component m(x) of $f^{-1}f(x)$ containing it. Such a component is either a point of D - Z, or a subset of $\overline{D}$ - D or a subset of Z. Since m($\overline{D}$ - D), then, cannot intersect m(D), we see that m(D) is open. Whyburn has proved ([4], 10.4, p. 14) that a compact mapping is quasi-interior if and only if it factors into a (topologically) unique monotone map followed by a light interior map. Although his result does not apply directly, inspection of his argument shows that Theorem 3 follows from it and from what we have proved.

We remark that all our results are valid for mappings of open subsets of differentiable manifolds into differentiable manifolds of the same dimension, though the details would take more space.

To return to the case of a function f(z) analytic in a plane domain D. Representing f(z) as u(x, y) + iv(x, y), we have $J(f(z)) = u_x^2 + u_y^2 = v_x^2 + v_y^2$, which vanishes only where $f'(z) = 0$. If these form a zero-dimensional set, the mapping is necessarily light, and so is interior. Or if we know that f(z) is light and is not constant over an open subset of D, then, again, f is interior. These properties of analytic functions do not have "elementary" proofs, however. The interested reader should compare this discussion with Section 3 of Whyburn's [4].

Bibliography

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