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On the Symmetric Enumeration Degrees

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Abstract A set A is *symmetric enumeration* (se-) reducible to a set B ($A \leq_{\text{se}} B$) if A is enumeration reducible to B *and* $\overline{A}$ is enumeration reducible to $\overline{B}$. This reducibility gives rise to a degree structure ($\mathcal{D}_{\text{se}}$) whose least element is the class of computable sets. We give a classification of $\leq_{\text{se}}$ in terms of other standard reducibilities and we show that the natural embedding of the Turing degrees ($\mathcal{D}_{\text{T}}$) into the enumeration degrees ($\mathcal{D}_{\text{e}}$) translates to an embedding (ι_{se}) into $\mathcal{D}_{\text{se}}$ that preserves least element, suprema, and infima. We define a weak and a strong jump and we observe that ι_{se} preserves the jump operator relative to the latter definition. We prove various (global) results concerning branching, exact pairs, minimal covers, and diamond embeddings in $\mathcal{D}_{\text{se}}$. We show that certain classes of se-degrees are first-order definable, in particular, the classes of semirecursive, $\Sigma_n \cup \Pi_n$, Δ_n (for any $n \in \omega$), and *embedded* Turing degrees. This last result allows us to conclude that the theory of $\mathcal{D}_{\text{se}}$ has the same 1-degree as the theory of Second-Order Arithmetic.

1 Introduction

The original motivation behind the definition of *symmetric enumeration* (se-) reducibility given below—an equivalent definition was given by Selman in [18]—was its role in providing a nontrivial generalization of the relativized Arithmetical Hierarchy. In effect, it was shown in [3], Section 6, that an appropriate hierarchy could be obtained by replacing the relations "c.e. in" and "Turing reducible to" in the underlying framework of the Arithmetical Hierarchy by the relations "enumeration reducible to" and "se-reducible to." Moreover, it was proved that not only is this hierarchy a refinement of the Arithmetical Hierarchy but also it is identical with the latter when relativized to sets belonging to embedded Turing degrees (in the sense of Proposition 4.8 below).

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At the same time, it emerged from our work that se-reducibility had distinctive properties with regard to other reducibilities. For example, we found that the standard deterministic positive reducibilities (and, in particular, $\leq_{\rm p}$) are subrelations of se-reducibility. Also it transpired that the embedding of the Turing degrees into the enumeration degrees translates to an embedding into the se-degrees with similar properties. These results are reiterated in the early sections of the present paper. However, looking beyond the basic theory, our main purpose here is to present an overview of the associated degree structure of this reducibility. We show how a number of structural results can be obtained using both relatively old (Section 6) and more recent (Section 8) methods that were originally developed in the context of the enumeration degrees. Underpinning these results in part is an inherent type of local similarity between the se-degrees and the Turing degrees (Section 7). It is this phenomenon, in conjunction with some of the structural insights already gained, that leads to a straightforward appraisal of various definability properties of the se-degree structure and, in particular, of the complexity of its first-order theory (Section 9).

2 Preliminaries

2.1 Background notation We let ω (ω^+) denote the set of (nonzero) natural numbers and $A, B, \ldots$ denote subsets of ω. Lowercase letters $n, x, \ldots$ and $f, g, \ldots$ represent numbers and functions (from ω to ω), respectively, whereas $\mathbf{A}, \mathbf{B}, \ldots$ represent classes of sets. $\overline{A}$ denotes the complement of A. The set $\{ n \cdot x + m \mid x \in A \}$ is written $nA + m$ and $2A \cup 2B + 1$ is written $A \oplus B$. We use $\langle\ ,\ \rangle$ to denote the standard diagonal coding function defined by $\langle x, y\rangle = 1/2(x^2 + y^2 + 2xy + 3x + y)$. The characteristic function of A is written c_A, and for any function f, its graph is written $\mathbb{F}$ (and so $\mathbb{C}_A$ stands for the graph of c_A). We assume the availability of effective enumerations of (oracle) Turing machines $\varphi_0, \varphi_1, \ldots$, and computably enumerable (c.e.) sets $W_0, W_1, \ldots$. We also assume $D_0, D_1, \ldots$ to be an enumeration of finite sets given by the binary decomposition of the natural numbers; that is, $D_0 = \varnothing$ and for $n > 0$, if (say) $n = \Sigma_{i \leq k} 2^{a_i}$, then $D_n = \{ a_i \mid i \leq k \}$. Note that, to simplify notation, we usually use D, D', and so on, to denote both the finite sets themselves and their indices. For example, if i, j are the indices of D, D' then $\langle D, D'\rangle$ is shorthand for $\langle i, j\rangle$.

2.2 Basic reducibilities We assume the standard multitape Turing machine model for computing partial functions and we suppose an oracle Turing machine to be equipped with a function oracle. We say that the set A is *Turing reducible* to the set B ($A \leq_{\rm T} B$) if there is an oracle machine φ that computes c_A when equipped with oracle c_B (written $c_A \simeq \varphi^B$). A is said to be *computably enumerable in* B (A *c.e. in* B) if A is the range of some function f computable in B or, equivalently, if $A = \{ x \mid \varphi^B(x)\downarrow \}$ for some oracle Turing machine φ. $\mathcal{K}_B$ denotes the set $\{ x \mid \varphi_x^B(x)\downarrow \}$. For Turing reductions we use $Q(\varphi, x, B)$ to denote the set of oracle queries made by φ^B on input x. We say that A is *many-one reducible* to B ($A \leq_{\rm m} B$) if there is a computable function f such that $A = f^{-1}(B)$. Furthermore, if f is one-one, A is said to be *one-one reducible* to B ($A \leq_1 B$). We say that A is *enumeration reducible* to B ($A \leq_{\rm e} B$) if there exists a computably enumerable set W such that, for all x,

$$x \in A \quad \text{iff} \quad \exists D(\,\langle x, D\rangle \in W \;\&\; D \subseteq B\,),$$

and in this case we also say that $A \leq_e B$ *via* W. Similarly—assuming $W_0, W_1, \ldots$ to be a fixed computable listing of all c.e. sets—the *nth enumeration operator* Φ_n is defined such that, for any set A,

$$\Phi_n(B) = \{ x \mid \exists D(\langle x, D \rangle \in W_n \ \& \ D \subseteq B) \}.$$

A is said to be *positive reducible* to B ($A \leq_p B$) if there exists a computable function $f : \omega \to \omega^+$ such that, for all $x \geq 0$, $x \in A \Leftrightarrow \exists y(y \in D_{f(x)} \ \& \ D_y \subseteq B)$. We say that A is *wtt-reducible* to B ($A \leq_{wtt} B$) if there exists a Turing machine φ and computable function f such that $c_A \simeq \varphi^B$ and such that, for all $x \geq 0$, $Q(\varphi, x, B) \subseteq \{0, \ldots, f(x)\}$.

$\deg_r(A)$ denotes the *degree* of A under the reducibility $\leq_r$, that is, the class $\{ B \mid B \equiv_r A \}$. We use $\boldsymbol{a}_r, \boldsymbol{b}_r, \ldots$ to denote the degrees derived according to this definition and $\mathcal{D}_r$ to denote the corresponding *degree structure*. Subscripts are dropped if the context is clear. A is said to be *r-hard* for a class $\mathbf{C}$ if $X \leq_r A$ for all X in $\mathbf{C}$ and A is said to be *r-complete* for $\mathbf{C}$ if A also belongs to $\mathbf{C}$. We use the shorthand $\mathbf{Comp}(A)$, $\mathbf{Enum}(A)$, and $\mathbf{Ce}(A)$ to denote the classes $\{ E \mid E \ \mathcal{R} \ A \}$ such that (respectively) $\mathcal{R}$ is $\leq_T$, $\leq_e$, or "*c.e. in*." Accordingly, we use **Comp** and **Ce** to denote the classes of computable and c.e. sets. Also we will employ the abbreviations r-reduction, r-degree, and so on, when appropriate.

2.3 String notation A string is a partial function $\sigma : \omega \to \{0, 1\}$ with finite domain. λ denotes the empty string and $|\sigma|$ the length of σ (i.e., the cardinality of its domain). For $(s, i) \in \{ (+, 1), (-, 0) \}$, we use σ^s to denote the set $\{ n \mid \sigma(n)\downarrow = i \}$ and $(\sigma \restriction A)^s$ to denote the set $\{ n \mid n \in A \ \& \ \sigma(n)\downarrow = i \}$ (and so $\sigma^s = (\sigma \restriction \omega)^s$ for $s \in \{+, -\}$). If the domain of σ ($\mathrm{Dom}(\sigma)$) is an initial segment of ω, σ is said to be an *initial segment*. Note that this means that if $|\sigma| = n + 1$, the domain of σ is $\{0, \ldots, n\}$. We use the shorthand $\sigma' = \sigma^\frown(i)$ to denote the extension of σ of length $|\sigma|+1$ such that $\sigma'(|\sigma|) = i$. For any two strings α and β such that α is a substring of β, $\beta - \alpha$ denotes the string formed from the difference of β and α, that is, such that $\mathrm{Dom}(\beta - \alpha) = \mathrm{Dom}(\beta) - \mathrm{Dom}(\alpha)$ and $\beta - \alpha(n) = \beta(n)$ for all $n \in \mathrm{Dom}(\beta - \alpha)$.

3 Introduction to Symmetric Enumeration Reducibility

Enumeration reducibility compares the positive information content of two sets. Symmetric enumeration reducibility, as we will see, compares both positive and negative information content. We will now introduce this reducibility and consider how it relates to other standard reducibilities. First, however, we draw the reader's attention to the fact that Selman exhibited some of the basic properties of this reducibility in Section 4 of [18]. In particular, Selman noted the inclusion $\leq_m \subseteq \leq_{se} \subseteq \leq_T$ and proved Theorem 3.8 (below) relative to the pair $(\leq_{tt}, \leq_{se})$.

Definition 3.1 For any sets A and B, A is defined to be *symmetric enumeration reducible* to B ($A \leq_{se} B$) if $A \leq_e B$ and $\overline{A} \leq_e \overline{B}$.

Notation For any set A, $\mathbf{s\text{-}Enum}(A)$ denotes the class $\{ E \mid E \leq_{se} A \}$.

Lemma 3.2 *Let A and B be sets such that $B \notin \{\emptyset, \omega\}$. Then there exists a computable function f such that $A = \Phi_{f(i,j)}(B)$ and $\overline{A} = \Phi_{f(i,j)}(\overline{B})$ whenever $A = \Phi_i(B)$ and $\overline{A} = \Phi_j(\overline{B})$ (i.e., whenever $A \leq_{se} B$ via operators i and j).*

Proof Choose $b \in B$ and $\overline{b} \in \overline{B}$. Define f so that, for any $i, j \in \omega$,

$$\begin{aligned} W_{f(i,j)} \quad &= \quad \{\, \langle x, D \cup \{b\} \rangle \mid \langle x, D \rangle \in W_i \,\} \\ & \quad \bigcup \{\, \langle x, D \cup \{\overline{b}\} \rangle \mid \langle x, D \rangle \in W_j \,\}. \end{aligned}$$

It is easily checked that if $A = \Phi_i(B)$ and $\overline{A} = \Phi_j(\overline{B})$, then $A = \Phi_{f(i,j)}(B)$ and $\overline{A} = \Phi_{f(i,j)}(\overline{B})$. $\square$

Corollary 3.3 *For any sets A, B such that $B \notin \{\emptyset, \omega\}$, $A \leq_{\mathrm{se}} B$ if and only if there exists an enumeration operator Φ such that $A = \Phi(B)$ and $\overline{A} = \Phi(\overline{B})$.*

Note 3.4 Clearly, $\leq_{\mathrm{se}}$ inherits reflexivity and transitivity from $\leq_{\mathrm{e}}$. It thus gives rise to a degree structure ($\mathcal{D}_{\mathrm{se}}$). The least upper bound of any two degrees $\boldsymbol{a}_{\mathrm{se}}, \boldsymbol{b}_{\mathrm{se}}$ (written $\boldsymbol{a}_{\mathrm{se}} \cup \boldsymbol{b}_{\mathrm{se}}$) always exists: it is the degree of $A \oplus B$ for any $A \in \boldsymbol{a}_{\mathrm{se}}$ and $B \in \boldsymbol{b}_{\mathrm{se}}$. Therefore, $\mathcal{D}_{\mathrm{se}}$ is an upper semilattice. The zero element ($\mathbf{0}_{\mathrm{se}}$) of $\mathcal{D}_{\mathrm{se}}$ is **Comp**. Each of these properties is easily checked.

Lemma 3.5 *For any sets A and B, if $A \leq_{\mathrm{se}} B$ then $A \leq_{\mathrm{e}} B$ and $A \leq_{\mathrm{T}} B$. In other words,*

$$\leq_{\mathrm{se}} \;\subseteq\; \leq_{\mathrm{e}} \bigcap \leq_{\mathrm{T}}.$$

Moreover, this inclusion is proper.

Proof Since $\leq_{\mathrm{se}}$ is a subrelation of $\leq_{\mathrm{e}}$ by definition, in order to prove the inclusion it suffices to note that, for any sets A and B, $\overline{A} \leq_{\mathrm{e}} \overline{B}$ implies that $\overline{A}$ *c.e. in* B. Also, $\mathbb{C}_{\mathcal{K}} \leq_{\mathrm{r}} \overline{\mathcal{K}}$ for $\mathrm{r} \in \{\mathrm{e}, \mathrm{T}\}$ whereas $\mathbb{C}_{\mathcal{K}} \not\leq_{\mathrm{se}} \overline{\mathcal{K}}$ (since this would imply $\overline{\mathcal{K}} \leq_{\mathrm{e}} \mathcal{K}$). Thus the inclusion is proper. $\square$

Theorem 3.6 $\leq_{\mathrm{p}} \subseteq \leq_{\mathrm{se}}$.

Proof Clearly, $\leq_{\mathrm{p}} \subseteq \leq_{\mathrm{e}}$. Also, for any sets A and B, if $A \leq_{\mathrm{p}} B$, then $\overline{A} \leq_{\mathrm{p}} \overline{B}$. Therefore, $\leq_{\mathrm{p}} \subseteq \leq_{\mathrm{se}}$. $\square$

Note 3.7 Theorem 3.6 implies that all conjunctive and disjunctive subreducibilities of $\leq_{\mathrm{T}}$ are contained in $\leq_{\mathrm{se}}$ and, in particular, that $\leq_1 \subseteq \leq_{\mathrm{m}} \subseteq \leq_{\mathrm{se}}$.

Theorem 3.8 *It is neither the case that $\leq_{\mathrm{wtt}} \subseteq \leq_{\mathrm{se}}$ nor the case that $\leq_{\mathrm{se}} \subseteq \leq_{\mathrm{wtt}}$.*

Proof The first inequality is witnessed by $\mathcal{K}$ in that $\overline{\mathcal{K}} \leq_{\mathrm{wtt}} \mathcal{K}$ (and in fact $\overline{\mathcal{K}} \leq_{\mathrm{btt}(1)} \mathcal{K}$) whereas $\overline{\mathcal{K}} \not\leq_{\mathrm{e}} \mathcal{K}$. The second inequality can be deduced from the well-known fact that $\leq_{\mathrm{T}} \not\subseteq \leq_{\mathrm{wtt}}$ as follows. Choose sets A and B such that $A \leq_{\mathrm{T}} B$ whereas $A \not\leq_{\mathrm{wtt}} B$. Then $A \oplus \overline{A} \leq_{\mathrm{T}} B \oplus \overline{B}$ and so, by Lemma 4.7 (and Lemma 4.1), $A \oplus \overline{A} \leq_{\mathrm{se}} B \oplus \overline{B}$. On the other hand, obviously $A \oplus \overline{A} \not\leq_{\mathrm{wtt}} B \oplus \overline{B}$. $\square$

4 Embedding the Turing Degrees

The isomorphic embedding ι_{e} of the Turing degrees ($\mathcal{D}_{\mathrm{T}}$) into the enumeration degrees ($\mathcal{D}_{\mathrm{e}}$) induced by the map $X \mapsto \mathbb{C}_X$ is essentially an embedding into $\mathcal{D}_{\mathrm{se}}$. Moreover, the range of this embedding contains *gaps* similar to those appearing in the range of ι_e. These results are presented below. We begin with an easy but useful lemma.

Lemma 4.1 *For any set A the following equivalences hold:*

$$\text{(a)}\ \mathbb{C}_A \equiv_{\mathrm{se}} A \oplus \overline{A} \qquad \text{(b)}\ \mathbb{C}_A \equiv_{\mathrm{se}} \overline{\mathbb{C}_A} \qquad \text{(c)}\ \mathbb{C}_A \equiv_{\mathrm{se}} \mathbb{C}_{\overline{A}}\,.$$

Notation We say that a set A is *characteristic* if $A = B \oplus \overline{B}$ for some set B. For the sake of simplicity, and in view of Lemma 4.1, we sometimes prefer to work with a *characteristic* set $(X \oplus \overline{X})$ rather than with the corresponding *characteristic function graph* $(\mathbb{C}_X)$.

Definition 4.2 An e-degree is said to be *total* if it contains the graph of a total (or, equivalently, characteristic) function. An se-degree is said to be *characteristic* if it contains the graph of a characteristic function (or, equivalently, a characteristic set).

Proposition 4.3 *For any se-degree $\boldsymbol{a}$ the following are equivalent:*

(a) *$\boldsymbol{a}$ is characteristic;*
(b) *for all A in $\boldsymbol{a}$, $A \equiv_{\mathrm{se}} \overline{A}$.*

Proof Apply Lemma 4.1 and use the transitivity of $\leq_{\mathrm{se}}$. □

Note 4.4 $\mathbf{0}_{\mathrm{se}}$ is characteristic.

Lemma 4.5 *Every total e-degree contains exactly one characteristic se-degree.*

Proof Suppose that $B, C \in \boldsymbol{a}_{\mathrm{e}}$ and that $B \equiv_{\mathrm{se}} \overline{B}$ and $C \equiv_{\mathrm{se}} \overline{C}$. This means that $\mathbb{C}_B \equiv_{\mathrm{e}} \mathbb{C}_C$, and by applying Lemma 4.1 it follows that $\mathbb{C}_B \equiv_{\mathrm{se}} \mathbb{C}_C$. Hence $B \equiv_{\mathrm{se}} C$. □

Lemma 4.6 *For any sets A and B, A c.e. in B if and only if $A \leq_{\mathrm{e}} \mathbb{C}_B$.*

Proof Obvious. □

Lemma 4.7 *For any sets A and B,*

$$A \leq_{\mathrm{T}} B \quad \textit{iff} \quad A \leq_{\mathrm{se}} \mathbb{C}_B \quad \textit{iff} \quad \mathbb{C}_A \leq_{\mathrm{se}} \mathbb{C}_B.$$

Proof Apply Lemma 4.6 in conjunction with Lemma 4.1. □

Proposition 4.8 *The embedding ι_{se} of the Turing degrees into the se-degrees induced by the map $X \mapsto \mathbb{C}_X$ is one-one structure preserving (i.e., isomorphic) and also preserves suprema, infima, and least element.*

Proof The only part of this proof that does not follow in a straightforward manner from Lemma 4.7 and the results listed in Note 3.4 is the assertion that ι_{se} preserves infima. To do this—given that the rest of the proposition holds—suppose that $\boldsymbol{a}_{\mathrm{T}}$, $\boldsymbol{b}_{\mathrm{T}}$, and $\boldsymbol{c}_{\mathrm{T}}$ are Turing degrees such that $\boldsymbol{a}_{\mathrm{T}} = \boldsymbol{b}_{\mathrm{T}} \cap \boldsymbol{c}_{\mathrm{T}}$ in $\mathcal{D}_{\mathrm{T}}$ and choose $A \in \boldsymbol{a}_{\mathrm{T}}$, $B \in \boldsymbol{b}_{\mathrm{T}}$, and $C \in \boldsymbol{c}_{\mathrm{T}}$. (Hence $B \oplus \overline{B} \in \iota_{\mathrm{se}}(\boldsymbol{b}_{\mathrm{T}})$ and so on.) Since ι_{se} is structure preserving, $\iota_{\mathrm{se}}(\boldsymbol{a}_{\mathrm{T}}) \leq \iota_{\mathrm{se}}(\boldsymbol{b}_{\mathrm{T}}), \iota_{\mathrm{se}}(\boldsymbol{c}_{\mathrm{T}})$ in $\mathcal{D}_{\mathrm{se}}$. Let $\boldsymbol{d}_{\mathrm{se}}$ be any se-degree such that $\boldsymbol{d}_{\mathrm{se}} \leq \iota_{\mathrm{se}}(\boldsymbol{b}_{\mathrm{T}}), \iota_{\mathrm{se}}(\boldsymbol{c}_{\mathrm{T}})$. Choose $E \in \boldsymbol{d}_{\mathrm{se}}$ and let $\boldsymbol{e}_{\mathrm{T}} = \deg_{\mathrm{T}}(E)$ and $\boldsymbol{e}_{\mathrm{se}} = \deg_{\mathrm{se}}(E \oplus \overline{E})$. Now, as $\boldsymbol{d}_{\mathrm{se}} \leq \iota_{\mathrm{se}}(\boldsymbol{b}_{\mathrm{T}}), \iota_{\mathrm{se}}(\boldsymbol{c}_{\mathrm{T}})$, we know that $E \leq_{\mathrm{se}} B \oplus \overline{B}, C \oplus \overline{C}$. It thus follows by definition of $\leq_{\mathrm{se}}$ and Lemma 4.1 that $E \oplus \overline{E} \leq_{\mathrm{se}} B \oplus \overline{B}, C \oplus \overline{C}$. In other words, $\boldsymbol{e}_{\mathrm{se}} \leq \iota_{\mathrm{se}}(\boldsymbol{b}_{\mathrm{T}}), \iota_{\mathrm{se}}(\boldsymbol{c}_{\mathrm{T}})$ and, by Lemma 4.7, $E \leq_{\mathrm{T}} B, C$. Hence, by hypothesis, $\boldsymbol{e}_{\mathrm{T}} \leq \boldsymbol{a}_{\mathrm{T}}$ in $\mathcal{D}_{\mathrm{T}}$. But $\boldsymbol{e}_{\mathrm{se}} = \iota_{\mathrm{se}}(\boldsymbol{e}_{\mathrm{T}})$ by definition, and so $\boldsymbol{e}_{\mathrm{se}} \leq \iota_{\mathrm{se}}(\boldsymbol{a}_{\mathrm{T}})$ in $\mathcal{D}_{\mathrm{se}}$ since ι_{se} is structure preserving. It now suffices to note that $\boldsymbol{d}_{\mathrm{se}} \leq \boldsymbol{e}_{\mathrm{se}}$. □

Note 4.9 It follows from Lemma 4.7 that every Turing degree $\boldsymbol{a}_{\mathrm{T}}$ contains exactly one characteristic se-degree $\boldsymbol{a}_{\mathrm{se}}$ (say). Also it is clear that $\boldsymbol{a}_{\mathrm{se}} = \imath_{\mathrm{se}}(\boldsymbol{a}_{\mathrm{T}})$ by definition. Moreover, as $X \leq_{\mathrm{se}} \mathbb{C}_X$ for any X, $\boldsymbol{a}_{\mathrm{se}}$ is the *top* se-degree in $\boldsymbol{a}_{\mathrm{T}}$ (i.e., $\boldsymbol{b}_{\mathrm{se}} \leq \boldsymbol{a}_{\mathrm{se}}$ for every $\boldsymbol{b}_{\mathrm{se}} \subseteq \boldsymbol{a}_{\mathrm{T}}$).

Definition 4.10 An se-degree $\boldsymbol{a}$ is said to be *quasi-minimal* if $\boldsymbol{a} > \boldsymbol{0}$ and $\forall \boldsymbol{d}(\,\boldsymbol{d} < \boldsymbol{a}\ \&\ \boldsymbol{d}$ characteristic $\Rightarrow \boldsymbol{d} = \boldsymbol{0}\,)$.

Theorem 4.11 *For any se-degree $\boldsymbol{b}$ there exists a degree $\boldsymbol{a}$ such that $\boldsymbol{b} < \boldsymbol{a}$ and such that, for any characteristic degree $\boldsymbol{c}$, if $\boldsymbol{c} \leq \boldsymbol{a}$ then $\boldsymbol{c} \leq \boldsymbol{b}$.*

Proof The proof is a straightforward modification of the corresponding result relative to $\mathcal{D}_{\mathrm{e}}$ due to Medvedev [13]. Indeed, suppose that B is any set. Then it suffices to construct a set A such that $B \leq_{\mathrm{se}} A$ and such that A satisfies the following requirements:

$$
\begin{aligned}
R_{3e} &: \quad A \neq \Phi_e(B) \\
R_{3e+1} &: \quad \Phi_e(A) \text{ characteristic } \Rightarrow \Phi_e(A) \leq_{\mathrm{e}} B \\
R_{3e+2} &: \quad \Phi_e(\overline{A}) \text{ characteristic } \Rightarrow \Phi_e(\overline{A}) \leq_{\mathrm{e}} \overline{B}.
\end{aligned}
$$

We ensure that $B \leq_{\mathrm{se}} A$ by encoding B into A in the following manner:

$$\forall x(\,x \in B \quad \text{iff} \quad 2x \in A\,)\,. \qquad (B\text{-coding})$$

Notation We say that an *initial segment* σ is *B-compatible* if, for all x such that $2x < |\sigma|$, $x \in B$ if and only if $2x \in \sigma^{+}$.

The construction The set A is constructed by finite initial segments $\{\sigma_n\}_{n \geq 0}$ such that $A = \bigcup \{\,\sigma_n^{+} \mid n \geq 0\,\}$.

Stage $s = 0$ $\quad \sigma_0 = \lambda$.

Stage $s + 1$ $\quad \sigma_s$ has already been defined.

Case 1 If $s = 3e$ then, letting $n_s := |\sigma_s|$, we satisfy R_{3e} by defining

$$
\sigma_{s+1} \quad := \quad \begin{cases} \sigma_s \frown (\,1 - \Phi_e(A)(n_s)\,) & \text{if } n_s \text{ is odd,} \\ \sigma_s \frown (\,B(n_s/2)\,) \frown (\,1 - \Phi_e(A)(n_s + 1)\,) & \text{if } n_s \text{ is even.} \end{cases}
$$

Case 2 If $s = 3e + 1$, then we try to satisfy R_{3e+1} vacuously by searching for a B-compatible initial segment $\sigma \supseteq \sigma_s$ such that, for some n : $2n$, $2n + 1 \in \Phi_e(\sigma^{+})$. If this search is successful, choose the least such σ and set $\sigma_{s+1} := \sigma$. Otherwise, set $\sigma_{s+1} := \sigma_s$.

Case 3 If $s = 3e + 2$, then we try to satisfy R_{3e+2} vacuously by searching for a B-compatible initial segment $\sigma \supseteq \sigma_s$ such that, for some n : $2n$, $2n + 1 \in \Phi_e(\sigma^{-})$. If this search is successful, choose the least such σ and set $\sigma_{s+1} := \sigma$. Otherwise, set $\sigma_{s+1} := \sigma_s$.

Analysis of the construction The construction obviously ensures that the constraint (B-coding) holds, which means that $B \leq_{\mathrm{se}} A$. Also the requirements $\{R_{3e}\}_{e \geq 0}$ prevent $A \leq_{\mathrm{se}} B$ and hence $B <_{\mathrm{se}} A$. So suppose that there exists a set E such that $E \oplus \overline{E} \leq_{\mathrm{se}} A$. Thus, by definition, $\Phi_i(A) = E \oplus \overline{E}$ and $\Phi_j(\overline{A}) = \overline{E} \oplus E$ for some $i, j \geq 0$. Now set $s := 3i+1$ and $t := 3j + 2$ and define

$$\begin{aligned} P_s &:= \{\, n \mid (\exists \sigma \supseteq \sigma_s)(\, n \in \Phi_i(\sigma^+) \;\&\; (\sigma \restriction 2\omega)^+ \subseteq B \oplus \varnothing \,)\,\}\,, \\ N_t &:= \{\, n \mid (\exists \sigma \supseteq \sigma_t)(\, n \in \Phi_j(\sigma^-) \;\&\; (\sigma \restriction 2\omega)^- \subseteq \overline{B} \oplus \varnothing \,)\,\}\,. \end{aligned}$$

Clearly, $P_s \leq_{\mathrm{e}} B$ and $N_t \leq_{\mathrm{e}} \overline{B}$ and also $\Phi_i(A) \subseteq P_s$ and $\Phi_j(\overline{A}) \subseteq N_t$. So now suppose that $N_t \not\subseteq \Phi_j(\overline{A})$. Without loss of generality, choose $2n+1 \in N_t - \Phi_j(\overline{A})$. Thus there exists $\beta \supseteq \sigma_t$ such that $2n+1 \in \Phi_j(\beta^-)$ and $(\beta \restriction 2\omega)^- \subseteq \overline{B} \oplus \varnothing$. Also, by hypothesis (that $\Phi_j(\overline{A})$ is characteristic), there exists B-compatible $\alpha \supseteq \sigma_t$ such that $2n \in \Phi_j(\alpha^-)$. Define initial segment γ of length $\max\{\,|\alpha)|, |\beta|\,\}$ such that, for all $m < |\gamma|$,

$$\gamma(m) = \begin{cases} 0 & \text{if } \alpha(m)\downarrow = 0 \;\vee\; \beta(m)\downarrow = 0 \;\vee\; c_{B \oplus \omega}(m) = 0 \\ 1 & \text{otherwise.} \end{cases}$$

Then γ is a B-compatible extension of σ_t and $2n, 2n+1 \in \Phi_j(\gamma^-)$. Thus at stage $(t+1)$ the construction would prevent $\Phi_j(\overline{A})$ from being characteristic in contradiction with the hypothesis. $P_s \subseteq \Phi_i(A)$ is proved in a similar way. □

Corollary 4.12 *There exists a quasi-minimal se-degree.*

Corollary 4.13 *For any quasi-minimal se-degree* $\boldsymbol{b}$ *there exists a quasi-minimal se-degree* $\boldsymbol{a}$ *such that* $\boldsymbol{b} < \boldsymbol{a}$.

5 Jump Operators

We now consider the problem of defining the *jump operator* with respect to se-reducibility. By analogy with the *Turing jump* we will require that such an operator be derived from a map that sends any set A to a set A' that is ordered strictly above A by $\leq_{\mathrm{se}}$ and that, in addition, possesses certain hardness properties (relative to A). We begin with the observation that an "inverse" function can be defined for $\mathcal{D}_{\mathrm{se}}$, since for any sets X and Y, $X \leq_{\mathrm{se}} Y$ if and only if $\overline{X} \leq_{\mathrm{se}} \overline{Y}$. We then proceed with a reminder of some standard results in the study of enumeration reducibility.

Definition 5.1 $\mathrm{inv} : \mathcal{D}_{\mathrm{se}} \to \mathcal{D}_{\mathrm{se}}$ is defined to be the function such that, for any $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{D}_{\mathrm{se}}$, $\mathrm{inv}(\boldsymbol{x}) = \boldsymbol{y}$ if and only if $\boldsymbol{y} = \deg_{\mathrm{se}}(\overline{X})$ for some (or equivalently *all*) $X \in \boldsymbol{x}$. For any se-degree $\boldsymbol{a}$, the notation $\overline{\boldsymbol{a}}$ is shorthand for $\mathrm{inv}(\boldsymbol{a})$. Note that $\boldsymbol{a} \cup \overline{\boldsymbol{a}} = \deg_{\mathrm{se}}(A \oplus \overline{A})$ for any $A \in \boldsymbol{a}$.

Notation For any set A, K_A denotes the set $\{\, x \mid x \in \Phi_x(A)\,\}$ and J_A denotes the set $K_A \oplus \overline{K_A}$. Similarly, $J_A^{(k)}$ denotes the iterated form of J_A defined by $J_A^{(0)} = A$ and $J_A^{(k+1)} = J_{J_A^{(k)}}$.

Lemma 5.2 *For any set A,* K_A *is 1-complete for* $\mathbf{Enum}(A)$.

Lemma 5.3 *For any sets A and B,* $A \leq_{\mathrm{e}} B$ *if and only if* $A \leq_1 K_B$ *if and only if* $K_A \leq_1 K_B$.

Note 5.4 A jump operator on the enumeration degrees is defined by Cooper and McEvoy in [12] as the function induced by $X \mapsto J_X$. It follows from Lemma 5.3 that this function also gives rise to a well-defined operator over the se-degrees. We employ the term *e-jump* to refer to this operator and we use $\boldsymbol{a}_{\mathrm{se}}^{\diamond}$ to denote the e-jump of $\boldsymbol{a}_{\mathrm{se}}$.

Notation For any set A, H_A denotes the set $K_A \oplus K_{\overline{A}}$.

Lemma 5.5 *For any sets A and B, if $A \leq_{\mathrm{se}} B$ then $H_A \leq_1 H_B$.*

Proof Let A and B be any sets such that $A \leq_{\mathrm{se}} B$. Then, by definition, $A \leq_{\mathrm{e}} B$ and $\overline{A} \leq_{\mathrm{e}} \overline{B}$. Now apply Lemma 5.3. □

Lemma 5.6 *For any set A, $A <_{\mathrm{se}} H_A$.*

Proof Let A be any set. Then by Lemma 5.2 we know that $A \leq_{\mathrm{se}} H_A$. Also notice that $H_A \leq_{\mathrm{se}} A$ would imply $\overline{K_{\overline{A}}} \leq_{\mathrm{e}} \overline{A}$ from which we derive a contradiction. □

Note 5.7 If the set A has characteristic degree, then $A \equiv_{\mathrm{se}} \overline{A}$ by Proposition 4.3 and so $K_{\overline{A}} \equiv_1 K_A$ by Lemma 5.3. Thus Lemma 5.2 implies that $H_A \equiv_1 K_A$.

Lemma 5.8 *For any set A, H_A is 1-hard for* **Enum**(A)*. Moreover, if* $\deg_{\mathrm{se}}(A)$ *is characteristic, then H_A is 1-complete for* **Enum**(A).

Proof Let A be any set. Then Lemma 5.2 implies that H_A is 1-hard for **Enum**(A). Now suppose that $\deg_{\mathrm{se}}(A)$ is characteristic. Then H_A is 1-complete for **Enum**(A) by Note 5.7 and Lemma 5.2. □

Definition 5.9 Let $\boldsymbol{a}_{\mathrm{se}}$ be any se-degree. The *weak* jump of $\boldsymbol{a}_{\mathrm{se}}$ (written $\boldsymbol{a}_{\mathrm{se}}^{*}$) is defined to be $\deg_{\mathrm{se}}(H_A)$ for any A in $\boldsymbol{a}_{\mathrm{se}}$. We use $\boldsymbol{a}_{\mathrm{se}}^{**}$ to denote the *double weak* jump of $\boldsymbol{a}_{\mathrm{se}}$ (i.e., $\deg_{\mathrm{se}}(H_{H_A})$).

Proposition 5.10 *Suppose that* $\mathrm{r} \in \{\mathrm{e}, \mathrm{T}\}$. *Let $\boldsymbol{a}_{\mathrm{se}}$ be any se-degree, let $\boldsymbol{a}_{\mathrm{r}}$ be the r-degree of which it is a subclass, and let $\boldsymbol{d}_{\mathrm{r}}$ be the r-degree that contains $\boldsymbol{a}_{\mathrm{se}}^{**}$; then $\boldsymbol{a}_{\mathrm{r}} < \boldsymbol{d}_{\mathrm{r}}$. In other words, the double weak jump is strictly increasing relative to the relation induced by $\leq_{\mathrm{r}}$ over $\mathcal{D}_{\mathrm{se}}$.*

Proof Suppose that $\boldsymbol{a}_{\mathrm{se}} \subseteq \boldsymbol{a}_{\mathrm{e}}, \boldsymbol{a}_{\mathrm{T}}$ and pick any A in $\boldsymbol{a}_{\mathrm{se}}$. Then H_A is 1-hard for **Enum**(A) by Lemma 5.8 and this implies that $\boldsymbol{b}_{\mathrm{se}} \leq \boldsymbol{a}_{\mathrm{se}}^{*}$ for any se-degree, $\boldsymbol{b}_{\mathrm{se}} \subseteq \boldsymbol{a}_{\mathrm{e}}$. Note that by Lemma 5.6, $\boldsymbol{a}_{\mathrm{se}}^{*} < \boldsymbol{a}_{\mathrm{se}}^{**}$ and so $\boldsymbol{a}_{\mathrm{se}}^{**} \nsubseteq \boldsymbol{a}_{\mathrm{e}}$. Now let $\boldsymbol{c}_{\mathrm{se}}$ be the (unique) characteristic degree contained in $\boldsymbol{a}_{\mathrm{T}}$. Then $\boldsymbol{d}_{\mathrm{se}} \leq \boldsymbol{c}_{\mathrm{se}}$ for any $\boldsymbol{d}_{\mathrm{se}} \subseteq \boldsymbol{a}_{\mathrm{T}}$ (see Note 4.9). Also $A \oplus \overline{A} \leq_1 H_A$ and so $\boldsymbol{c}_{\mathrm{se}} \leq \boldsymbol{a}_{\mathrm{se}}^{*}$. Therefore, as above, $\boldsymbol{a}_{\mathrm{se}}^{**} \nsubseteq \boldsymbol{a}_{\mathrm{T}}$. □

Notation For any set A, S_A denotes the set $H_A \oplus \overline{H_A}$. Similarly, $S_A^{(k)}$ denotes the iterated form of S_A defined by $S_A^{(0)} = A$ and $S_A^{(k+1)} = S_{S_A^{(k)}}$.

Note 5.11 It follows from Lemma 5.5 and Lemma 5.6 that S induces a well-defined and strictly increasing operator over the se-degrees. Notice that, for any A, $J_A \leq_1 S_A \leq_1 S_{A\oplus\overline{A}}$ and if A has characteristic degree, $S_A \equiv_{\mathrm{m}} J_A$ (and so $S_{X\oplus\overline{X}} \equiv_{\mathrm{m}} J_{X\oplus\overline{X}}$ for all X).

Definition 5.12 Let $\boldsymbol{a}_{\mathrm{se}}$ be any se-degree. The (strong) jump of $\boldsymbol{a}_{\mathrm{se}}$ (written $\boldsymbol{a}_{\mathrm{se}}'$) is defined to be $\deg_{\mathrm{se}}(S_A)$ for any A in $\boldsymbol{a}_{\mathrm{se}}$. Thus $\boldsymbol{a}_{\mathrm{se}}' =_{\mathrm{def}} \boldsymbol{a}_{\mathrm{se}}^{*} \cup \mathrm{inv}(\boldsymbol{a}_{\mathrm{se}}^{*})$. The iterated jump of $\boldsymbol{a}_{\mathrm{se}}$ is written $\boldsymbol{a}_{\mathrm{se}}^{(k)}$ and is defined by $\boldsymbol{a}_{\mathrm{se}}^{(0)} = \boldsymbol{a}_{\mathrm{se}}$ and $\boldsymbol{a}_{\mathrm{se}}^{(k+1)} = (\boldsymbol{a}_{\mathrm{se}}^{(k)})'$.

Note 5.13 By Note 5.11 the jump is strictly increasing relative to the relation induced by $\leq_e$ over $\mathcal{D}_{se}$. On the other hand, the weak and strong jumps of any set are clearly contained in the same Turing degree. Notice that both jumps are defined in terms of putative symmetric enumeration operators in the sense of Lemma 3.2. Accordingly, they both reflect the separation of positive and negative information intrinsic to $\leq_{se}$. See Section 9 and Remark 5.14 below for further motivation behind the definition of the (strong) jump.

Notation Let $\boldsymbol{a}_{se}$ be any se-degree and A a set in $\boldsymbol{a}_{se}$. We refer to $\deg_{se}(S_{A\oplus\overline{A}})$ as the *embedded Turing jump* of $\boldsymbol{a}_{se}$ (written $\boldsymbol{a}^{\dagger}_{se}$).

Remark 5.14 The canonical embedding $\iota_{se} : \mathcal{D}_T \to \mathcal{D}_{se}$ (see Proposition 4.8) preserves the jump operation. Indeed, choose any Turing degree $\boldsymbol{a}_T$ and $A \in \boldsymbol{a}_T$. Then $H_{A\oplus\overline{A}} \in \boldsymbol{a}'_T$ as $\mathcal{K}_A \equiv_1 K_{A\oplus\overline{A}} \equiv_1 H_{A\oplus\overline{A}}$. Let $\boldsymbol{a}_{se} = \deg_{se}(A \oplus \overline{A})$ and note that, by definition, $\iota_{se}(\boldsymbol{a}_T) = \boldsymbol{a}_{se}$. Also,

$$\iota_{se}(\boldsymbol{a}'_T) = \iota_{se}(\deg_T(H_{A\oplus\overline{A}})) = \deg_{se}(H_{A\oplus\overline{A}} \oplus \overline{H_{A\oplus\overline{A}}}) = \boldsymbol{a}'_{se} .$$

Remark 5.15 The embedding $\varsigma_{se} : \mathcal{D}_e \to \mathcal{D}_{se}$ induced by the map $X \mapsto K_X$ is structure preserving, sends $\mathbf{0}_e$ to $\mathbf{0}^*_{se}$, and preserves infima.

Remark 5.16 Define the *Generalized Symmetric Enumeration* (GSE) Hierarchy relative to set A to be

$$\{\, \Sigma_n^{\mathrm{GSE},A}, \Pi_n^{\mathrm{GSE},A}, \Delta_n^{\mathrm{GSE},A} \ : \ n \geq 0 \,\}$$

where

$$\Sigma_0^{\mathrm{GSE},A} = \Pi_0^{\mathrm{GSE},A} = \Delta_0^{\mathrm{GSE},A} = \textbf{s-Enum}(A)$$

and, for $n \geq 0$, $\Sigma_{n+1}^{\mathrm{GSE},A} = \textbf{Enum}(S_A^{(n)})$, $\Pi_{n+1}^{\mathrm{GSE},A} = co\text{-}\Sigma_{n+1}^{\mathrm{GSE},A}$, and $\Delta_{n+1}^{\mathrm{GSE},A} = \textbf{s-Enum}(S_A^{(n)})$. If $A = \varnothing$, call this simply the *GSE* Hierarchy. Now we know that if $\deg_{se}(A)$ is characteristic, then for all $n \geq 0$, $\Sigma_{n+1}^A = \textbf{Enum}(S_A^{(n)})$ and $\Delta_{n+1}^A = \textbf{s-Enum}(S_A^{(n)})$ (see Corollary 7.2 below). In other words, if $\deg_{se}(A)$ is characteristic, the GSE Hierarchy and the Arithmetical Hierarchy relativized to A are identical. Thus, similarly to the SE Hierarchy—see [3], Section 6—the relativized GSE Hierarchy is a refinement of the relativized Arithmetical Hierarchy.

Remark 5.17 Let A and B be any sets. A is *partial many-one* reducible [2] to B ($A \leq_{pm} B$) if there exists a partial computable function $g(x)$ such that $x \in A$ if and only if $g(x)\downarrow \in B$. Let $\{\, f_n \mid n \in \omega \,\}$ be a computable enumeration of all unary partial computable functions. Define L_A to be the set $\{\, x \mid f_x(x)\downarrow \in A \,\}$ and define the weak jump of $\deg_m(A)$ to be the m-degree of the set $F_A = L_A \oplus L_{\overline{A}}$. Note that the function $h(x) = f_x(x)$ witnesses the reduction $L_A \leq_{pm} A$. Using standard methods it can be shown that $A \leq_{pm} B$ if and only if $A \leq_1 L_B$ and also that if $A \leq_m B$, then $F_A \leq_1 F_B$ whereas $F_A \not\leq_m A$. Moreover, Δ_2 is downward closed under $\leq_{pm}$ [2]. Thus, as in the proof of [14], Proposition XI.6.13, we can easily show that there is no maximal Δ_2 m-degree: if $A \in \Delta_2$, then $F_A \in \Delta_2$ since $F_A \leq_{pm} A \oplus \overline{A}$, whereas $A <_{pm} F_A$. Notice also that, by the same argument, for any set $C \in \Sigma_2 - \Delta_2$, $\deg_m(C)$ and $\deg_m(\overline{C})$ form an exact pair for the Δ_2 m-degrees and thus witness the fact that $\mathcal{D}_m$ is not a lattice. Finally, note that $T_A = F_A \oplus \overline{F_A}$ is arguably an appropriate definition for the derivation of a (strong) jump over $\mathcal{D}_m$.

6 Basic Properties of $\mathcal{D}_{\mathrm{se}}$

We know that $\mathcal{D}_{\mathrm{se}}$ is an upper semilattice and that the zero se-degree ($\mathbf{0}_{\mathrm{se}}$) is the class of computable sets (see Note 3.4). Also, the existence of an isomorphic embedding (ι_{se}) of the Turing degree structure $\mathcal{D}_{\mathrm{T}}$ into $\mathcal{D}_{\mathrm{se}}$ (Proposition 4.8) tells us—using results from [16] and [10]—that any countable partial ordering is embeddable in $\mathcal{D}_{\mathrm{se}}$ and that, in consequence, the one quantifier theory of $\mathcal{D}_{\mathrm{se}}$ is decidable. Of course, ι_{se} also preserves infima and suprema so any lattice embedding into $\mathcal{D}_{\mathrm{T}}$ is also a lattice embedding into $\mathcal{D}_{\mathrm{se}}$. In particular, as both M_3 and N_5 are embeddable in $\mathcal{D}_{\mathrm{T}}$ we know that $\mathcal{D}_{\mathrm{se}}$ is nondistributive. These observations suggest a certain resemblance between $\mathcal{D}_{\mathrm{se}}$ on the one side and $\mathcal{D}_{\mathrm{T}}$ and $\mathcal{D}_{\mathrm{e}}$ on the other. We now consider other basic properties of $\mathcal{D}_{\mathrm{se}}$ that further underline the similarities between these structures.

Definition 6.1 A degree $\boldsymbol{c}$ is said to be *branching* if there exist degrees $\boldsymbol{b}, \boldsymbol{a} \neq \boldsymbol{c}$ such that $\boldsymbol{b} \cap \boldsymbol{a} = \boldsymbol{c}$. If $\boldsymbol{c} = \mathbf{0}$ we say that $\boldsymbol{b}$ and $\boldsymbol{a}$ form a *minimal pair*.

Proposition 6.2 *For any se-degrees $\boldsymbol{b}$, $\boldsymbol{c}$ such that $\boldsymbol{c} \leq \boldsymbol{b}$ there exists an se-degree $\boldsymbol{a} \neq \boldsymbol{c}$ such that $\boldsymbol{b} \cap \boldsymbol{a} = \boldsymbol{c}$. Thus every se-degree is* branching.

Remark 6.3 The methods in the proof are adapted from those used by Rozinas [15] to prove the same result for the e-degrees.

Proof Choose $C \in \boldsymbol{c}$ and $B \in \boldsymbol{b}$ and let $\Phi_0, \Phi_1, \ldots$ be the computable listing of enumeration operators stipulated in Section 2. We construct a set A satisfying, for all $e, i \geq 0$, the requirements,

$$\begin{aligned}
R_{3e} \quad &: \quad A \neq \Phi_e(C) \\
R_{3\langle e,i\rangle+1} \quad &: \quad \Phi_e(A \oplus C) = \Phi_i(B) \;\Rightarrow\; \Phi_e(A \oplus C) \leq_{\mathrm{e}} C \\
R_{3\langle e,i\rangle+2} \quad &: \quad \Phi_e(\overline{A} \oplus \overline{C}) = \Phi_i(\overline{B}) \;\Rightarrow\; \Phi_e(\overline{A} \oplus \overline{C}) \leq_{\mathrm{e}} \overline{C}.
\end{aligned}$$

Note that the requirements R_{3e} ensure that $C <_{\mathrm{se}} A \oplus C$. Now consider any set $E \leq_{\mathrm{se}} B$ such that $E \leq_{\mathrm{se}} A \oplus C$. Then $E \leq_{\mathrm{e}} A \oplus C$ and $\overline{E} \leq_{\mathrm{e}} \overline{A} \oplus \overline{C}$ by definition of se-reducibility. So requirements $R_{3\langle e,i\rangle+1}$ force $E \leq_{\mathrm{e}} C$ and requirements $R_{3\langle e,i\rangle+2}$ force $\overline{E} \leq_{\mathrm{e}} \overline{C}$ or, in other words, $E \leq_{\mathrm{se}} C$.

The construction A is constructed by finite initial segments $\{\alpha_n\}_{n \geq 0}$ such that $A = \bigcup \{\alpha_n^+ \mid n \geq 0\}$.

Stage $s = 0$ $\quad \alpha_0 = \lambda$.

Stage $s + 1$ $\quad \alpha_s$ has already been defined. There are three cases to consider.

Case 1 $s = 3e$ for some $e \geq 0$. Let $a_s = |\alpha_s|$. Then we satisfy R_{3e} by defining α_{s+1} to be the extension of α_s of length $a_s + 1$ such that

$$\alpha_{s+1}(a_s) = 1 - \Phi_e(C)(a_s).$$

Case 2 $s = 3\langle e, i\rangle + 1$ for some $e, i \geq 0$. Then we try to vacuously satisfy $R_{3\langle e,i\rangle+1}$ by forcing an inequality in its premise. To do this we search for $x \geq 0$ and $\alpha \supseteq \alpha_s$ such that

$$x \in \Phi_e(\alpha^+ \oplus C) \;\&\; x \notin \Phi_i(B).$$

If this search is successful we pick the least such α and set $\alpha_{s+1} := \alpha$; otherwise, we set $\alpha_{s+1} := \alpha_s$.

Case 3 $s = \langle e, i\rangle+2$ for some $e, i \geq 0$. Then we try to vacuously satisfy $R_{3\langle e,i\rangle+2}$ by forcing an inequality in its premise. To do this we search for $x \geq 0$ and $\alpha \supseteq \alpha_s$ such that

$$x \in \Phi_e(\alpha^- \oplus \overline{C})\ \&\ x \notin \Phi_i(\overline{B}).$$

If this search is successful we pick the least such α and set $\alpha_{s+1} := \alpha$; otherwise, we set $\alpha_{s+1} := \alpha_s$.

Analysis of the construction The construction of A obviously ensures, via Case 1 above, that R_{3e} is satisfied for all $e \geq 0$. So we need only to show that $R_{3\langle e,i\rangle+1}$ and $R_{3\langle e,i\rangle+2}$ are both satisfied for all $e, i \geq 0$.

Claim 6.4 *For all $e, i \geq 0$ and $1 \leq k \leq 2$, $R_{3\langle e,i\rangle+k}$ is satisfied.*

Proof Fix e and i. We prove that $R_{3\langle e,i\rangle+2}$ is satisfied. (The case $k = 1$ is similar.) Accordingly, suppose that $\Phi_e(\overline{A} \oplus \overline{C}) = \Phi_i(\overline{B})$. Let $s = 3\langle e, i\rangle+2$. We show that, for all $x \geq 0$,

$$x \in \Phi_e(\overline{A} \oplus \overline{C}) \quad \text{iff} \quad (\exists \alpha \supseteq \alpha_s)(x \in \Phi_e(\alpha^- \oplus \overline{C}))$$

since this implies that $\Phi_e(\overline{A} \oplus \overline{C}) \leq_{\mathrm{e}} \overline{C}$.

($\Rightarrow$) Obvious.

($\Leftarrow$) Suppose that there is an $\alpha \supseteq \alpha_s$ such that $x \in \Phi_e(\alpha^- \oplus \overline{C})$ but that $x \notin \Phi_e(\overline{A} \oplus \overline{C})$. Then $x \notin \Phi_i(\overline{B})$ since $\Phi_e(\overline{A} \oplus \overline{C}) = \Phi_i(\overline{B})$ by hypothesis. Thus the construction at stage $3\langle e, i\rangle+2$ would ensure that $\Phi_e(\overline{A} \oplus \overline{C}) \neq \Phi_i(\overline{B})$, contradicting the hypothesis. $\lozenge$

This concludes the proof. $\square$

Corollary 6.5 *Each nonzero se-degree is part of a minimal pair.*

Definition 6.6 (Kleene and Post [6]) Two degrees $\boldsymbol{a}$ and $\boldsymbol{b}$ form an exact pair for a set of degrees $\mathcal{C}$ if the following two conditions hold.

(1) Both $\boldsymbol{a}$ and $\boldsymbol{b}$ are above all degrees in $\mathcal{C}$; that is,

$$(\forall \boldsymbol{c} \in \mathcal{C})(\boldsymbol{c} \leq \boldsymbol{a}\ \&\ \boldsymbol{c} \leq \boldsymbol{b}).$$

(2) Any degree $\boldsymbol{x}$ that is below $\boldsymbol{a}$ and $\boldsymbol{b}$ is also below some degree in $\mathcal{C}$; that is,

$$\boldsymbol{x} \leq \boldsymbol{a}\ \&\ \boldsymbol{x} \leq \boldsymbol{b} \ \Rightarrow\ (\exists \boldsymbol{c} \in \mathcal{C})(\boldsymbol{x} \leq \boldsymbol{c}).$$

Notation For any set A and $n \in \omega$ we define $A^{[n]} = \{\langle x, n\rangle \mid \langle x, n\rangle \in A\}$ and $A^{[\leq n]} = \bigcup\{A^{[m]} \mid m \leq n\}$. We combine this notation with that already described for strings in Section 2 (page 177). So, for example, for any string σ and $n \geq 0$,

$$(\sigma \upharpoonright \omega^{[\leq n]})^+ =_{\mathrm{def}} \{\langle x, m\rangle \mid \sigma(\langle x, m\rangle)\downarrow= 1\ \&\ 0 \leq m \leq n\ \&\ 0 \leq x\}.$$

For any countable class of sets $\{B_k\}_{k\geq 0}$ and $n \in \omega$, $\oplus_{m\leq n} B_m$ denotes the set $\{\langle x, m\rangle \mid m \leq n\ \&\ x \in B_m\}$.

Theorem 6.7 *Every countable set of se-degrees in which every pair of elements is bounded has an exact pair.*

Proof Suppose that $\{B_n\}_{n\geq 0}$ is a countable class of sets such that, for all $n, n' \geq 0$, there exists $m \geq 0$ such that $B_n \oplus B_{n'} \leq_{\mathrm{se}} B_m$ (†). Then we will construct sets A and B such that

(1) $B_m \leq_{\rm se} A, B$ for all $m \geq 0$;
(2) for any set E, $E \leq_{\rm se} A, B \Rightarrow E \leq_{\rm se} \oplus_{m \leq n} B_m$, for some $n \geq 0$.

Note that, for all $n \geq 0$, $\oplus_{m\leq n} B_m \leq_{\rm se} B_{n'}$ for some $n' \geq 0$ by assumption (†), and so the sets A and B witness the truth of the theorem. We first set

$$B \ := \ \{ \langle x, m\rangle \mid x, m \in \omega \ \& \ x \in B_m \}.$$

Remark 6.8 For any $n \in \omega$, $B^{[n]}$ is essentially a *copy* of B_n, $B^{[\leq n]} = \oplus_{m\leq n} B_m$, and $\overline{B}^{[\leq n]} = \oplus_{m\leq n} \overline{B}_m$.

Suppose that $\Phi_0, \Phi_1, \ldots$ is the computable listing of enumeration operators stipulated in Section 2. Accordingly, it will suffice to construct A so as to satisfy, for all $e, i, j \geq 0$, condition C_e and requirements $R_{2\langle i,j\rangle}$ and $R_{2\langle i,j\rangle+1}$ defined as follows:

$$\begin{array}{lll} C_e & : \ B_e \leq_{\rm se} A & \\ R_{2\langle i,j\rangle} & : \ \Phi_i(A) = \Phi_j(B) \ \Rightarrow \ \Phi_i(A) \leq_{\rm e} B^{[\leq n]} & \text{for some } n \geq 0 \\ R_{2\langle i,j\rangle+1} & : \ \Phi_i(\overline{A}) = \Phi_j(\overline{B}) \ \Rightarrow \ \Phi_i(\overline{A}) \leq_{\rm e} \overline{B}^{[\leq n']} & \text{for some } n' \geq 0\,. \end{array}$$

Indeed, let E be any set such that $E \leq_{\rm se} A, B$; then the even requirements imply that $E \leq_{\rm e} \oplus_{m\leq n} B_m$ and the odd requirements imply that $\overline{E} \leq_{\rm e} \oplus_{m\leq n'} \overline{B}_m$ for some $n, n' \geq 0$. Let $\hat{n} = \max\{n, n'\}$ and choose p such that $B_m \leq_{\rm se} B_p$ for all $m \leq \hat{n}$. Note that this is possible by assumption (†). Then $\oplus_{m\leq n} B_m \leq_{\rm e} B_p$ and $\oplus_{m\leq n'} \overline{B}_m \leq_{\rm e} \overline{B}_p$, which implies that $E \leq_{\rm e} B_p$ and $\overline{E} \leq_{\rm e} \overline{B}_p$. Thus $E \leq_{\rm se} B_p$.

On the other hand, condition C_e will be satisfied by coding B_e directly into the eth column of A. In effect, we ensure that, for all but finitely many $z \geq 0$,

$$z \in B_e \quad \text{iff} \quad \langle z, e\rangle \in A.$$

Thus $B_e \leq_1 A$.

The construction A is constructed by finite initial segments $\{\alpha_n\}_{n\geq 0}$ such that $A = \bigcup \{\alpha_n^+ \mid n \geq 0\}$.

Stage $s = 0$ $\alpha_0 = \lambda$.

Stage $s + 1$ α_s has already been defined.

Notation We say that an initial segment $\alpha \supseteq \alpha_s$ is B-s-compatible if, for all $n \geq 0$ and $e \leq s$,

$$|\alpha_s| \leq \langle n, e\rangle < |\alpha| \ \Rightarrow \ \alpha(\langle n, e\rangle) = B(\langle n, e\rangle)\,.$$

There are two cases to consider depending on whether s is even or odd.

Case 1 $s = 2\langle i, j\rangle$ for some $i, j \geq 0$. Then we try to vacuously satisfy $R_{2\langle i,j\rangle}$ by forcing an inequality. To do this, we search for $x \geq 0$ and B-s-compatible $\alpha \supseteq \alpha_s$ such that

$$x \in \Phi_i(\alpha^+) \quad \text{whereas} \quad x \notin \Phi_j(B).$$

If this search is successful, we pick the least such α and we set $\alpha_{s+1} := \alpha^\frown(B(|\alpha|))$; otherwise, we set $\alpha_{s+1} := \alpha_s{}^\frown(B(|\alpha_s|))$.

Case 2 $s = 2\langle i, j\rangle + 1$ for some $i, j \geq 0$. Then we try to vacuously satisfy $R_{2\langle i,j\rangle+1}$ by searching for $x \geq 0$ and B-s-compatible $\alpha \supseteq \alpha_s$ such that

$$x \in \Phi_i(\alpha^-) \quad \text{whereas} \quad x \notin \Phi_j(\overline{B}).$$

If this search is successful, we pick the least such α and we set $\alpha_{s+1} := \alpha^\frown(B(|\alpha|))$; otherwise, we set $\alpha_{s+1} := \alpha_s{}^\frown(B(|\alpha_s|))$.

Analysis of the construction First, for any e, it is easy to see that C_e is satisfied since the construction obviously forces $A(\langle z, e\rangle) = B_e(z)$ for all but finitely many z. So we just need to show that both the requirements $R_{2\langle i,j\rangle}$ and $R_{2\langle i,j\rangle+1}$ are satisfied for all $i, j \geq 0$.

Claim 6.9 *For all $i, j \geq 0$ and $0 \leq k \leq 1$, $R_{2\langle i,j\rangle+k}$ is satisfied.*

Proof Fix i and j, let $(k, \widetilde{A}, \widetilde{B}, *) \in \{(0, A, B, +), (1, \overline{A}, \overline{B}, -)\}$, and suppose that $\Phi_i(\widetilde{A}) = \Phi_j(\widetilde{B})$. Let $s = 2\langle i, j\rangle + k$ and define the set

$$P_s \;:=\; \{\, x \mid (\exists \alpha \supseteq \alpha_s)(\, x \in \Phi_i(\alpha^*) \;\&\; (\,(\alpha - \alpha_s)\restriction \omega^{[\leq s]}\,)^* \subseteq \widetilde{B}^{[\leq s]}\;)\,\}\,.$$

Clearly, $P_s \leq_{\mathrm{e}} \widetilde{B}^{[\leq s]}$ and so, to show that $R_{2\langle i,j\rangle+k}$ is satisfied, it suffices to prove that, for all $x \geq 0$,

$$x \in \Phi_i(\widetilde{A}) \;\Leftrightarrow\; x \in P_s.$$

($\Rightarrow$) If $x \in \Phi_i(\widetilde{A})$ then $x \in \Phi(\alpha^*)$ for some $\alpha \subseteq c_A$ such that $\alpha \supseteq \alpha_s$. Pick $t \geq s+1$ large enough so that $\alpha \subseteq \alpha_t$. Then α_t is B-s-compatible since, for all $r \geq s$, α_{r+1} is B-r-compatible. However, this implies that $((\alpha_t - \alpha_s)\restriction \omega^{[\leq s]})^* \subseteq \widetilde{B}^{[\leq s]}$ and so $((\alpha - \alpha_s)\restriction \omega^{[\leq s]})^* \subseteq \widetilde{B}^{[\leq s]}$, since $((\alpha - \alpha_s)\restriction \omega^{[\leq s]})^* \subseteq ((\alpha_t - \alpha_s)\restriction \omega^{[\leq s]})^*$. Thus $x \in P_s$.

($\Leftarrow$) Suppose that $x \in P_s$ but that $x \notin \Phi_i(\widetilde{A})$. Then $x \notin \Phi_j(\widetilde{B})$ since $\Phi_i(\widetilde{A}) = \Phi_j(\widetilde{B})$ by hypothesis. Now, by definition of P_s, we know that $x \in \Phi_i(\alpha^*)$ for some $\alpha \supseteq \alpha_s$ such that $((\alpha - \alpha_s)\restriction \omega^{[\leq s]})^* \subseteq \widetilde{B}^{[\leq s]}$. So define $\hat{\alpha}$ of length $|\alpha|$ such that, for all $y < |\alpha|$,

$$\hat{\alpha}(y) = \begin{cases} \alpha_s(y) & \text{if } y < |\alpha_s| \\ (B^{[\leq s]})(y) & \text{if } y \geq |\alpha_s| \text{ and } y \in \omega^{[\leq s]} \\ \alpha(y) & \text{otherwise.} \end{cases} \tag{1}$$

It is easy to see that $\alpha^* \subseteq \hat{\alpha}^*$ and that $\hat{\alpha}$ is B-s-compatible. Therefore, $\hat{\alpha}$ would bear witness to the fact that $\Phi_i(\widetilde{A}) \neq \Phi_j(\widetilde{B})$ at stage $s + 1$, contradicting the hypothesis. $\Diamond$

This concludes the proof. $\square$

Proposition 6.10 *$\mathcal{D}_{\mathrm{se}}$ is not a lattice.*

Proof Consider any strictly ascending sequence $\mathcal{S}$ of se-degrees. Then by Theorem 6.7, $\mathcal{S}$ has an exact pair $\boldsymbol{a}$ and $\boldsymbol{b}$. Thus $\boldsymbol{a}$ and $\boldsymbol{b}$ do not have a greatest lower bound. $\square$

Remark 6.11 It is readily seen that if Turing degrees $\boldsymbol{a}_{\mathrm{T}}$ and $\boldsymbol{b}_{\mathrm{T}}$ do not have an infinum, then the images $\imath_{\mathrm{se}}(\boldsymbol{a}_{\mathrm{T}})$ and $\imath_{\mathrm{se}}(\boldsymbol{b}_{\mathrm{T}})$ under the canonical embedding of $\mathcal{D}_{\mathrm{T}}$ in $\mathcal{D}_{\mathrm{se}}$ (Proposition 4.8) also do not have an infinum. Therefore, Proposition 6.10 follows from Spector's (exact pair) Theorem for $\mathcal{D}_{\mathrm{T}}$ ([6], [9], [20]). Similarly, Proposition 6.10 may also be seen as a corollary to Proposition 7.7 below.

7 CEA and Co-CEA Substructures of $\mathcal{D}_{\mathrm{se}}$

By Proposition 4.8, the substructure of $\mathcal{D}_{\mathrm{se}}$ induced by the set of characteristic degrees is an isomorphic copy of $\mathcal{D}_{\mathrm{T}}$. In this sense each characteristic se-degree is in effect an *embedded* Turing degree. We now show that, for any given Turing degree $\boldsymbol{a}_{\mathrm{T}}$, there is a specific substructure of $\mathcal{D}_{\mathrm{T}}$ local to $\boldsymbol{a}_{\mathrm{T}}$ which has two isomorphic copies local to the embedded image of $\boldsymbol{a}_{\mathrm{T}}$ (under ι_{se}) in $\mathcal{D}_{\mathrm{se}}$. In consequence, in Sections 8 and 9, we will be able to apply results from the literature on $\mathcal{D}_{\mathrm{T}}$ (via Proposition 7.7) to prove structural and definability properties of $\mathcal{D}_{\mathrm{se}}$. First, however, we show (Corollary 7.2) that standard arithmetical notions are well defined relative to the embedded Turing degrees in $\mathcal{D}_{\mathrm{se}}$.

Lemma 7.1 (McEvoy [11]) *Suppose that A is a total set (i.e., $\overline{A} \leq_{\mathrm{e}} A$). Then for all $n \geq 0$, $\Sigma^A_{n+1} = \mathbf{Enum}(J^{(n)}_A)$.*

Corollary 7.2 *Suppose that A is a set of characteristic se-degree (i.e., $A \equiv_{\mathrm{se}} \overline{A}$). Then for all $n \geq 0$,*

(a) $\Sigma^A_{n+1} = \mathbf{Enum}(S^{(n)}_A)$,
(b) $\Delta^A_{n+1} = \mathbf{s\text{-}Enum}(S^{(n)}_A)$.

Proof By Note 5.11 and a simple induction, $S^{(n)}_A \equiv_{\mathrm{se}} J^{(n)}_A$ for all $n \geq 0$. Thus (a) is immediate by Lemma 7.1. To prove (b) note first that $S^{(0)}_A =_{\mathrm{def}} A$ (and $A \equiv_{\mathrm{se}} \overline{A}$ by hypothesis) and that $S^{(m+1)}_A$ is characteristic for all $m \geq 0$. Thus, for all $n \geq 0$,

$$\begin{aligned}
\Delta^A_{n+1} &= \{\, B \mid B \leq_{\mathrm{e}} S^{(n)}_A \ \&\ \overline{B} \leq_{\mathrm{e}} S^{(n)}_A \,\} \\
&= \{\, B \mid B \leq_{\mathrm{e}} S^{(n)}_A \ \&\ \overline{B} \leq_{\mathrm{e}} \overline{S^{(n)}_A} \,\} \\
&=_{\mathrm{def}} \mathbf{s\text{-}Enum}(S^{(n)}_A)\,.
\end{aligned}$$

□

Notation Let $\Gamma \in \{\Sigma, \Pi, \Delta\}$. Suppose that $\boldsymbol{u}$ is a characteristic se-degree. Then $\Gamma^{\boldsymbol{u}}_n$ denotes the class $\{\, \boldsymbol{a} \mid (\exists A \in \boldsymbol{a})(\exists U \in \boldsymbol{u})[\, A \in \Gamma^U_n \,]\,\}$. We will use the notation $\Sigma^{\boldsymbol{u}}_n \cup \Pi^{\boldsymbol{u}}_n$ with obvious meaning and the shorthand Γ_n for the class $\Gamma^{\mathbf{0}}_n$. If $\boldsymbol{v}$ is a Turing degree, we use $\Sigma^{\boldsymbol{v}}_n$ and $\Delta^{\boldsymbol{v}}_n$ in a similar manner (in the context of $\mathcal{D}_{\mathrm{T}}$).

Remark 7.3 Suppose that $\boldsymbol{u}$ is a characteristic se-degree. Since for any sets X and Y, $X \leq_{\mathrm{se}} Y$ if and only if $\overline{X} \leq_{\mathrm{se}} \overline{Y}$, it is easily seen that for any $n \geq 0$ and $\Gamma \in \{\Sigma, \Pi, \Delta\}$, $\boldsymbol{a} \in \Gamma^{\boldsymbol{u}}_n$ if and only if $A \in \Gamma^U_n$ for *all* $A \in \boldsymbol{a}$ and $U \in \boldsymbol{u}$.

Definition 7.4 Let $\boldsymbol{a}_{\mathrm{T}}$ be any Turing degree and $\boldsymbol{b}_{\mathrm{se}}$ any characteristic se-degree. Then $\mathcal{CEA}_{\mathrm{T}}(\boldsymbol{a}_{\mathrm{T}})$ is defined to be the substructure of $\mathcal{D}_{\mathrm{T}}$ generated by the set

$$\{\, \boldsymbol{d}_{\mathrm{T}} \mid \boldsymbol{a}_{\mathrm{T}} \leq \boldsymbol{d}_{\mathrm{T}} \ \&\ \boldsymbol{d}_{\mathrm{T}} \in \Sigma^{\boldsymbol{a}_{\mathrm{T}}}_1 \,\}\,.$$

Likewise, $\mathcal{CEA}_{\mathrm{se}}(\boldsymbol{b}_{\mathrm{se}})$ and co-$\mathcal{CEA}_{\mathrm{se}}(\boldsymbol{b}_{\mathrm{se}})$ are defined to be the substructures of $\mathcal{D}_{\mathrm{se}}$ generated by the sets

$$\{\, \boldsymbol{d}_{\mathrm{se}} \mid \boldsymbol{b}_{\mathrm{se}} \leq \boldsymbol{d}_{\mathrm{se}} \ \&\ \boldsymbol{d}_{\mathrm{se}} \in \Gamma^{\boldsymbol{b}_{\mathrm{se}}}_1 \,\}$$

for $\Gamma \in \{\Sigma, \Pi\}$, respectively. We use $\mathcal{E}_{\mathrm{T}}$, $\mathcal{E}_{\mathrm{se}}$, and co-$\mathcal{E}_{\mathrm{se}}$ as shorthand for the structures $\mathcal{CEA}_{\mathrm{T}}(\mathbf{0}_{\mathrm{T}})$, $\mathcal{CEA}_{\mathrm{se}}(\mathbf{0}_{\mathrm{se}})$, and co-$\mathcal{CEA}_{\mathrm{se}}(\mathbf{0}_{\mathrm{se}})$.

Proposition 7.5 *Let A be any set and let $\boldsymbol{a}_{\mathrm{T}} = \deg_{\mathrm{T}}(A)$ and $\boldsymbol{a}_{\mathrm{se}} = \deg_{\mathrm{se}}(A \oplus \overline{A})$ (i.e., the unique characteristic se-degree contained in $\boldsymbol{a}_{\mathrm{T}}$). Then*

$$\mathcal{CEA}_{\mathrm{T}}(\boldsymbol{a}_{\mathrm{T}}) \;\cong\; \mathcal{CEA}_{\mathrm{se}}(\boldsymbol{a}_{\mathrm{se}}) \;\cong\; \text{co-}\mathcal{CEA}_{\mathrm{se}}(\boldsymbol{a}_{\mathrm{se}})\,.$$

Proof The isomorphism $\mathcal{CEA}_{\mathrm{se}}(\boldsymbol{a}_{\mathrm{se}}) \cong \text{co-}\mathcal{CEA}_{\mathrm{se}}(\boldsymbol{a}_{\mathrm{se}})$ is witnessed by the restriction to $\mathcal{CEA}_{\mathrm{se}}(\boldsymbol{a}_{\mathrm{se}})$ of the inverse map $\mathrm{inv} : \mathcal{D}_{\mathrm{se}} \to \mathcal{D}_{\mathrm{se}}$ (see Definition 5.1). Thus it suffices to prove that $\mathcal{CEA}_{\mathrm{T}}(\boldsymbol{a}_{\mathrm{T}}) \cong \mathcal{CEA}_{\mathrm{se}}(\boldsymbol{a}_{\mathrm{se}})$. Consider any sets A, B, C such that $A \in \boldsymbol{a}_{\mathrm{T}}$, $A \leq_{\mathrm{T}} B, C$, and $B, C \in \Sigma_1^A$. Note that this last condition implies that $B, C \leq_{\mathrm{e}} A \oplus \overline{A}$. Then

$$\begin{array}{lll}
B \leq_{\mathrm{T}} C & & \\
\quad \text{iff} & B \oplus A \leq_{\mathrm{T}} C \oplus A & \\
\quad \text{iff} & (B \oplus \overline{B}) \oplus (A \oplus \overline{A}) \leq_{\mathrm{e}} (C \oplus \overline{C}) \oplus (A \oplus \overline{A}) & \text{by Lemma 4.6,} \\
\quad \text{iff} & \overline{B} \oplus (A \oplus \overline{A}) \leq_{\mathrm{e}} \overline{C} \oplus (A \oplus \overline{A}) & \text{as } B, C \leq_{\mathrm{e}} A \oplus \overline{A}, \\
\quad \text{iff} & \overline{B} \oplus (A \oplus \overline{A}) \leq_{\mathrm{se}} \overline{C} \oplus (A \oplus \overline{A}) & \text{since } B \leq_{\mathrm{e}} A \oplus \overline{A}, \\
\quad \text{iff} & B \oplus (\overline{A} \oplus A) \leq_{\mathrm{se}} C \oplus (\overline{A} \oplus A) & \text{by symmetry of } \leq_{\mathrm{se}}, \\
\quad \text{iff} & B \oplus (A \oplus \overline{A}) \leq_{\mathrm{se}} C \oplus (A \oplus \overline{A}) & \text{as } A \oplus \overline{A} \equiv_{\mathrm{se}} \overline{A} \oplus A.
\end{array}$$

Moreover, for any set $\widehat{B}$ such that $A \oplus \overline{A} \leq_{\mathrm{se}} \widehat{B}$, obviously $\widehat{B} \equiv_{\mathrm{se}} \widehat{B} \oplus (A \oplus \overline{A})$ and $A \leq_{\mathrm{T}} \widehat{B}$. Thus the map $F : \deg_{\mathrm{T}}(X) \mapsto \deg_{\mathrm{se}}(X \oplus (A \oplus \overline{A}))$ witnesses the isomorphism $\mathcal{CEA}_{\mathrm{T}}(\boldsymbol{a}_{\mathrm{T}}) \cong \mathcal{CEA}_{\mathrm{se}}(\boldsymbol{a}_{\mathrm{se}})$. □

Corollary 7.6 $\mathcal{E}_{\mathrm{T}} \cong \mathcal{E}_{\mathrm{se}} \cong \text{co-}\mathcal{E}_{\mathrm{se}}$.

Proposition 7.7 *Let $\boldsymbol{u}$ be a characteristic se-degree. Then the two structures $\mathcal{CEA}_{\mathrm{se}}(\boldsymbol{u})$ and co-$\mathcal{CEA}_{\mathrm{se}}(\boldsymbol{u})$ are nontrivial, dense, nondistributive upper semilattices with bottom element $\boldsymbol{u}$ and top element $\boldsymbol{u}^*$ and $\mathrm{inv}(\boldsymbol{u}^*)$, respectively. Neither structure is a lattice.*

Proof Choose $U \in \boldsymbol{u}$ and let $\boldsymbol{u}_{\mathrm{T}} = \deg_{\mathrm{T}}(U)$. Notice that $K_U \equiv_{\mathrm{se}} H_U$ as $\boldsymbol{u}$ is characteristic, and hence $\boldsymbol{u}^*$ and $\mathrm{inv}(\boldsymbol{u}^*)$ are in $\mathcal{CEA}_{\mathrm{se}}(\boldsymbol{u})$ and co-$\mathcal{CEA}_{\mathrm{se}}(\boldsymbol{u})$, respectively. Also $U \leq_1 K_U$ and $U \equiv_{\mathrm{se}} \overline{U} \leq_1 \overline{K_U}$ whereas $K_U \not\leq_{\mathrm{se}} U$ (as $K_U \equiv_{\mathrm{se}} H_U$) and $\overline{K_U} \not\leq_{\mathrm{se}} U$ (as $\overline{K_U} \not\leq_{\mathrm{e}} U$). Therefore, $\boldsymbol{u} < \boldsymbol{u}^*$ and $\boldsymbol{u} < \mathrm{inv}(\boldsymbol{u}^*)$. Nontriviality is immediate. Note that $\mathcal{CEA}_{\mathrm{T}}(\boldsymbol{u}_{\mathrm{T}})$ is dense by the relativized version of Sacks density theorem for $\mathcal{E}_{\mathrm{T}}$ [17]. It follows, by Proposition 7.5, that both $\mathcal{CEA}_{\mathrm{se}}(\boldsymbol{u}_{\mathrm{T}})$ and co-$\mathcal{CEA}_{\mathrm{se}}(\boldsymbol{u}_{\mathrm{T}})$ are dense. Likewise, both structures are nondistributive since N_5 is embeddable into $\mathcal{CEA}_{\mathrm{T}}(\boldsymbol{u}_{\mathrm{T}})$ [8] and neither structure is a lattice since $\mathcal{CEA}_{\mathrm{T}}(\boldsymbol{u}_{\mathrm{T}})$ contains a pair of degrees without infinum ([7], [21]).

If any set X is c.e. in U then $X \leq_1 K_U$ and if X is co-c.e. in U then $X \leq_1 \overline{K_U}$. So $\boldsymbol{u}^*$ and $\mathrm{inv}(\boldsymbol{u}^*)$ are the top elements of $\mathcal{CEA}_{\mathrm{se}}(\boldsymbol{u})$ and co-$\mathcal{CEA}_{\mathrm{se}}(\boldsymbol{u})$, respectively. □

Remark 7.8 Every total enumeration degree contains infinitely many se-degrees. Indeed, if $\boldsymbol{a}_{\mathrm{e}}$ is a total enumeration degree then $\boldsymbol{a}_{\mathrm{e}}$ not only contains a (unique) characteristic se-degree $\boldsymbol{a}_{\mathrm{se}}$ (say) but also its weak jump $\boldsymbol{a}^*_{\mathrm{se}}$. Thus $\boldsymbol{a}_{\mathrm{e}}$ also contains the set $\{\, \boldsymbol{b}_{\mathrm{se}} \mid \boldsymbol{a}_{\mathrm{se}} < \boldsymbol{b}_{\mathrm{se}} < \boldsymbol{a}^*_{\mathrm{se}} \,\}$ which we know to be infinite by Proposition 7.7.

8 Diamond Embeddings and Minimal Covers

Kalimullin defined the notion of a *U-e-ideal pair* in [5] and used it to show that the (enumeration) jump is definable in $\mathcal{D}_e$. It turns out that Kalimullin's notion can be symmetrized (Definition 8.2) and used as a tool in the context of the se-degrees. In effect, by defining the notion of a *U-se-ideal pair*, and applying results from [4], we are able to prove a diamond theorem for $\mathcal{D}_{se}$ similar to the result proved for $\mathcal{D}_e$ by Arslanov, Kalimullin, and Cooper (see [1], Theorem 6). We also show that every nonzero Turing degree contains at least two minimal se-degrees and we generalize this result.

Reminder For any sets X, Y, $\overline{X \oplus Y} = \overline{X} \oplus \overline{Y}$.

Definition 8.1 (Kalimullin [5])

(a) A pair of sets A and B is *e-ideal* if there is a c.e. set W such that $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$.

(b) For any set U, a pair of sets A and B is *U-e-ideal* if there is a set $W \leq_e U$ such that $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$.

Definition 8.2

(a) A pair of sets A and B is *se-ideal* if both (A, B) is e-ideal and $(\overline{A}, \overline{B})$ is e-ideal.

(b) For any set U, a pair of sets A and B is *U-se-ideal* if (A, B) is U-e-ideal and $(\overline{A}, \overline{B})$ is $\overline{U}$-e-ideal.

Proposition 8.3 *For any sets A, B and U, if $A \leq_e U$ and $\overline{B} \leq_e \overline{U}$, then the pair of sets (A, B) is U-se-ideal.*

Proof Suppose that $A \leq_e U$ and $\overline{B} \leq_e \overline{U}$. Define $M = A \times \omega$ and $N = \omega \times \overline{B}$. Then $M \leq_e U$ and $N \leq_e \overline{U}$. Also, for any sets X, Y, $A \times X \subseteq M$ and $\overline{A} \times \overline{X} \subseteq \overline{M}$ whereas $Y \times \overline{B} \subseteq N$ and $\overline{Y} \times B \subseteq \overline{N}$. Thus $A \times B \subseteq M$, $\overline{A} \times \overline{B} \subseteq \overline{M}$ and $\overline{A} \times \overline{B} \subseteq N$, $A \times B \subseteq \overline{N}$. □

Observe that the notion of a U-se-ideal pair is not ordered. So it would be redundant to add the case $\overline{A} \leq_e \overline{U}$ and $B \leq_e U$ in the formulation of Proposition 8.3. Similar considerations apply to the results below.

Corollary 8.4 *If A is a c.e. set and B is a co-c.e. set, then (A, B) is se-ideal.*

Lemma 8.5 *If A and U are sets such that $A \leq_{se} U$ then, for every set B, the pair of sets (A, B) is U-se-ideal.*

Proof Similar to proof of Proposition 8.3 but with $M = A \times \omega$ and $N = \overline{A} \times \omega$. □

Lemma 8.6 *For any sets A and U, if the pair (A, A) is U-se-ideal, then $A \leq_{se} U$.*

Proof Let $M \leq_e U$ and $N \leq_e \overline{U}$ be sets such that $A \times A \subseteq M$ and $\overline{A} \times \overline{A} \subseteq \overline{M}$ whereas $\overline{A} \times \overline{A} \subseteq N$ and $A \times A \subseteq \overline{N}$. Then clearly the function $f(x) = \langle x, x \rangle$ witnesses both $A \leq_1 M$ and $\overline{A} \leq_1 N$. Hence $A \leq_e U$ and $\overline{A} \leq_e \overline{U}$. □

Definition 8.7 (Jockusch [4]) A set A is *semirecursive* if there is a computable function f of two variables such that, for every x and y,

(1) $f(x, y) \in \{x, y\}$,
(2) $\{x, y\} \cap A \neq \varnothing \Rightarrow f(x, y) \in A$.

In this case f is called a *selector function* for A.

Remark 8.8 A is semirecursive if and only if $\overline{A}$ is semirecursive.

Lemma 8.9 *If A is semirecursive, the pair $(A, \overline{A})$ is se-ideal.*

Proof Suppose that f is a selector function for A. Define

$$W \;=\; \{\, \langle x, y\rangle \mid f(x, y) = x \,\}\,.$$

Then both W and $\overline{W}$ are c.e. and $A \times \overline{A} \subseteq W$ and $\overline{A} \times A \subseteq \overline{W}$. It follows that $(A, \overline{A})$ is e-ideal via W whereas $(\overline{A}, A)$ is e-ideal via $\overline{W}$. □

Theorem 8.10 (Jockusch [4]) *For any noncomputable set A there is a semirecursive set $B \equiv_{\mathrm{T}} A$ such that neither B nor $\overline{B}$ is computably enumerable.*

Lemma 8.11 *For any sets A, B and U, if the pair A, B forms a U-se-ideal pair, and $C \leq_{\mathrm{se}} A$, then the pair of sets C, B also forms a U-se-ideal pair.*

Proof Suppose that $M \leq_{\mathrm{e}} U$ and $N \leq_{\mathrm{e}} \overline{U}$ are sets witnessing the fact that (A, B) is U-se-ideal; that is,

$$A \times B \subseteq M \quad \text{and} \quad \overline{A} \times \overline{B} \subseteq \overline{M},$$
$$\overline{A} \times \overline{B} \subseteq N \quad \text{and} \quad A \times B \subseteq \overline{N}.$$

Let Φ and Ψ be enumeration operators such that $C = \Phi(A)$ and $\overline{C} = \Psi(\overline{A})$. Define

$$M' \;=\; \{\, \langle n, m\rangle \mid \exists D[\, n \in \Phi(D) \;\&\; (\forall z \in D)[\, \langle z, m\rangle \in M\,]\,]\,\}$$
$$N' \;=\; \{\, \langle n, m\rangle \mid \exists D[\, n \in \Psi(D) \;\&\; (\forall z \in D)[\, \langle z, m\rangle \in N\,]\,]\,\}$$

where D (as usual) ranges over finite sets. Notice that $M' \leq_{\mathrm{e}} M$ and $N' \leq_{\mathrm{e}} N$; for example, the c.e. set $\{\, \langle\, \langle n, m\rangle\,,\, \{\langle z, m\rangle \mid z \in D\}\,\rangle \mid n \in \Phi(D)\,\}$ witnesses the reduction $M' \leq_{\mathrm{e}} M$.

Claim 8.12 $C \times B \subseteq M'$ *and* $\overline{C} \times \overline{B} \subseteq \overline{M'}$.

Claim 8.13 $\overline{C} \times \overline{B} \subseteq N'$ *and* $C \times B \subseteq \overline{N'}$.

Proof We prove Claim 8.12. Claim 8.13 is proved in a similar manner.

1. Suppose that $\langle n, m\rangle \in C \times B$. Then $n \in \Phi(D)$ for some finite set $D \subseteq A$ and so, for all $z \in D$, $\langle z, m\rangle \in A \times B \subseteq M$. Hence $\langle n, m\rangle \in M'$.
2. Suppose that $\langle n, m\rangle \in \overline{C} \times \overline{B}$. Consider any finite set D such that $n \in \Phi(D)$. Then, as $C = \Phi(A)$ there exists some $z \in D$ such that $z \in \overline{A}$ and so $\langle z, m\rangle \in \overline{A} \times \overline{B} \subseteq \overline{M}$. Hence $\langle n, m\rangle \notin M'$. ◊

Thus sets C, B form a U-se-ideal pair. □

Remark 8.14 Lemma 8.11 is also a corollary of Theorem 8.21 below.

Corollary 8.15 *The notion of a U-se-ideal pair is invariant under se-equivalence (for any set U).*

Definition 8.16 We say that a pair of se-degrees $\boldsymbol{a}$ and $\boldsymbol{b}$ is $\boldsymbol{u}$*-se-ideal* for an se-degree $\boldsymbol{u}$ if the pair (A, B) is U-se-ideal for some—or equivalently *any*—sets $A \in \boldsymbol{a}$, $B \in \boldsymbol{b}$, and $U \in \boldsymbol{u}$.

Lemma 8.17 *For any se-degrees $\boldsymbol{a}$ and $\boldsymbol{u}$ the set*

$$\mathcal{I}(\boldsymbol{u},\boldsymbol{a}) = \{\, \boldsymbol{b} \in \mathcal{D}_{\mathrm{se}} \mid (\boldsymbol{a},\boldsymbol{b}) \text{ is } \boldsymbol{u}\text{-se-ideal} \,\}$$

is an ideal in $\mathcal{D}_{\mathrm{se}}$.

Proof Suppose that $(\boldsymbol{a},\boldsymbol{b})$ is $\boldsymbol{u}$-se-ideal and $\boldsymbol{d} \leq \boldsymbol{b}$. Then it follows from Lemma 8.11 that $(\boldsymbol{a},\boldsymbol{d})$ is $\boldsymbol{u}$-se-ideal. Now suppose that $(\boldsymbol{a},\boldsymbol{c})$ is also $\boldsymbol{u}$-se-ideal (i.e., both $\boldsymbol{b}$ and $\boldsymbol{c}$ are in $\mathcal{I}(\boldsymbol{u},\boldsymbol{a})$). Choose sets $A \in \boldsymbol{a}$, $B \in \boldsymbol{b}$, $C \in \boldsymbol{c}$, and $U \in \boldsymbol{u}$. By definition there exist sets $M_b, M_c \leq_{\mathrm{e}} U$ and $N_b, N_c \leq_{\mathrm{e}} \overline{U}$ such that

$$A \times B \subseteq M_b\,,\ \overline{A} \times \overline{B} \subseteq \overline{M_b} \quad \text{and} \quad \overline{A} \times \overline{B} \subseteq N_b\,,\ A \times B \subseteq \overline{N_b}\,,$$
$$A \times C \subseteq M_c\,,\ \overline{A} \times \overline{C} \subseteq \overline{M_c} \quad \text{and} \quad \overline{A} \times \overline{C} \subseteq N_c\,,\ A \times C \subseteq \overline{N_c}\,.$$

Now define

$$M = \{\, \langle n, 2m\rangle \mid \langle n,m\rangle \in M_b \,\} \bigcup \{\, \langle n, 2m+1\rangle \mid \langle n,m\rangle \in M_c \,\}\,,$$
$$N = \{\, \langle n, 2m\rangle \mid \langle n,m\rangle \in N_b \,\} \bigcup \{\, \langle n, 2m+1\rangle \mid \langle n,m\rangle \in N_c \,\}\,,$$

and notice that $M \leq_{\mathrm{e}} U$ and $N \leq_{\mathrm{e}} \overline{U}$. Also it is straightforward to check that

$$A \times (B \oplus C) \subseteq M \quad \text{and} \quad \overline{A} \times (\overline{B} \oplus \overline{C}) \subseteq \overline{M},$$
$$\overline{A} \times (\overline{B} \oplus \overline{C}) \subseteq N \quad \text{and} \quad A \times (B \oplus C) \subseteq \overline{N}.$$

Therefore, the pair $(\boldsymbol{a}, \boldsymbol{b} \cup \boldsymbol{c})$ is $\boldsymbol{u}$-se-ideal. □

Theorem 8.18 (Kalimullin [5]) *Let A, B be a pair of sets that is not U-e-ideal and let $\{F_x, E_x\}_{x\in\omega}$ be a computable enumeration of all pairs of finite sets. Then there exist sets $X, Y \leq_{\mathrm{T}} A \oplus B \oplus K_U$ such that*

$$Y = \{\, z \mid z \in X \ \&\ F_z \subseteq A \,\} = \{\, z \mid z \in X \ \&\ E_z \subseteq B \,\}$$

(so $Y \leq_{\mathrm{e}} X \oplus A$ and $Y \leq_{\mathrm{e}} X \oplus B$) and $Y \not\leq_{\mathrm{e}} X \oplus U$.

Proof See Theorem 2.5 and its proof in [5]. □

Corollary 8.19 *Let A, B be a pair of sets that is not U-se-ideal. Then there exist sets $X, Y \leq_{\mathrm{T}} A \oplus B \oplus H_U$ such that $Y \leq_{\mathrm{se}} X \oplus A$ and $Y \leq_{\mathrm{se}} X \oplus B$ whereas $Y \not\leq_{\mathrm{se}} X \oplus U$.*

Remark 8.20 $X, Y \leq_{\mathrm{T}} A \oplus B \oplus H_U$ implies that $X, Y \leq_{\mathrm{se}} (A \oplus \overline{A}) \oplus (B \oplus \overline{B}) \oplus S_U$.

Proof Since the pair (A, B) is not U-se-ideal we know (by definition) that *either* (A, B) is not U-e-ideal *or* $(\overline{A}, \overline{B})$ is not $\overline{U}$-e-ideal. We consider both cases.

Case 1 (A, B) is not U-e-ideal. Then by Theorem 8.18—and assuming $\{F_x, E_x\}_{x\in\omega}$ to be an enumeration of pairs of finite sets—there exist sets X, Y computable in $A \oplus B \oplus K_U$ such that

$$Y = \{\, z \mid z \in X \ \&\ F_z \subseteq A \,\} = \{\, z \mid z \in X \ \&\ E_z \subseteq B \,\}$$

and $Y \not\leq_{\mathrm{e}} X \oplus U$. Now clearly, $Y \leq_{\mathrm{p}} X \oplus A$ and $Y \leq_{\mathrm{p}} X \oplus B$. Thus, by Theorem 3.6, $Y \leq_{\mathrm{se}} X \oplus A$ and $Y \leq_{\mathrm{se}} X \oplus B$. On the other hand, $Y \not\leq_{\mathrm{e}} X \oplus U$ obviously implies $Y \not\leq_{\mathrm{se}} X \oplus U$.

Case 2 $(\overline{A}, \overline{B})$ is not $\overline{U}$-e-ideal. Then, by the same argument as that applied to Case 1, there exist sets Z, V computable in $\overline{A} \oplus \overline{B} \oplus K_{\overline{U}}$ such that $V \leq_{\mathrm{se}} Z \oplus \overline{A}$ and $V \leq_{\mathrm{se}} Z \oplus \overline{B}$ but $V \not\leq_{\mathrm{se}} Z \oplus \overline{U}$. However, if we let $X = \overline{Z}$ and $Y = \overline{V}$, then the latter is equivalent to $Y \leq_{\mathrm{se}} X \oplus A$, $Y \leq_{\mathrm{se}} X \oplus B$, and $Y \not\leq_{\mathrm{se}} X \oplus U$ (by definition of $\leq_{\mathrm{se}}$). □

Theorem 8.21 *For any sets A, B, U the conditions (a) – (c) are equivalent.*

(a) The pair (A, B) is U-se-ideal.

(b) There exist computable functions $f(x, y)$ and $\hat{f}(x, y)$ such that, for any set $X \subseteq \omega$ and for every $x, y \in \omega$,

$$\Phi_x(A \oplus X) \cap \Phi_y(B \oplus X) \subseteq \Phi_{f(x,y)}(U \oplus X) \subseteq \Phi_x(A \oplus X) \cup \Phi_y(B \oplus X)$$

and

$$\Phi_x(\overline{A} \oplus \overline{X}) \cap \Phi_y(\overline{B} \oplus \overline{X}) \subseteq \Phi_{\hat{f}(x,y)}(\overline{U} \oplus \overline{X}) \subseteq \Phi_x(\overline{A} \oplus \overline{X}) \cup \Phi_y(\overline{B} \oplus \overline{X}).$$

(c) For every set $X \subseteq \omega$, the se-degree $\deg_{\mathrm{se}}(U \oplus X)$ is the infinum of $\deg_{\mathrm{se}}(A \oplus (U \oplus X))$ and $\deg_{\mathrm{se}}(B \oplus (U \oplus X))$.

Remark 8.22 This theorem and its proof are adapted from Theorem 2.6 of [5].

Proof (a) ⇒ (b) Since (A, B) is U-se-ideal there exist sets $M \leq_{\mathrm{e}} U$ and $N \leq_{\mathrm{e}} \overline{U}$ such that

$$A \times B \subseteq M \quad \text{and} \quad \overline{A} \times \overline{B} \subseteq \overline{M},$$
$$\overline{A} \times \overline{B} \subseteq N \quad \text{and} \quad A \times B \subseteq \overline{N}.$$

Suppose that $M = \Phi_M(U)$ and $N = \Phi_N(\overline{U})$. Then there exist computable functions $f(x, y)$ and $\hat{f}(x, y)$ such that

$$\begin{aligned} W_{f(x,y)} = \big\{ &\langle n, D \oplus E\rangle \mid \exists D' \exists D'' \big(n \in \Phi_x(D' \oplus E) \cap \Phi_y(D'' \oplus E)\ \&\\ &(\forall z \in D')(\forall w \in D'')[\, \langle z, w\rangle \in \Phi_M(D)\,] \big) \big\},\\ W_{\hat{f}(x,y)} = \big\{ &\langle n, D \oplus E\rangle \mid \exists D' \exists D'' \big(n \in \Phi_x(D' \oplus E) \cap \Phi_y(D'' \oplus E)\ \&\\ &(\forall z \in D')(\forall w \in D'')[\, \langle z, w\rangle \in \Phi_N(D)\,] \big) \big\}. \end{aligned}$$

where D', D'' (and, of course, D, E) range over finite sets. We can now check that the associated enumeration operators $\Phi_{f(x,y)}$ and $\Phi_{\hat{f}(x,y)}$ satisfy condition (b). The argument for $\Phi_{f(x,y)}$ is below; that for $\Phi_{\hat{f}(x,y)}$ is similar.

(1) Suppose that $n \in \Phi_x(A \oplus X) \cap \Phi_y(B \oplus X)$. Then $n \in \Phi_x(D' \oplus E) \cap \Phi_y(D'' \oplus E)$ for some (finite sets) $E \subseteq X$, $D' \subseteq A$, $D'' \subseteq B$. Thus, for any $z \in D'$ and $w \in D''$, $\langle z, w\rangle \in A \times B \subseteq M = \Phi_M(U)$. It easily follows that there exists a finite set $D \subseteq U$ such that $\langle z, w\rangle \in \Phi_M(D)$ for all such z, w. Thus $n \in \Phi_{f(x,y)}(U \oplus X)$.

(2) Suppose that $n \in \Phi_{f(x,y)}(U \oplus X)$. Then $n \in \Phi_x(D' \oplus X) \cap \Phi_y(D'' \oplus X)$ for some D', D'' such that for any $z \in D'$ and $w \in D''$, $\langle z, w\rangle \in \Phi_M(U) = M$. Suppose for a contradiction that $n \notin \Phi_x(A \oplus X) \cup \Phi_y(B \oplus X)$. Then there must exist numbers $z' \in D'$ and $w' \in D''$ such that $z' \in \overline{A}$ and $w' \in \overline{B}$ and this means that $\langle z', w'\rangle \in \overline{M}$ (contradiction).

(b) ⇒ (c) Let X be any set. It is obvious that $U \oplus X \leq_{\mathrm{se}} A \oplus (U \oplus X)$ and $U \oplus X \leq_{\mathrm{se}} B \oplus (U \oplus X)$. Consider any set C such that $C \leq_{\mathrm{se}} A \oplus (U \oplus X)$ and $C \leq_{\mathrm{se}} B \oplus (U \oplus X)$. Then there exist numbers x, y, x', y' such that

$$C = \Phi_x(A \oplus (U \oplus X)) = \Phi_y(B \oplus (U \oplus X))$$

and

$$\overline{C} \;=\; \Phi_{x'}(\,\overline{A} \oplus (\overline{U} \oplus \overline{X})) \;=\; \Phi_{y'}(\,\overline{B} \oplus (\overline{U} \oplus \overline{X}))\,.$$

Now since condition (b) holds by hypothesis,

$$\begin{aligned} C \;&=\; \Phi_{f(x,y)}(\,U \oplus (U \oplus X)\,)\,,\\ \overline{C} \;&=\; \Phi_{\hat{f}(x',y')}(\,\overline{U} \oplus (\overline{U} \oplus \overline{X})\,)\,. \end{aligned}$$

Thus $C \leq_{\mathrm{se}} U \oplus X$.

(c) $\Rightarrow$ (a) Suppose that the pair (A, B) is not U-se-ideal. Then it follows from Corollary 8.19 there exist sets X, Y such that $Y \leq_{\mathrm{se}} A \oplus (U \oplus X)$, $Y \leq_{\mathrm{se}} B \oplus (U \oplus X)$, and $Y \not\leq_{\mathrm{se}} U \oplus X$. Therefore, $\deg_{\mathrm{se}}(U \oplus X)$ is not the infinum of $\deg_{\mathrm{se}}(A \oplus (U \oplus X))$ and $\deg_{\mathrm{se}}(B \oplus (U \oplus X))$. ☐

Note 8.23 By Theorem 8.21 (a) $\Leftrightarrow$ (c) we know that the pair (A, B) is U-se-ideal if and only if for any set $X \;{}_{\mathrm{se}}\!\geq U$,

$$\deg_{\mathrm{se}}(X) \;=\; \deg_{\mathrm{se}}(A \oplus X) \,\cap\, \deg_{\mathrm{se}}(B \oplus X)\,. \tag{2}$$

Note that if (A, B) is se-ideal then (2) holds for *any* X (and so also, if neither A nor B is computable, $\deg_{\mathrm{se}}(A)$ and $\deg_{\mathrm{se}}(B)$ form a minimal pair).

Corollary 8.24 *A pair of se-degrees* $\boldsymbol{a}$,$\boldsymbol{b}$ *is* $\boldsymbol{u}$*-se-ideal if and only if*

$$(\forall z \geq \boldsymbol{u})[\,(\boldsymbol{a} \cup z\,) \cap (\boldsymbol{b} \cup z\,) \;=\; z\,]\,.$$

Note 8.25 Corollary 8.24 implies that, for any degree $\boldsymbol{u}$, the relation "$(\boldsymbol{x}\,, \boldsymbol{y})$ is a $\boldsymbol{u}$-se-ideal pair" is first-order definable with parameter $\boldsymbol{u}$ in $\mathcal{D}_{\mathrm{se}}$. In particular, it implies that the first-order predicate $\forall z\,[\,(\boldsymbol{x} \cup z\,) \cap (\,\boldsymbol{y} \cup z\,) \;=\; z\,]$ defines "$(\boldsymbol{x}\,, \boldsymbol{y})$ is an se-ideal pair" in $\mathcal{D}_{\mathrm{se}}$.

Theorem 8.26 (Diamond embeddings) *Let* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *be se-degrees such that* $\boldsymbol{b}$ *is characteristic and* $\boldsymbol{a} < \boldsymbol{b}$. *Then the diamond lattice is embeddable in the se-degrees with* $\boldsymbol{b}$ *as the greatest element and* $\boldsymbol{a}$ *as the least element provided that there is a characteristic degree* $\boldsymbol{a} \leq \boldsymbol{c} < \boldsymbol{b}$.

Proof Choose $A \in \boldsymbol{a}$, $C \oplus \overline{C} \in \boldsymbol{c}$, and $B \oplus \overline{B} \in \boldsymbol{b}$. By Theorem 8.10 there exists a semirecursive set V such that $V \equiv_{\mathrm{T}} B$. Equivalently, $V \oplus \overline{V} \equiv_{\mathrm{se}} B \oplus \overline{B}$. Set $\boldsymbol{u} = \deg_{\mathrm{se}}(V \oplus A)$ and $\boldsymbol{v} = \deg_{\mathrm{se}}(\overline{V} \oplus A)$.

1. Note first that $\boldsymbol{u} \cup \boldsymbol{v} = \deg_{\mathrm{se}}(V \oplus \overline{V} \oplus A) = \boldsymbol{b}$.
2. If $V \leq_{\mathrm{se}} A$ then $V \leq_{\mathrm{se}} C \oplus \overline{C}$ and so $V \oplus \overline{V} \leq_{\mathrm{se}} C \oplus \overline{C}$. Likewise, $\overline{V} \leq_{\mathrm{se}} A$ implies $V \oplus \overline{V} \leq_{\mathrm{se}} C \oplus \overline{C}$. Hence, in either case we would have $B \oplus \overline{B} \leq_{\mathrm{se}} C \oplus \overline{C}$ in contradiction with the hypothesis. Therefore, $\boldsymbol{u} > \boldsymbol{a}$ and $\boldsymbol{v} > \boldsymbol{a}$. Now the pair $(V, \overline{V})$ is se-ideal by Lemma 8.9. Thus it follows from Theorem 8.21 (see Note 8.23) that $\boldsymbol{u} \cap \boldsymbol{v} = \boldsymbol{a}$. ☐

Corollary 8.27 *For any nonzero characteristic degree* $\boldsymbol{a}$, *the diamond lattice is embeddable in the se-degrees with* $\boldsymbol{a}$ *as the greatest element and* $\mathbf{0}_{\mathrm{se}}$ *as the least element.*

Remark 8.28 In addition to Corollary 8.27 it follows from the proof of Theorem 8.26 that for any noncomputable set B there exists a (semirecursive) set $X \equiv_{\mathrm{T}} B$ such that $\deg_{\mathrm{se}}(X)$ and $\deg_{\mathrm{se}}(\overline{X})$ form a minimal pair in $\mathcal{D}_{\mathrm{se}}$.

We now move on to the second topic of this section: minimal degrees.

Theorem 8.29 *Let $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{u}$ be se-degrees such that*

1. *$\boldsymbol{u}$ is characteristic,*
2. *$\boldsymbol{a} \in \Sigma_1^{\boldsymbol{u}}$ and $\boldsymbol{b} \in \Pi_1^{\boldsymbol{u}}$.*

Then the pair $(\boldsymbol{a}, \boldsymbol{b})$ is $\boldsymbol{u}$-se-ideal.

Proof Choose $A \in \boldsymbol{a}$, $B \in \boldsymbol{b}$, and $U \in \boldsymbol{u}$. Then $A \leq_e U$ and $\overline{B} \leq_e U \equiv_e \overline{U}$ as $\boldsymbol{u}$ is characteristic (and the fact that, by Corollary 7.2 with $n = 0$, $X \in \Sigma_1^U$ if and only if $X \leq_e U$ for any X). Now apply Proposition 8.3. □

Theorem 8.30 (Kalimullin [5]) *If A, B, M are any sets such that*

$$A \times B \subseteq M \quad \text{and} \quad \overline{A} \times \overline{B} \subseteq \overline{M} \tag{3}$$

and such that $B \not\leq_e M$, then $A \leq_e \overline{B} \oplus M$.

Proof For each $x \in \omega$, let $M_x = \{ y \mid \langle x, y \rangle \in M \}$. Clearly, $M_x \leq_e M$. Now (3) implies that if $x \in A$ then $B \subseteq M_x$, whereas if $x \in \overline{A}$ then $M_x \subseteq B$ (since $\overline{B} \subseteq \overline{M_x}$). Also, as $B \not\leq_e M_x$ by assumption, each of these inclusions is proper. It therefore follows that

$$x \in A \quad \text{iff} \quad M_x - B \neq \varnothing \quad \text{iff} \quad (\exists y \notin B)\big(\langle x, y \rangle \in M\big)\,.$$

This means that $A \leq_e \overline{B} \oplus M$. □

Corollary 8.31 *Let (A, B) be a U-se-ideal pair such that $B \not\leq_e U$ and $\overline{B} \not\leq_e \overline{U}$. Then $A \leq_{se} \overline{B} \oplus U$.*

Proof Suppose that (A, B) is U-se-ideal *via* the sets $M \leq_e U$ and $N \leq_e \overline{U}$, that is, that

$$\begin{array}{lcl} A \times B \subseteq M & \text{and} & \overline{A} \times \overline{B} \subseteq \overline{M} \\ \overline{A} \times \overline{B} \subseteq N & \text{and} & A \times B \subseteq \overline{N}\,. \end{array}$$

First note the following two points.

1. $B \not\leq_e U$ implies that $B \not\leq_e M$ and so, by Theorem 8.30, $A \leq_e \overline{B} \oplus M$.
2. $\overline{B} \not\leq_e \overline{U}$ implies that $\overline{B} \not\leq_e N$ and so, by Theorem 8.30, $\overline{A} \leq_e B \oplus N$.

Now notice that $\overline{B} \oplus M \leq_e \overline{B} \oplus U$ and $B \oplus N \leq_e B \oplus \overline{U}$. Therefore, $A \leq_{se} \overline{B} \oplus U$. □

Corollary 8.32 *Let (A, B) be an se-ideal pair such that neither B nor $\overline{B}$ is c.e. Then $A \leq_{se} \overline{B}$.*

Proposition 8.33 *Let $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{u}$ be se-degrees such that*

1. *$\boldsymbol{u}$ is characteristic,*
2. *$\boldsymbol{b} \not\leq \boldsymbol{u}$,*
3. *$(\boldsymbol{a}, \boldsymbol{b})$ is $\boldsymbol{u}$-se-ideal,*
4. *$\boldsymbol{b} \in \Sigma_1^{\boldsymbol{u}}$ ($\boldsymbol{b} \in \Pi_1^{\boldsymbol{u}}$).*

Then $\boldsymbol{a} \in \Pi_1^{\boldsymbol{u}}$ ($\boldsymbol{a} \in \Sigma_1^{\boldsymbol{u}}$).

Proof Choose sets $A \in \boldsymbol{a}$, $B \in \boldsymbol{b}$, and $U \in \boldsymbol{u}$.

Remark 8.34 By assumption (A, B) is U-se-ideal, $B \not\leq_{\mathrm{se}} U$, and $U \equiv_{\mathrm{se}} \overline{U}$. Notice also that $U \equiv_{\mathrm{se}} \overline{U}$ implies that for any set X, $X \in \Sigma_1^U$ ($X \in \Pi_1^U$) if and only if $X \leq_{\mathrm{e}} U$ ($\overline{X} \leq_{\mathrm{e}} \overline{U}$).

Suppose that M, N are sets via which (A, B) is U-se-ideal, that is, such that $M \leq_{\mathrm{e}} U$, $N \leq_{\mathrm{e}} \overline{U}$, and

$$A \times B \subseteq M \quad \text{and} \quad \overline{A} \times \overline{B} \subseteq \overline{M}$$
$$\overline{A} \times \overline{B} \subseteq N \quad \text{and} \quad A \times B \subseteq \overline{N}\,.$$

We consider each of the two possible cases in turn.

Case 1 Suppose that $\boldsymbol{b} \in \Sigma_1^{\boldsymbol{u}}$. By Remark 8.34, $B \leq_{\mathrm{e}} U$. So, since $B \not\leq_{\mathrm{se}} U$, we know that $\overline{B} \not\leq_{\mathrm{e}} \overline{U}$ and thus $\overline{B} \not\leq_{\mathrm{e}} N$. Therefore, by Theorem 8.30, $\overline{A} \leq_{\mathrm{e}} B \oplus N$. But $B \oplus N \leq_{\mathrm{e}} U \oplus \overline{U} \leq_{\mathrm{e}} \overline{U}$. So $\overline{A} \leq_{\mathrm{e}} \overline{U}$, which implies (see Remark 8.34) that $\boldsymbol{a} \in \Pi_1^{\boldsymbol{u}}$.

Case 2 Suppose that $\boldsymbol{b} \in \Pi_1^{\boldsymbol{u}}$. By Remark 8.34, $\overline{B} \leq_{\mathrm{e}} \overline{U}$. So, as $B \not\leq_{\mathrm{se}} U$, we know that $B \not\leq_{\mathrm{e}} U$ and this implies that $B \not\leq_{\mathrm{e}} M$. Therefore, by Theorem 8.30, $A \leq_{\mathrm{e}} \overline{B} \oplus M$. But $\overline{B} \oplus M \leq_{\mathrm{e}} \overline{U} \oplus U \leq_{\mathrm{e}} U$. So $A \leq_{\mathrm{e}} U$, which implies (see Remark 8.34) that $\boldsymbol{a} \in \Sigma_1^{\boldsymbol{u}}$. ☐

Definition 8.35 Let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{u}$, and $\boldsymbol{v}$ be se-degrees such that $\boldsymbol{u}$ and $\boldsymbol{v}$ are characteristic and $(\boldsymbol{a}, \boldsymbol{b})$ is $\boldsymbol{u}$-se-ideal. Then $(\boldsymbol{a}, \boldsymbol{b})$ is said to be "$\Sigma_1^{\boldsymbol{v}}$" if either $\boldsymbol{a} \in \Sigma_1^{\boldsymbol{v}}$ or $\boldsymbol{b} \in \Sigma_1^{\boldsymbol{v}}$. Otherwise, $(\boldsymbol{a}, \boldsymbol{b})$ is said to be "non-$\Sigma_1^{\boldsymbol{v}}$".

Note 8.36 Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be characteristic se-degrees. Then we know from Proposition 8.33 that the $\boldsymbol{u}$-se-ideal pair $(\boldsymbol{a}, \boldsymbol{b})$ is non-$\Sigma_1^{\boldsymbol{u}}$ if and only if neither $\boldsymbol{a}$ nor $\boldsymbol{b}$ is in $\Sigma_1^{\boldsymbol{u}} \cup \Pi_1^{\boldsymbol{u}}$. It also follows that if $\boldsymbol{u} \leq \boldsymbol{v}$ then $(\boldsymbol{a}, \boldsymbol{b})$ is a non-$\Sigma_1^{\boldsymbol{v}}$ $\boldsymbol{v}$-se-ideal pair if and only if neither $\boldsymbol{a}$ nor $\boldsymbol{b}$ is in $\Sigma_1^{\boldsymbol{v}} \cup \Pi_1^{\boldsymbol{v}}$ (since $\boldsymbol{u} \leq \boldsymbol{v}$ implies that $(\boldsymbol{a}, \boldsymbol{b})$ is $\boldsymbol{v}$-se-ideal).

Lemma 8.37 *If $(\boldsymbol{a}, \boldsymbol{b})$ is a non-Σ_1 se-ideal pair then $\boldsymbol{b} = \overline{\boldsymbol{a}}$. Thus $\boldsymbol{a}$, $\boldsymbol{b}$ are contained in the same Turing degree (i.e., $\deg_{\mathrm{T}}(A)$ for $A \in \boldsymbol{a}$) and $\boldsymbol{a} \cup \boldsymbol{b}$ is characteristic.*

Proof Pick any set $A \in \boldsymbol{a}$ and $B \in \boldsymbol{b}$. Then, by Corollary 8.32, $B \leq_{\mathrm{se}} \overline{A}$ and $A \leq_{\mathrm{se}} \overline{B}$. However, the latter is equivalent to $\overline{A} \leq_{\mathrm{se}} B$ and so $B \equiv_{\mathrm{se}} \overline{A}$. ☐

Proposition 8.38 (Minimal degrees) *If $(\boldsymbol{a}, \boldsymbol{b})$ is a non-Σ_1 se-ideal pair then both $\boldsymbol{a}$ and $\boldsymbol{b}$ are minimal degrees in $\mathcal{D}_{\mathrm{se}}$.*

Proof Suppose that se-degree $\boldsymbol{c}$ is such that $\boldsymbol{0} < \boldsymbol{c} \leq \boldsymbol{b}$. Then $(\boldsymbol{a}, \boldsymbol{c})$ is se-ideal by Lemma 8.17. Now $\boldsymbol{c}$ is neither Σ_1 nor Π_1 since this would imply, by Proposition 8.33, that $\boldsymbol{a}$ is either Π_1 or Σ_1, respectively, in contradiction with the hypothesis. Thus $(\boldsymbol{a}, \boldsymbol{c})$ is non-Σ_1 and so $\boldsymbol{c} = \overline{\boldsymbol{a}} = \boldsymbol{b}$ by Lemma 8.37. A similar argument applies to $\boldsymbol{a}$. ☐

Corollary 8.39 *Every nonzero Turing degree contains at least two minimal se-degrees.*

Proof By Lemma 8.9 and Theorem 8.10 every nonzero Turing degree contains at least one non-Σ_1 se-ideal pair. ☐

Proposition 8.40 *Let $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{u}$ be se-degrees such that $\boldsymbol{u}$ is characteristic and $(\boldsymbol{a}, \boldsymbol{b})$ is an se-ideal pair that is non-$\Sigma_1^{\boldsymbol{u}}$. Then $\boldsymbol{a} \cup \boldsymbol{u}$ and $\boldsymbol{b} \cup \boldsymbol{u}$ are (distinct) minimal covers for $\boldsymbol{u}$.*

Proof As $\mathbf{0} \leq \boldsymbol{u}$ trivially, $(\boldsymbol{a}, \boldsymbol{b})$ is $\boldsymbol{u}$-se-ideal. By Lemma 8.17, $(\boldsymbol{a} \cup \boldsymbol{u}, \boldsymbol{b} \cup \boldsymbol{u})$ is also a $\boldsymbol{u}$-se-ideal pair. Note that by assumption neither $\boldsymbol{a}$ nor $\boldsymbol{b}$ is in $\Sigma_1^{\boldsymbol{u}} \cup \Pi_1^{\boldsymbol{u}}$ (see Note 8.36). Consider any se-degree $\boldsymbol{c}$ such that $\boldsymbol{c} \leq \boldsymbol{b} \cup \boldsymbol{u}$. Then, by Lemma 8.17, $(\boldsymbol{a} \cup \boldsymbol{u}, \boldsymbol{c})$ is $\boldsymbol{u}$-se-ideal. There are two cases.

Case 1 $\boldsymbol{c} \notin \Sigma_1^{\boldsymbol{u}} \cup \Pi_1^{\boldsymbol{u}}$. Then choose $A \in \boldsymbol{a}$, $B \in \boldsymbol{b}$, $C \in \boldsymbol{c}$, and $U \in \boldsymbol{u}$ and note that $U \equiv_{\mathrm{se}} \overline{U}$ as $\boldsymbol{u}$ is characteristic. By Corollary 8.31, $B \oplus U \leq_{\mathrm{se}} (\overline{A} \oplus \overline{U}) \oplus U$ and $\overline{A} \oplus \overline{U} \leq_{\mathrm{se}} C \oplus \overline{U}$. However, $(\overline{A} \oplus \overline{U}) \oplus U \leq_{\mathrm{se}} \overline{A} \oplus \overline{U}$ and $C \oplus \overline{U} \equiv_{\mathrm{se}} C \oplus U$ (as $\boldsymbol{u}$ is characteristic). Therefore, $\boldsymbol{b} \cup \boldsymbol{u} \leq \boldsymbol{c} \cup \boldsymbol{u}$. Thus, if $\boldsymbol{u} \leq \boldsymbol{c}$, then $\boldsymbol{b} \cup \boldsymbol{u} = \boldsymbol{c}$.

Case 2 $\boldsymbol{c} \in \Sigma_1^{\boldsymbol{u}}$ or $\boldsymbol{c} \in \Pi_1^{\boldsymbol{u}}$. It cannot be the case that $\boldsymbol{c} \not\leq \boldsymbol{u}$ since this would imply, by Proposition 8.33, that either $\boldsymbol{a} \in \Pi_1^{\boldsymbol{u}}$ or $\boldsymbol{a} \in \Sigma_1^{\boldsymbol{u}}$ (respectively). Hence $\boldsymbol{c} \leq \boldsymbol{u}$.

We conclude that $\boldsymbol{b} \cup \boldsymbol{u}$ is a minimal cover for $\boldsymbol{u}$. A similar argument proves that $\boldsymbol{a} \cup \boldsymbol{u}$ is also a minimal cover for $\boldsymbol{u}$. These two degrees are distinct as $(\boldsymbol{a} \cup \boldsymbol{u}, \boldsymbol{b} \cup \boldsymbol{u})$ is a $\boldsymbol{u}$-se-ideal pair. □

Corollary 8.41 *Let $\boldsymbol{a}_{\mathrm{T}}$ and $\boldsymbol{b}_{\mathrm{T}}$ be Turing degrees such that $\boldsymbol{a}_{\mathrm{T}} < \boldsymbol{b}_{\mathrm{T}}$ and let $\boldsymbol{a}_{\mathrm{se}}$ be the (unique) characteristic se-degree contained in $\boldsymbol{a}_{\mathrm{T}}$. Then $\boldsymbol{b}_{\mathrm{T}}$ contains at least two minimal covers for $\boldsymbol{a}_{\mathrm{se}}$.*

Proof Pick any $A \in \boldsymbol{a}_{\mathrm{T}}$ and note that $\boldsymbol{a}_{\mathrm{se}} = \deg_{\mathrm{se}}(A \oplus \overline{A})$. By Theorem 5 of [1] there exists a semirecursive set $B \in \boldsymbol{b}_{\mathrm{T}}$ such that neither B nor $\overline{B}$ is c.e. in A. Let $\boldsymbol{b}_{\mathrm{se}} = \deg_{\mathrm{se}}(B)$ and $\boldsymbol{c}_{\mathrm{se}} = \deg_{\mathrm{se}}(\overline{B})$. Then $(\boldsymbol{b}_{\mathrm{se}}, \boldsymbol{c}_{\mathrm{se}})$ is an se-ideal pair which is non-$\Sigma_1^{\boldsymbol{a}_{\mathrm{se}}}$. By Proposition 8.40, $\boldsymbol{b}_{\mathrm{se}} \cup \boldsymbol{a}_{\mathrm{se}}$ and $\boldsymbol{c}_{\mathrm{se}} \cup \boldsymbol{a}_{\mathrm{se}}$ are minimal covers for $\boldsymbol{a}_{\mathrm{se}}$. Clearly, both these se-degrees are contained in $\boldsymbol{b}_{\mathrm{T}}$. □

Remark 8.42 Note that Corollary 8.39 means that any set of nonzero Turing degrees $\mathcal{A}$ (say) gives rise to an antichain of se-degrees $\mathcal{B}$ such that (for example) each $\boldsymbol{a}_{\mathrm{T}} \in \mathcal{A}$ contains exactly one $\boldsymbol{b}_{\mathrm{se}} \in \mathcal{B}$ and such that each $\boldsymbol{b}_{\mathrm{se}} \in \mathcal{B}$ is contained in some $\boldsymbol{a}_{\mathrm{T}} \in \mathcal{A}$. In contrast the set $\mathcal{A}$, of course, also gives rise to

$$\mathcal{C} = \{\, \boldsymbol{c}_{\mathrm{se}} \mid \boldsymbol{c}_{\mathrm{se}} \text{ characteristic and } \boldsymbol{c}_{\mathrm{se}} \subseteq \boldsymbol{a}_{\mathrm{T}} \text{ for some } \boldsymbol{a}_{\mathrm{T}} \in \mathcal{A} \,\}$$

which once again has the property that any $\boldsymbol{a}_{\mathrm{T}} \in \mathcal{A}$ contains exactly one $\boldsymbol{c}_{\mathrm{se}} \in \mathcal{C}$. However, in this case, for any $\boldsymbol{a}_{\mathrm{T}}, \boldsymbol{b}_{\mathrm{T}} \in \mathcal{A}$ and $\boldsymbol{a}_{\mathrm{se}}, \boldsymbol{b}_{\mathrm{se}} \in \mathcal{C}$ such that $\boldsymbol{a}_{\mathrm{se}} \subseteq \boldsymbol{a}_{\mathrm{T}}$ and $\boldsymbol{b}_{\mathrm{se}} \subseteq \boldsymbol{b}_{\mathrm{T}}$, we know that $\boldsymbol{a}_{\mathrm{T}} \leq \boldsymbol{b}_{\mathrm{T}}$ if and only if $\boldsymbol{a}_{\mathrm{se}} \leq \boldsymbol{b}_{\mathrm{se}}$.

Remark 8.43 The first part of Remark 8.42 applies to any reducibility subsumed by $\leq_{\mathrm{se}}$. So, for example, any set of nonzero Turing degrees gives rise to an antichain of m-degrees in the manner described above.

Remark 8.44 It follows from Proposition 8.38 that every non-nonzero Turing degree contains at least three se-degrees: two incomparable se-degrees forming a non-Σ_1 se-ideal pair and their (characteristic) join.

9 Automorphisms and Definability

By combining results from Section 7 and Section 8 we are now in a position to demonstrate some of the definability properties of $\mathcal{D}_{\mathrm{se}}$. As a consequence we are able to identify the degree of complexity of the first-order theory of $\mathcal{D}_{\mathrm{se}}$. We also prove some negative results.

Reminder An automorphism base for $\mathcal{D}_{se}$ is any set of se-degrees $\mathcal{A}$ such that the behavior of any automorphism of $\mathcal{D}_{se}$ is completely determined by its behavior on elements of $\mathcal{A}$.

Proposition 9.1 *The map* $\mathrm{inv} : \mathcal{D}_{se} \to \mathcal{D}_{se}$ *is a nontrivial automorphism.*

Proof It follows easily from the fact that for any sets A, B, $A \leq_{se} B$ if and only if $\overline{A} \leq_{se} \overline{B}$ that inv is an automorphism of $\mathcal{D}_{se}$. It is clearly nontrivial since $\deg_{se}(C)$ and $\deg_{se}(\overline{C})$ are distinct whenever $\deg_{se}(C)$ is noncharacteristic. □

Corollary 9.2 *The characteristic degrees do not form an automorphism base for* $\mathcal{D}_{se}$.

Proof It suffices to note that $\mathrm{inv} : \boldsymbol{c} \mapsto \boldsymbol{c}$ whenever $\boldsymbol{c}$ is characteristic. □

Remark 9.3 Contrast the situation in $\mathcal{D}_e$ where the embedded Turing e-degrees (i.e., the *total* e-degrees) do form an automorphism base.

Lemma 9.4 *If* $(\boldsymbol{a}, \boldsymbol{b})$ *is an se-ideal pair such that* $\boldsymbol{a}, \boldsymbol{b} > \boldsymbol{0}$*, then both* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *are quasi-minimal.*

Proof Suppose without loss of generality that $\boldsymbol{c} \leq \boldsymbol{a}$ is characteristic. There are two cases to consider.

Case 1 $(\boldsymbol{a}, \boldsymbol{b})$ is Σ_1. Then $\boldsymbol{a}$ is Σ_1 or Π_1 and it easily follows that $\boldsymbol{c} = \boldsymbol{0}$.

Case 2 $(\boldsymbol{a}, \boldsymbol{b})$ is non-Σ_1. By Proposition 8.38, $\boldsymbol{a}$ is minimal and so $\boldsymbol{c} = \boldsymbol{0}$ or $\boldsymbol{c} = \boldsymbol{a}$. Suppose, for a contradiction, that $\boldsymbol{c} = \boldsymbol{a}$. Then $\overline{\boldsymbol{c}} = \boldsymbol{b}$ by Lemma 8.37. However, $\boldsymbol{c} = \overline{\boldsymbol{c}}$ as $\boldsymbol{c}$ is characteristic and so the pair $(\boldsymbol{c}, \boldsymbol{c})$ is se-ideal. But then it follows from Lemma 8.6 that $\boldsymbol{c} = \boldsymbol{0}$ (contradiction). □

Note 9.5 Using an easy modification of the proof of Lemma 9.4 it can be shown that, if $\boldsymbol{u}$ is characteristic, $\boldsymbol{a}, \boldsymbol{b} > \boldsymbol{u}$, and $(\boldsymbol{a}, \boldsymbol{b})$ is $\boldsymbol{u}$-se-ideal, then both $\boldsymbol{a}$ and $\boldsymbol{b}$ are $\boldsymbol{u}$-quasi-minimal.

Lemma 9.6 *The class of noncharacteristic se-degrees forms an automorphism base for* $\mathcal{D}_{se}$.

Proof It suffices to show that the characteristic degrees are *generated* by the noncharacteristic degrees. Note first that, by Corollary 8.24 (with $\boldsymbol{u} = \boldsymbol{0}$), we know that $\boldsymbol{0} = \boldsymbol{a} \cap \boldsymbol{b}$ for any se-ideal pair $(\boldsymbol{a}, \boldsymbol{b})$. On the other hand, if $\boldsymbol{c} > \boldsymbol{0}$ is characteristic then there exists a non-Σ_1 se-ideal pair such that $\boldsymbol{a} = \boldsymbol{b} \cup \boldsymbol{c}$ (see the proof of Theorem 9.15 below). □

Notation We say that an se-degree $\boldsymbol{a}$ is *semirecursive* if it contains a semirecursive set.

Lemma 9.7 *An se-degree* $\boldsymbol{a} > \boldsymbol{0}$ *is semirecursive if and only if there exists an se-degree* $\boldsymbol{b} > \boldsymbol{0}$ *such that* $(\boldsymbol{a}, \boldsymbol{b})$ *is se-ideal.*

Proof ($\Rightarrow$) Suppose that $\boldsymbol{a}$ is semirecursive. Then $\overline{\boldsymbol{a}} > \boldsymbol{0}$ (by symmetry of $\leq_{se}$) and $(\boldsymbol{a}, \overline{\boldsymbol{a}})$ is se-ideal by Lemma 8.9.

($\Leftarrow$) Suppose that there exists $\boldsymbol{b} > \boldsymbol{0}$ such that $(\boldsymbol{a}, \boldsymbol{b})$ is se-ideal. There are two cases to consider.

Case 1 $(\boldsymbol{a}, \boldsymbol{b})$ is Σ_1 se-ideal. Then without loss of generality we can suppose, by Proposition 8.33, that $\boldsymbol{a}$ is Σ_1 and $\boldsymbol{b}$ is Π_1. Now the isomorphisms of Corollary 7.6 imply that every nonzero c.e. Turing degree $\boldsymbol{a}_{\mathrm{T}}$ contains precisely one c.e. se-degree $\boldsymbol{d}$ and one co-c.e. se-degree $\boldsymbol{e} = \overline{\boldsymbol{d}}$ (and the isomorphisms send $\boldsymbol{a}_{\mathrm{T}}$ to $\boldsymbol{d}$ and $\boldsymbol{d}$ to $\boldsymbol{e}$). Also, by Corollary 3.3 of [4], every nonzero c.e. Turing degree contains a hypersimple semirecursive set. It follows that both $\boldsymbol{a}$ and $\boldsymbol{b}$ (the latter by Remark 8.8 on page 191) both contain semirecursive sets.

Case 2 $(\boldsymbol{a}, \boldsymbol{b})$ is non-Σ_1 se-ideal. By Lemma 8.37, $\boldsymbol{b} = \overline{\boldsymbol{a}}$ and so $\boldsymbol{a}, \boldsymbol{b}$ are contained in the same Turing degree $\boldsymbol{a}_{\mathrm{T}}$ (say). Choose $A \in \boldsymbol{a}$ (and so $\overline{A} \in \boldsymbol{b}$). By Theorem 3.6 of [4], there exists a semirecursive set $C \leq_{\mathrm{p}} A$ (thus $C \leq_{\mathrm{se}} A$ and $\overline{C} \leq_{\mathrm{se}} \overline{A}$) such that $C \in \boldsymbol{a}_{\mathrm{T}}$. Let $\boldsymbol{c} = \deg_{\mathrm{se}}(C)$. Then $\boldsymbol{c}$ and $\overline{\boldsymbol{c}}$ are semirecursive, $(\boldsymbol{c}, \overline{\boldsymbol{c}})$ is se-ideal (by Lemma 8.9), and $\boldsymbol{c} \leq \boldsymbol{a}$ whereas $\overline{\boldsymbol{c}} \leq \boldsymbol{b}$ (as $\overline{\boldsymbol{a}} = \boldsymbol{b}$). Therefore, since $\boldsymbol{c}, \overline{\boldsymbol{c}} > \boldsymbol{0}$ ($\boldsymbol{a}_{\mathrm{T}}$ being nonzero), Proposition 8.38 implies that $\boldsymbol{c} = \boldsymbol{a}$ and $\overline{\boldsymbol{c}} = \boldsymbol{b}$. □

Corollary 9.8 *The class of se-degrees $\mathcal{SR} = \{\, \boldsymbol{a} \mid \boldsymbol{a}$ is semirecursive $\}$ is first-order definable in $\mathcal{D}_{\mathrm{se}}$.*

Proof Lemma 9.7 in conjunction with Corollary 8.24 implies that the set $\mathcal{SR}^{>\boldsymbol{0}} = \{\, \boldsymbol{a} \mid \boldsymbol{a} > \boldsymbol{0} \;\&\; \boldsymbol{a}$ is semirecursive $\}$ is first-order definable in $\mathcal{D}_{\mathrm{se}}$. Also, of course, for any se-degree $\boldsymbol{d}$, $\boldsymbol{d} \in \mathcal{SR}$ if and only if $\boldsymbol{d} = \boldsymbol{0} \vee \boldsymbol{d} \in \mathcal{SR}^{>\boldsymbol{0}}$. □

Proposition 9.9 *Suppose that $\boldsymbol{a}$ and $\boldsymbol{u}$ are se-degrees such that $\boldsymbol{u}$ is characteristic and $\boldsymbol{a} \not\leq \boldsymbol{u}$. Then $\boldsymbol{a} \in \Sigma_1^{\boldsymbol{u}} \cup \Pi_1^{\boldsymbol{u}}$ if and only if there exist se-degrees $\boldsymbol{b}$,$\boldsymbol{c}$ such that both the pairs $(\boldsymbol{a} \cup \boldsymbol{u}, \boldsymbol{b} \cup \boldsymbol{u})$ and $(\boldsymbol{a} \cup \boldsymbol{u}, \boldsymbol{c} \cup \boldsymbol{u})$ are $\boldsymbol{u}$-se-ideal and $\boldsymbol{u} < \boldsymbol{b} \cup \boldsymbol{u} < \boldsymbol{c} \cup \boldsymbol{u}$.*

Proof We consider ($\Rightarrow$) and then ($\Leftarrow$) of the proposition.

($\Rightarrow$) Suppose that $\boldsymbol{a} \in \Sigma_1^{\boldsymbol{u}} \cup \Pi_1^{\boldsymbol{u}}$. Without loss of generality, we can assume that $\boldsymbol{a} \in \Sigma_1^{\boldsymbol{u}}$ (since the case $\boldsymbol{a} \in \Pi_1^{\boldsymbol{u}}$ follows using a similar argument). Let $\boldsymbol{c} = \mathrm{inv}(\boldsymbol{u}^*)$ and note that $\boldsymbol{c} > \boldsymbol{u}$ (see proof of Proposition 7.7). By Proposition 7.7 there exists an se-degree $\boldsymbol{b} \in \Pi_1^{\boldsymbol{u}}$ such that $\boldsymbol{u} < \boldsymbol{b} < \boldsymbol{c}$. It thus suffices to note that $\boldsymbol{b} = \boldsymbol{b} \cup \boldsymbol{u}$ and $\boldsymbol{c} = \boldsymbol{c} \cup \boldsymbol{u}$ and that, by Proposition 8.3 and Lemma 8.17, the pairs $(\boldsymbol{a} \cup \boldsymbol{u}, \boldsymbol{b} \cup \boldsymbol{u})$ and $(\boldsymbol{a} \cup \boldsymbol{u}, \boldsymbol{c} \cup \boldsymbol{u})$ are $\boldsymbol{u}$-se-ideal.

($\Leftarrow$) Suppose that $\boldsymbol{a} \notin \Sigma_1^{\boldsymbol{u}} \cup \Pi_1^{\boldsymbol{u}}$ and suppose that there exists se-degree $\boldsymbol{c}$ such that $(\boldsymbol{a} \cup \boldsymbol{u}, \boldsymbol{c} \cup \boldsymbol{u})$ is $\boldsymbol{u}$-se-ideal. Then it is neither the case that $\boldsymbol{c} \in \Sigma_1^{\boldsymbol{u}}$ nor the case that $\boldsymbol{c} \in \Pi_1^{\boldsymbol{u}}$ (since this would imply that $\boldsymbol{a} \in \Pi_1^{\boldsymbol{u}}$ or $\boldsymbol{a} \in \Sigma_1^{\boldsymbol{u}}$, respectively). Thus $(\boldsymbol{a} \cup \boldsymbol{u}, \boldsymbol{c} \cup \boldsymbol{u})$ is non-$\Sigma_1^{\boldsymbol{u}}$ and it follows by Proposition 8.40 that $\boldsymbol{c} \cup \boldsymbol{u}$ is a minimal cover for $\boldsymbol{u}$. Hence there exists no $\boldsymbol{b}$ such that $\boldsymbol{u} < \boldsymbol{b} \cup \boldsymbol{u} < \boldsymbol{c} \cup \boldsymbol{u}$. □

Corollary 9.10 *For any characteristic se-degree $\boldsymbol{u}$, the class of $\Sigma_1^{\boldsymbol{u}} \cup \Pi_1^{\boldsymbol{u}}$ se-degrees is first-order definable in $\mathcal{D}_{\mathrm{se}}$ with parameter $\boldsymbol{u}$.*

Corollary 9.11 *The class of $\Sigma_1 \cup \Pi_1$ se-degrees is first-order definable in $\mathcal{D}_{\mathrm{se}}$.*

Theorem 9.12 *If $\boldsymbol{u}$ is characteristic then $\boldsymbol{u}'$ is first-order definable in $\mathcal{D}_{\mathrm{se}}$ with parameter $\boldsymbol{u}$.*

Proof Note first that $\boldsymbol{u}' = \boldsymbol{u}^* \cup \mathrm{inv}(\boldsymbol{u}^*)$ and that, as explained in the proof of Proposition 7.7, the se-degrees $\boldsymbol{u}^*$ and $\mathrm{inv}(\boldsymbol{u}^*)$ are the top elements of $\mathcal{CEA}_{\mathrm{se}}(\boldsymbol{u})$ and co-$\mathcal{CEA}_{\mathrm{se}}(\boldsymbol{u})$, respectively. Hence, it follows from Corollary 9.10, Theorem 8.29, and Proposition 8.33 that se-degree $\boldsymbol{w} = \boldsymbol{u}'$ if and only if there exist se-degrees $\boldsymbol{x}$ and $\boldsymbol{y}$ satisfying conditions (1) – (4) below.

1. $\boldsymbol{x}, \boldsymbol{y} > \boldsymbol{u}$ and $\boldsymbol{x}, \boldsymbol{y} \in \Sigma_1^{\boldsymbol{u}} \cup \Pi_1^{\boldsymbol{u}}$.
2. $(\boldsymbol{x}, \boldsymbol{y})$ is $\boldsymbol{u}$-se-ideal.
3. For any $\boldsymbol{x}_1$, $\boldsymbol{y}_1$ satisfying conditions (1) and (2) it is either the case that $\boldsymbol{x}_1 \leq \boldsymbol{x}$ and $\boldsymbol{y}_1 \leq \boldsymbol{y}$ or the case that $\boldsymbol{y}_1 \leq \boldsymbol{x}$ and $\boldsymbol{x}_1 \leq \boldsymbol{y}$.
4. $\boldsymbol{w} = \boldsymbol{x} \cup \boldsymbol{y}$.

We can therefore conclude from Corollary 8.24 and Corollary 9.10 that $\boldsymbol{u}'$ is first-order definable in $\mathcal{D}_{\text{se}}$ with parameter $\boldsymbol{u}$. □

Theorem 9.13 *Let $\boldsymbol{u}$ be any characteristic se-degree. Then, for all $n \geq 0$,*

1. $\boldsymbol{u}^{(n)}$ *(the* n*th jump of $\boldsymbol{u}$),*
2. *the class of $\Sigma_n^{\boldsymbol{u}} \cup \Pi_n^{\boldsymbol{u}}$ se-degrees,*
3. *the class of $\Delta_n^{\boldsymbol{u}}$ se-degrees*

are each first-order definable in $\mathcal{D}_{\text{se}}$ with parameter $\boldsymbol{u}$.

Proof For (1) – (3) the case $n = 0$ is obvious. (1) then follows by Theorem 9.12 and induction on $n \geq 1$ (using the fact that $\boldsymbol{u}^{(n)}$ is characteristic). Also, by Corollary 7.2,

$$\Sigma_{n+1}^{\boldsymbol{u}} \cup \Pi_{n+1}^{\boldsymbol{u}} \;=\; \Sigma_1^{\boldsymbol{u}^{(n)}} \cup \Pi_1^{\boldsymbol{u}^{(n)}},$$

whereas $\Delta_{n+1}^{\boldsymbol{u}} = \{\boldsymbol{a} \mid \boldsymbol{a} \leq \boldsymbol{u}^{(n)}\}$. Thus (2) follows by Corollary 9.10 and (1), and (3) follows directly from (1). □

Corollary 9.14 *For all $n \geq 0$,*

1. $\mathbf{0}^{(n)}$*,*
2. *the class of $\Sigma_n \cup \Pi_n$ se-degrees,*
3. *the class of Δ_n se-degrees*

are each first-order definable in $\mathcal{D}_{\text{se}}$.

Theorem 9.15 *The class $\mathcal{CHAR} = \{\boldsymbol{a} \mid \boldsymbol{a}$ is characteristic$\}$ is first-order definable in $\mathcal{D}_{\text{se}}$.*

Remark 9.16 In other words, the *embedded* Turing degrees are definable in $\mathcal{D}_{\text{se}}$.

Proof We first show that a nonzero se-degree $\boldsymbol{a}$ is characteristic if and only if there exists a non-Σ_1 se-ideal pair $(\boldsymbol{b}, \boldsymbol{c})$ such that $\boldsymbol{a} = \boldsymbol{b} \cup \boldsymbol{c}$.

1. Suppose that $\boldsymbol{a}$ is characteristic. Choose $A \in \boldsymbol{a}$. By Theorem 8.10 there exists semirecursive $B \equiv_{\text{T}} A$ (and so $B \oplus \overline{B} \equiv_{\text{se}} A \oplus \overline{A} \equiv_{\text{se}} A$) such that neither B nor $\overline{B}$ is c.e. Let $\boldsymbol{b} = \deg_{\text{se}}(B)$ and $\boldsymbol{c} = \deg_{\text{se}}(\overline{B})$ (i.e., $\boldsymbol{c} = \overline{\boldsymbol{b}}$). By Lemma 8.9 and Definition 8.35, $(\boldsymbol{b}, \boldsymbol{c})$ is a non-Σ_1 se-ideal pair. Also, clearly, $\boldsymbol{a} = \boldsymbol{b} \cup \boldsymbol{c}$.
2. Suppose that $(\boldsymbol{b}, \boldsymbol{c})$ is a non-Σ_1 se-ideal pair such that $\boldsymbol{a} = \boldsymbol{b} \cup \boldsymbol{c}$. Then by Lemma 8.37, $\boldsymbol{c} = \overline{\boldsymbol{b}}$. Thus $\boldsymbol{a}$ is characteristic.

It now suffices to point out that by Note 8.25 and Corollary 9.14 (and Note 8.36) the class of non-Σ_1 se-ideal pairs is first-order definable in $\mathcal{D}_{\text{se}}$. □

Reminder $\iota_{\text{se}} : \mathcal{D}_{\text{T}} \to \mathcal{D}_{\text{se}}$ is the canonical embedding defined in Proposition 4.8.

Remark 9.17 Suppose $\boldsymbol{u}_{\mathrm{T}}$ is a Turing degree and let $\boldsymbol{u}_{\mathrm{se}} = \iota_{\mathrm{se}}(\boldsymbol{u}_{\mathrm{T}})$, that is, the unique characteristic se-degree contained in $\boldsymbol{u}_{\mathrm{T}}$. Choose any Turing degree $\boldsymbol{a}_{\mathrm{T}}$ and let $\boldsymbol{a}_{\mathrm{se}} = \iota_{\mathrm{se}}(\boldsymbol{a}_{\mathrm{T}})$. Then we can show that $\boldsymbol{a}_{\mathrm{T}} \in \Sigma_1^{\boldsymbol{u}_{\mathrm{T}}}$ if and only if

$$\exists \boldsymbol{x}_{\mathrm{se}} \exists \boldsymbol{y}_{\mathrm{se}} [\, \boldsymbol{x}_{\mathrm{se}}, \boldsymbol{y}_{\mathrm{se}} \in \Sigma_1^{\iota_{\mathrm{se}}(\boldsymbol{u}_{\mathrm{T}})} \cup \Pi_1^{\iota_{\mathrm{se}}(\boldsymbol{u}_{\mathrm{T}})} \;\&\; \iota_{\mathrm{se}}(\boldsymbol{a}_{\mathrm{T}}) = \boldsymbol{x}_{\mathrm{se}} \cup \boldsymbol{y}_{\mathrm{se}} \,].$$

Indeed, we can argue as follows.

1. Suppose that $\boldsymbol{a}_{\mathrm{T}} \in \Sigma_1^{\boldsymbol{u}_{\mathrm{T}}}$. Then there exists $B \in \boldsymbol{a}_{\mathrm{T}}$ and $U \oplus \overline{U} \in \boldsymbol{u}_{\mathrm{T}}$ such that $B \in \Sigma_1^{U \oplus \overline{U}}$. Let $\boldsymbol{b}_{\mathrm{se}} = \deg_{\mathrm{se}}(B)$. Then $\boldsymbol{b}_{\mathrm{se}} \in \Sigma_1^{\boldsymbol{u}_{\mathrm{se}}}$, $\overline{\boldsymbol{b}}_{\mathrm{se}} \in \Pi_1^{\boldsymbol{u}_{\mathrm{se}}}$, and $\boldsymbol{a}_{\mathrm{se}} = \boldsymbol{b}_{\mathrm{se}} \cup \overline{\boldsymbol{b}}_{\mathrm{se}}$.
2. On the other hand, suppose that there exist $\boldsymbol{c}_{\mathrm{se}}, \boldsymbol{d}_{\mathrm{se}} \in \Sigma_1^{\boldsymbol{u}_{\mathrm{se}}} \cup \Pi_1^{\boldsymbol{u}_{\mathrm{se}}}$ such that $\boldsymbol{a}_{\mathrm{se}} = \boldsymbol{c}_{\mathrm{se}} \cup \boldsymbol{d}_{\mathrm{se}}$. If $\boldsymbol{a}_{\mathrm{se}} \leq \boldsymbol{u}_{\mathrm{se}}$ then $\boldsymbol{a}_{\mathrm{T}} \leq \boldsymbol{u}_{\mathrm{T}}$ and so $\boldsymbol{a}_{\mathrm{T}} \in \Sigma_1^{\boldsymbol{u}_{\mathrm{T}}}$ trivially. If $\boldsymbol{a}_{\mathrm{se}} \not\leq \boldsymbol{u}_{\mathrm{se}}$ then either $\boldsymbol{c}_{\mathrm{se}} \in \Sigma_1^{\boldsymbol{u}_{\mathrm{se}}}$ and $\boldsymbol{d}_{\mathrm{se}} \in \Pi_1^{\boldsymbol{u}_{\mathrm{se}}}$ (Case 1) or vice versa (Case 2). Without loss of generality, suppose that Case 1 holds and note that $\boldsymbol{a}_{\mathrm{se}} = \overline{\boldsymbol{a}}_{\mathrm{se}} = \overline{\boldsymbol{c}}_{\mathrm{se}} \cup \overline{\boldsymbol{d}}_{\mathrm{se}}$. Let $\boldsymbol{e}_{\mathrm{se}} = \boldsymbol{c}_{\mathrm{se}} \cup \overline{\boldsymbol{d}}_{\mathrm{se}}$. Then $\overline{\boldsymbol{e}}_{\mathrm{se}} = \overline{\boldsymbol{c}}_{\mathrm{se}} \cup \boldsymbol{d}_{\mathrm{se}}$ and we know that $\boldsymbol{e}_{\mathrm{se}} \in \Sigma_1^{\boldsymbol{u}_{\mathrm{se}}}$ (*) and $\overline{\boldsymbol{e}}_{\mathrm{se}} \in \Pi_1^{\boldsymbol{u}_{\mathrm{se}}}$. Choose $E \in \boldsymbol{e}_{\mathrm{se}}$ and notice that (*) implies that $E \in \Sigma_1^{U \oplus \overline{U}}$ for any $U \in \boldsymbol{u}_{\mathrm{T}}$. Now $\boldsymbol{a}_{\mathrm{se}} = \boldsymbol{e}_{\mathrm{se}} \cup \overline{\boldsymbol{e}}_{\mathrm{se}}$, and so $E \oplus \overline{E} \in \boldsymbol{a}_{\mathrm{se}} \subseteq \boldsymbol{a}_{\mathrm{T}}$. In other words, $\boldsymbol{a}_{\mathrm{T}} \in \Sigma_1^{\boldsymbol{u}_{\mathrm{T}}}$.

Hence the class of *embedded* $\Sigma_1^{\boldsymbol{u}_{\mathrm{T}}}$ Turing degrees is first-order definable in $\mathcal{D}_{\mathrm{se}}$ with parameter $\iota_{\mathrm{se}}(\boldsymbol{u}_{\mathrm{T}})$. Moreover, it follows from this result, in conjunction with Remark 5.14 on page 183, Theorem 9.12, Theorem 9.15, and the observations made in the proof of Theorem 9.13(2), that the class of *embedded* $\Sigma_n^{\boldsymbol{u}_{\mathrm{T}}}$ Turing degrees is first-order definable in $\mathcal{D}_{\mathrm{se}}$ with parameter $\iota_{\mathrm{se}}(\boldsymbol{u}_{\mathrm{T}})$ for all $n \geq 0$. The same can then easily be shown for the class of $\Delta_n^{\boldsymbol{u}_{\mathrm{T}}}$ *embedded* Turing degrees. In particular, this means that both the class of *embedded* Σ_n Turing degrees and the class of *embedded* Δ_n Turing degrees are first-order definable in $\mathcal{D}_{\mathrm{se}}$ for all $n \geq 0$.

Theorem 9.18 *The first-order theory of $\mathcal{D}_{\mathrm{se}}$ has the same 1-degree (and isomorphism type) as the theory of Second-Order Arithmetic.*

Proof Assume that $\{F_n\}_{n \in \omega}$ is a fixed computable enumeration of first-order sentences in the language $\{\leq\}$. Also assume a fixed computable enumeration of second-order sentences in the language of arithmetic. Let $\mathrm{Th}(\mathcal{D}_{\mathrm{r}}), \mathrm{Th}(\mathrm{SOA})) \subseteq \omega$ be the sets of numbers corresponding to the first-order theory of $\mathcal{D}_{\mathrm{r}}$ (with $\mathrm{r} \in \{\mathrm{T}, \mathrm{se}\}$) and the theory of Second-Order Arithmetic, respectively, in the context of the given enumerations. It is easily seen that $\mathrm{Th}(\mathcal{D}_{\mathrm{se}}) \leq_1 \mathrm{Th}(\mathrm{SOA}))$ as every sentence of the theory of $\mathcal{D}_{\mathrm{se}}$ has a natural (and obviously computable) interpretation as a sentence about sets of integers.

On the other hand, as the first-order theory of $\mathcal{D}_{\mathrm{T}}$ has the same 1-degree as the theory of Second-Order Arithmetic [19], there exists a 1-1 computable function f witnessing the reduction $\mathrm{Th}(\mathrm{SOA})) \leq_1 \mathrm{Th}(\mathcal{D}_{\mathrm{T}})$. Suppose that $\mathrm{char}(x)$ is a first-order predicate (which can easily be written down using the above results) such that an se-degree $\boldsymbol{c}$ is characteristic if and only if $\mathcal{D}_{\mathrm{se}} \models \mathrm{char}(\boldsymbol{c})$. Also, for any first-order sentence F define F^* to be the translation of F obtained by replacing any atomic subformula "$x \leq y$" (say) of F by the formula "$\mathrm{char}(x) \;\&\; \mathrm{char}(y) \;\&\; x \leq y$". This translation clearly induces a 1-1 computable function g such that $F_{g(n)} = F_n^*$ for all $n \in \omega$. Moreover, it follows from Proposition 4.8 that $\mathcal{D}_{\mathrm{T}} \models F_n$ if and only if $\mathcal{D}_{\mathrm{se}} \models F_{g(n)}$ for all $n \in \omega$. Hence g witnesses the reduction $\mathrm{Th}(\mathcal{D}_{\mathrm{T}}) \leq_1 \mathrm{Th}(\mathcal{D}_{\mathrm{se}})$ and $g \circ f$ witnesses the reduction $\mathrm{Th}(\mathrm{SOA}) \leq_1 \mathrm{Th}(\mathcal{D}_{\mathrm{se}})$. □

Reminder For any se-degree $\boldsymbol{a}$ and set $A \in \boldsymbol{a}$, the *e-jump* ($\boldsymbol{a}^{\diamond}$) and the *embedded Turing jump* ($\boldsymbol{a}^{\dagger}$) are defined to be $\deg_{\mathrm{se}}(J_A)$ and $\deg_{\mathrm{se}}(S_{A \oplus \overline{A}})$, respectively.

Lemma 9.19 *Let A be a set of characteristic se-degree. Then* (a) $H_A \not\equiv_{\mathrm{se}} \overline{H_A}$ *and* (b) $J_{H_A} \not\equiv_{\mathrm{se}} J_{\overline{H_A}}$.

Proof (a) $H_A \equiv_1 K_A$ (since $A \equiv_{\mathrm{se}} \overline{A}$) and so $H_A \in \deg_{\mathrm{e}}(A)$. Hence $\deg_{\mathrm{se}}(H_A)$ is not characteristic (i.e., $H_A \not\equiv_{\mathrm{se}} \overline{H_A}$) as otherwise we obtain that $H_A \equiv_{\mathrm{se}} A$ (by Lemma 4.5) in contradiction with Lemma 5.6.

(b) Since $\overline{A} \leq_{\mathrm{e}} A$ (and $H_A \equiv_1 K_A$) we know that $A \leq_1 \overline{H_A}$. It follows that $\overline{H_A} \equiv_{\mathrm{e}} J_A$ which in turn implies that $J_{\overline{H_A}} \equiv_{\mathrm{se}} J_A^{(2)}$. On the other hand, as $H_A \equiv_{\mathrm{e}} A$ we know that $J_{H_A} \equiv_{\mathrm{se}} J_A$. $\square$

Proposition 9.20 *Neither the weak jump nor the e-jump is first-order definable in $\mathcal{D}_{\mathrm{se}}$.*

Proof Let $\boldsymbol{a}$ be a characteristic se-degree. Then $\mathrm{inv}(\boldsymbol{a}) = \boldsymbol{a}$ whereas we know from Lemma 9.19(a) that $\mathrm{inv}(\boldsymbol{a}^*) \neq \boldsymbol{a}^*$. Now let $\boldsymbol{b} = \boldsymbol{a}^*$ and $\boldsymbol{c} = \mathrm{inv}(\boldsymbol{a}^*)$. Then $\mathrm{inv}(\boldsymbol{b}) = \boldsymbol{c}$ whereas $\mathrm{inv}(\boldsymbol{b}^{\diamond}) = \boldsymbol{b}^{\diamond}$ as $\boldsymbol{b}^{\diamond}$ is characteristic. So, by Lemma 9.19(b), $\mathrm{inv}(\boldsymbol{b}^{\diamond}) \neq \boldsymbol{c}^{\diamond}$. Thus, as inv is an automorphism of $\mathcal{D}_{\mathrm{se}}$, neither the weak jump nor the e-jump is definable in $\mathcal{D}_{\mathrm{se}}$. $\square$

Notation Define $\imath : \mathcal{D}_{\mathrm{se}} \to \mathcal{D}_{\mathrm{se}}$ to be the operator induced by the map $X \mapsto \mathbb{C}_X$.

Note 9.21 It follows from Theorem 9.15 that the operator $\imath$ is first-order definable in $\mathcal{D}_{\mathrm{se}}$. In effect, for any set A (obviously) $\deg_{\mathrm{se}}(A \oplus \overline{A}) \geq \deg_{\mathrm{se}}(A)$, whereas if $\boldsymbol{c}$ is any characteristic se-degree such that $\boldsymbol{c} \geq \deg_{\mathrm{se}}(A)$, then $\boldsymbol{c} \geq \deg_{\mathrm{se}}(A \oplus \overline{A})$.

Lemma 9.22 *The embedded Turing jump is first-order definable in $\mathcal{D}_{\mathrm{se}}$.*

Proof For any se-degree $\boldsymbol{a}$, $\boldsymbol{a}^{\dagger} = (\imath(\boldsymbol{a}))'$ and so the lemma follows by Theorem 9.12 and Note 9.21. $\square$

Remark 9.23 We conjecture that the (strong) jump is also first-order definable in $\mathcal{D}_{\mathrm{se}}$ and we draw the reader's attention to the three observations below.

1. In contrast with the situation for the weak jump and the e-jump (see Proposition 9.20) it is easily shown that, for any se-degree $\boldsymbol{a}$, $\boldsymbol{a}' = (\mathrm{inv}(\boldsymbol{a}))'$ whereas $\mathrm{inv}(\boldsymbol{a}') = \boldsymbol{a}'$.
2. Kalimullin's proof of the definability of the enumeration jump hinges on Theorem 3.1 of [5]. However, we can also prove, in the context of $\mathcal{D}_{\mathrm{se}}$, that the jump of any se-degree $\boldsymbol{u}$ satisfies (I) $\Rightarrow$ (II) of Kalimullin's Theorem. (To see this, use a similar argument to that which yields Corollary 2.8 of [5] to show that if (A, B) is a U-se-ideal pair such that $A \nleq_{\mathrm{se}} U$ and $B \nleq_{\mathrm{se}} U$ then
$$\overline{A} \leq_{\mathrm{se}} B \oplus \overline{U} \oplus \overline{K_U} \quad \text{and} \quad B \leq_{\mathrm{se}} \overline{A} \oplus U \oplus \overline{K_{\overline{U}}}$$
and hence that $\overline{A} \oplus S_U \equiv_{\mathrm{se}} B \oplus S_U$.) We are therefore left with the question of whether the implication (II) $\Rightarrow$ (I) of the theorem holds in the se-degrees.
3. Consider the first-order predicate $P(\boldsymbol{u}, z) = \forall \boldsymbol{a} \forall \boldsymbol{b} [(\boldsymbol{a}, \boldsymbol{b})$ is not $\boldsymbol{u}$-se-ideal $\Rightarrow (\exists \boldsymbol{x} \leq \imath(\boldsymbol{a}) \cup \imath(\boldsymbol{b}) \cup z)[\boldsymbol{x} \cup \boldsymbol{a} \neq \boldsymbol{x} \cup \boldsymbol{b}]]$ and note that it follows from Corollary 8.19 (see Remark 8.20) that, for any se-degree $\boldsymbol{u}$, $\mathcal{D}_{\mathrm{se}} \models P(\boldsymbol{u}, \boldsymbol{u}')$.

Thus the natural question to ask here is whether it can be shown that $\boldsymbol{c} \geq \boldsymbol{u}'$ for any characteristic $\boldsymbol{c} \geq \boldsymbol{u}$ satisfying $\mathcal{D}_{\mathrm{se}} \models P(\boldsymbol{u}, \boldsymbol{c})$.

Note that (a) shows how the obvious obstacle to definability of the jump does not apply in this case, whereas (b) and (c) indicate the manner in which this question might be addressed.

Remark 9.24 Let $\boldsymbol{a}$ be any non-$\Sigma_1 \cup \Pi_1$ se-degree. By Theorem 4.11 there exists a noncharacteristic se-degree $\boldsymbol{b}$ such that $\boldsymbol{a} < \boldsymbol{b}$. Clearly, $\boldsymbol{b} \notin \mathcal{SR}$ (since $\boldsymbol{b}$ is non-$\Sigma_1 \cup \Pi_1$ and nonminimal). Thus $\overline{\mathcal{CHAR} \cup \mathcal{SR}}$ is nonempty. Also, if $\boldsymbol{a}$ is quasi-minimal then so also is $\boldsymbol{b}$. Hence not all quasi-minimal se-degrees are semirecursive.

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