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Cellularity of Pseudo-Tree Algebras

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Abstract Recall that for any Boolean algebra (BA) A, the *cellularity* of A is $c(A) = \sup\{|X| : X \text{ is a pairwise-disjoint subset of } A\}$. A *pseudo-tree* is a partially ordered set $(T, \leq)$ such that for every t in T, the set $\{r \in T : r \leq t\}$ is a linear order. The *pseudo-tree algebra* on T, denoted Treealg(T), is the subalgebra of $\mathcal{P}(T)$ generated by the cones $\{r \in T : r \geq t\}$, for t in T. We characterize the cellularity of pseudo-tree algebras in terms of cardinal functions on the underlying pseudo-trees. For T a pseudo-tree, $c(\text{Treealg}(T))$ is the maximum of four cardinals c_T, ι_T, φ_T, and μ_T: roughly, c_T measures the "tallness" of the pseudo-tree T; ι_T the "breadth"; φ_T the number of "finite branchings"; and μ_T the number of places where T "does not branch." We give examples to demonstrate that all four of these cardinals are needed.

1 Definitions and Introductions

We use standard notation for Boolean algebras; see Koppelberg [4]. For facts about pseudo-tree algebras, see Koppelberg and Monk [5] or Monk [7]. Note that a pseudo-tree is a generalization of a tree: for T a tree, the sets $(T \downarrow t) = \{r \in T : r \leq t\}$ are required to be well-ordered. Also, recall that if A is an infinite BA, then $c(A) \geq \omega$; see [4] for a proof. For any sets X and Y, "$X \subseteq Y$" means that X is any subset of Y; "$X \subset Y$" means that X is a proper subset of Y.

The cellularity of a tree algebra was characterized by Brenner and Monk; but since the characterization depends on enumerating the immediate successors of elements of the tree, it does not hold for pseudo-tree algebras (see [7]). Monk [7] posed the problem: Describe cellularity for pseudo-tree algebras. We do this by characterizing $c(\text{Treealg}(T))$ in terms of four cardinal functions that reflect the structure of the underlying pseudo-tree.

Definition 1.1 Recall that the *interval algebra* Intalg(L) on a linear order L is defined as follows: if L does not have a first element, add one. Extend the linear

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order of L to $L \cup \{\infty\}$, where ∞ is an element not contained in L, by letting $x < \infty$ for $x \in L$. Intalg(L) is the algebra of sets over L consisting of finite unions of half-open intervals $[x, y) = \{z \in L : x \leq z < y\}$ (for $x, y \in L \cup \infty$).

We will make use of the following normal form lemma (see [4] and [5]). Here E and F denote two special types of products over the canonical generators of T:

$$E = \{(T \uparrow t) \setminus \bigcup_{s \in S} (T \uparrow s) : S \text{ is a finite antichain in } T, \text{ and } t < s \text{ for } s \in S\}$$

and

$$F = \{T \setminus \bigcup_{s \in S} (T \uparrow s) : S \text{ is a finite antichain in } T\}.$$

(An *antichain* in T is a set of pairwise-incomparable elements of T.)

Lemma 1.2

1. *The elements of E are nonzero.*
2. *If T has a single root, then every nonzero element of F is in E.*
3. *Every element b of A is a sum of pairwise-disjoint nonzero elements*

$$b = e_0 + \cdots + e_{n-1} + f_0 + \cdots + f_{m-1},$$

where $e_i \in E (i < n)$ and $f_j \in F (j < m)$.

Definition 1.3 Let $e \in E$ and $b \in A$. A representation of e,

$$e = (T \uparrow t) \setminus \bigcup_{s \in S} (T \uparrow s),$$

is in *normal form* if S is a finite antichain in T and $t < s$ for all $s \in S$. A representation of b,

$$b = e_1 + \cdots + e_n,$$

is in *normal form* if the e_i are pairwise-disjoint elements of E, say

$$e_i = (T \uparrow t_i) \setminus \bigcup_{s \in S(i)} (T \uparrow s)$$

in normal form, and $t_i \notin S(j)$ for $i \neq j$.

It is convenient, and does no harm, to assume that all of our pseudo-trees have single roots (see [4] and [5]).

Lemma 1.4 *Let T be a pseudo-tree with a single root. Then every $b \in A$ can be written in normal form.*

Lemma 1.5 *For every pseudo-tree T there is a pseudo-tree T^* with a single root such that* Treealg(T) *is isomorphic to* Treealg(T^*).

For what follows, let T be an infinite pseudo-tree with a single root and set A = Treealg(T). By a *branch* of T we mean a maximal chain in T, and we set $\mathcal{B} = \{B \subseteq T : B \text{ is a branch}\}$. We will write $t \perp s$ when t, s are incomparable elements of T.

Definition 1.6 A *fan element* of T is an $a \in T$ such that there exists a set $F = \text{fan}(a)$ with the following properties:

1. F is a finite set of pairwise-incomparable elements each greater than a, and $|F| \geq 2$;

2. for every $c > a$, c is comparable to a unique $b \in F$.

Definition 1.7 A *pure chain* in T is a chain $C \subseteq T$ such that the following conditions hold:

1. $|C| \geq 2$;
2. if $a, b \in C$, $a < b$, and $a < c \in T$, then c is comparable to b.

$C \subseteq T$ is a *maximal pure chain* if C is a pure chain and if whenever $C' \supset C$, C' is not a pure chain.

Remark 1.8 The definition of a maximal pure chain is essentially contained in the discussion of "reduced trees" in Fraïssé [2].

We define four cardinal functions $c_T, \iota_T, \varphi_T, \mu_T$ on a pseudo-tree T.

Definition 1.9 For any pseudo-tree T,

1. the *cellularity* of T is
   $c_T = \sup\{c(\text{Intalg}(C)) : C \text{ is a chain in } T\}$
   (note that we are using "cellularity" in a special sense here);
2. the *incomparability* of T is
   $\iota_T = \sup\{|S| : S \text{ is an antichain in } T\}$
   ("incomparability" is also being used in a special sense);
3. the *number of fan elements* of T is
   $\varphi_T = |\{a \in T : a \text{ is a fan element of } T\}|$; and
4. the *number of maximal pure chains* of T is
   $\mu_T = |\{C \subseteq T : C \text{ is a maximal pure chain}\}|$.

Note that, by König's theorem, at least one of c_T, ι_T is infinite if T is an infinite pseudo-tree. We prove that $c(A)$ is the maximum of the four cardinals c_T, ι_T, φ_T, and μ_T, and that all four cardinals are necessary (that is, they are nonredundant).

2 Helpful Lemmas

Let $X \subseteq A$ be pairwise-disjoint, and suppose without loss of generality that $|X| \geq \omega$. We will show, in Theorem 3.5, that $|X| \leq \max\{c_T, \iota_T, \varphi_T, \mu_T\}$ (so that $c(A) \leq \max\{c_T, \iota_T, \varphi_T, \mu_T\}$). By Lemma 1.4, we may suppose that every $x \in X$ is of the form

$$x = (T \uparrow t_x) \setminus \bigcup_{s \in F_x} (T \uparrow s)$$

where F_x is a finite antichain of elements $s > t_x$.

Let $Y = \{t_x : x \in X\}$. Note that if $x, y \in X$ and $x \neq y$, then $t_x \neq t_y$ (otherwise both x and y would contain the element $t_x = t_y$, and so x and y would not be disjoint). Thus $|Y| = |X|$. Note also that for all $x, y \in X$ with $x \neq y$, either t_x and t_y are incomparable, or $s \leq t_y$ for some $s \in F_x$, or $s \leq t_x$ for some $s \in F_y$. Let

$$c' = \sup\{|S| : S \subseteq Y \text{ is a chain}\}.$$

We will use, in proving that $|X| \leq \max\{c_T, \iota_T, \varphi_T, \mu_T\}$, the following inequality.

Lemma 2.1 $c' \leq c_T$.

Proof Let $Y' \subseteq Y$ be a chain. Then the elements of Y' all lie on a single branch B of T. List the elements of Y' as $Y' = \{t_{x_\alpha} : \alpha < \gamma\}$, for some γ. Let $t_{x_\alpha}, t_{x_\beta} \in Y'$ with $t_{x_\alpha} \neq t_{x_\beta}$. Then one of two things happens:

(i) $s \leq t_{x_\beta}$ for some $s \in F_{x_\alpha}$ (if $t_{x_\alpha} < t_{x_\beta}$), or
(ii) $s \leq t_{x_\alpha}$ for some $s \in F_{x_\beta}$ (if $t_{x_\beta} < t_{x_\alpha}$).

Let t^* be the maximum element of Y', if such an element exists; if not, define t^* to be any set not in T. Suppose $t_{x_\alpha} \in Y' \setminus \{t^*\}$. The associated element of X is $x_\alpha = (T \uparrow t_{x_\alpha}) \setminus \bigcup_{s \in F_{x_\alpha}} (T \uparrow s)$, where F_{x_α} is a finite antichain of elements $s > t_{x_\alpha}$. As F_{x_α} is an antichain, no two elements of F_{x_α} lie on the same branch; by (i) above, at least one element of F_{x_α} lies on the branch B. So for every $t_{x_\alpha} \in Y' \setminus \{t^*\}$, let s_{x_α} be the unique element of F_{x_α} on B. Consider the element $[t_{x_\alpha}, s_{x_\alpha})$ of Intalg(B).

Claim 2.2 *If $x_\alpha, x_\beta \in \{x \in X : t_x \in Y' \setminus \{t^*\}\}$ and $x_\alpha \neq x_\beta$, then $[t_{x_\alpha}, s_{x_\alpha}) \cap [t_{x_\beta}, s_{x_\beta}) = \varnothing$.*

Proof of Claim Let $x_\alpha, x_\beta \in \{x \in X : t_x \in Y' \setminus \{t^*\}\}$ and $x_\alpha \neq x_\beta$, and suppose without loss of generality that $t_{x_\alpha} < t_{x_\beta}$. We know that $s_{x_\alpha} > t_{x_\alpha}$ and $s_{x_\alpha} \in B$. Suppose $s_{x_\alpha} > t_{x_\beta}$. Then $t_{x_\beta} \in x_\alpha \cap x_\beta$, but this is a contradiction as x_α and x_β should be disjoint. Therefore $s_{x_\alpha} \leq t_{x_\beta}$, and so as elements of Intalg(B), $[t_{x_\alpha}, s_{x_\alpha}) \cap [t_{x_\beta}, s_{x_\beta}) = \varnothing$. Thus the claim is proved.

By this claim, we have found a pairwise-disjoint subset

$$Z = \{[t_{x_\alpha}, s_{x_\alpha}) : t_{x_\alpha} \in Y'\}$$

of Intalg(B) of the same size as Y'. Then $|Y'| \leq c_T$. Therefore $c' \leq c_T$. □

Lemma 2.3 *In proving that $|X| \leq \max\{c_T, \imath_T, \varphi_T, \mu_T\}$, we may assume that for each $x \in X$ and for any two distinct $s, w \in F_x$, t_x is the largest element of T such that $t_x < s$ and $t_x < w$.*

Proof Let $x \in X$ and suppose $|X| = \mu$. Say that $x \in X$ satisfies property $(\star)$ if for any two distinct $s, w \in F_x$, t_x is the greatest element of T that is less than both s and w. If x already has the property $(\star)$, do nothing to x. Assume that $2 \leq |F_x| < \omega$ and that x does not satisfy $(\star)$. We define a sequence of antichains $F_0, F_1, \ldots$ of T and a sequence of elements $y_0, y_1, \ldots$ of A. Set $F_0 = F_x$ and $y_0 = x$. Suppose F_k and y_k have been defined, and suppose y_k does not have the property $(\star)$. Then there are distinct $s, w \in F_k$ and a $u_k \in T$ such that $t_x < u_k < s, w$. Let $G = \{v \in F_k : u_k \not< v\}$. Set $F_{k+1} = G \cup \{u_k\}$, and set $y_{k+1} = (T \uparrow t_x) \setminus \bigcup_{r \in F_{k+1}} (T \uparrow r)$. Then $y_{k+1} \neq \varnothing$ and $y_{k+1} \subseteq y_k$, so replacing y_k with y_{k+1} in X, X will still be a pairwise-disjoint subset of A of size μ. Since $|F_{k+1}| < |F_k|$, and since $|F_k| = 1$ implies that y_k has the property $(\star)$, this process eventually stops. Thus, applying this process to each $x \in X$, we get the statement in the lemma. □

The following fact, whose proof is left to the reader, is useful in proving that $c(A) \geq \mu_T$.

Lemma 2.4 *If $C \subseteq T$ is a maximal pure chain, $a, c \in C$, and $a < b < c$, then $b \in C$.* □

3 c(Treealg(T)) for T a Pseudo-tree

The reader may readily verify that $c(A) \geq c_T$ and that $c(A) \geq \imath_T$. We provide proofs that $c(A) \geq \theta_T$ and that $c(A) \geq \mu_T$, since these proofs are slightly technical.

Lemma 3.1 $c(A) \geq \varphi_T$.

Proof Let $a_1, a_2 \in T$ be fan elements with $a_1 \neq a_2$. Write the fans of a_1 and a_2, respectively, as $\text{fan}(a_1) = \{f_i^1 : i < n\}$ and $\text{fan}(a_2) = \{f_j^2 : j < m\}$. Let $x = (T \uparrow a_1) \setminus \bigcup_{i<n}(T \uparrow f_i^1)$ and $y = (T \uparrow a_2) \setminus \bigcup_{j<m}(T \uparrow f_j^2)$. We claim that $x \cap y = \varnothing$. If a_1 and a_2 are incomparable, this is clear; so say $a_1 < a_2$. Then as a_1 is a fan element, a_2 is comparable with a unique $f_i^1 \in \text{fan}(a_1)$.

Claim 3.2 *It cannot be that $a_2 < f_i^1$ for any $i < n$.*

Proof of Claim Suppose otherwise; say $a_2 < f_0^1$. As a_2 is a fan element, f_0^1 is comparable with a unique $f_j^2 \in \text{fan}(a_2)$; say f_0^1 is comparable with f_0^2. Consider f_1^2. Now $a_1 < a_2 < f_1^2$, so as a_1 is a fan element, f_1^2 is comparable to a unique $f_j^1 \in \text{fan}(a_1)$. Then a_2 is comparable to f_j^1: if $f_1^2 \leq f_j^1$, then $a_2 < f_j^1$; if $f_1^2 > f_j^1$ then $a_2, f_j^1 < f_1^2$. Hence $j = 0$. But then f_0^1 is comparable to both f_0^2 and f_1^2, which is a contradiction. Thus the claim is proved.

By the claim, it must be that $a_2 \geq f_i^1$ for some $f_i^1 \in \text{fan}(a_1)$. Then $y \subseteq (T \uparrow f_i^1)$, so $x \cap y = \varnothing$. □

Lemma 3.3 *Any two distinct maximal pure chains are disjoint, and hence $c(A) \geq \mu_T$.*

Proof Let C_1 and C_2 be distinct maximal pure chains; suppose there is an $a \in C_1 \cap C_2$, and suppose (by way of contradiction) that $C_1 \neq C_2$.

Claim 3.4 *$C_1 \cup C_2$ is a chain.*

Proof of Claim Let $c \in C_1$ and $d \in C_2$; we show that c and d are comparable. If $c \leq a \leq d$, $d \leq a \leq c$, or $c, d \leq a$, this is clear. So suppose $a < c, d$. Then $a, c \in C_1$, $a < c$, and $d > a$, so c and d are comparable as C_1 is a maximal pure chain. Thus the claim is proved.

Since $C_1 \neq C_2$, it follows by the above claim that $C_1 \cup C_2$ is not a pure chain. Hence there are $b, c \in C_1 \cup C_2$ and $d \in T$ such that $b < c$ and $d > b$ but $c \perp d$. By the pureness of C_1 and of C_2, it cannot be that both b and c are in C_1 or that both b and c are in C_2. So say $b \in C_1 \setminus C_2$ and $c \in C_2 \setminus C_1$. Since a is comparable to both b and c, by Lemma 2.4 we must have $b < a < c$. Since $a, b \in C_1$ and $d > b$, d and a are comparable (since C_1 is a pure chain). If $d \leq a$, then $d < c$, which is a contradiction since $d \perp c$. So $a < d$. Then since $a, c \in C_2$ and $d > a$, d is comparable to c, and this is again a contradiction. □

Theorem 3.5 $c(A) \leq \max\{c_T, \imath_T, \varphi_T, \mu_T\}$.

Proof Let $X \subseteq A$ be pairwise-disjoint. By Lemmas 1.4 and 2.3, we may suppose that each $x \in X$ is of the form $x = (T \uparrow t_x) \setminus \bigcup_{s \in F_x}(T \uparrow s)$ where F_x is a finite antichain of elements above t_x and where for any two distinct elements $s, w \in F_x$, t_x is the greatest element of T that is less than both s and w. We may also suppose that if $F_x \neq \varnothing$, then for every $u \in T$, if $u > t_x$ then u is comparable to some element of F_x. (If there were a $u > t_x$ such that $u \perp s$ for every $s \in F_x$, then we could replace

x with $(T \uparrow u)$ in the antichain X.) Write $X = X_0 \cup X_1 \cup X_2$, where

$$\begin{aligned} X_0 &= \{x \in X : |F_x| = 0\}, \\ X_1 &= \{x \in X : |F_x| = 1\}, \quad \text{and} \\ X_2 &= \{x \in X : 2 \leq |F_x| < \omega\}. \end{aligned}$$

First consider X_0. If $x, y \in X_0$ and $x \neq y$, then clearly $t_x \perp t_y$, else X would not be pairwise-disjoint in A. Thus $|X_0| \leq \iota_T$.

Next, consider X_1. Each $x \in X_1$ is of the form $x = (T \uparrow t_x) \setminus (T \uparrow s_x)$ for some $s_x > t_x$. Since (by assumption) every $u > t_x$ is comparable with s_x, $\{t_x, s_x\}$ is a pure chain. Let C be a maximal pure chain containing $\{t_x, s_x\}$. Then by Lemma 2.4, every element of x is contained in C. Then every $x \in X_1$ is a subset of a maximal pure chain C, and $x \subseteq C$ if and only if $t_x \in C$. Let $Y = \{t_x : x \in X_1\}$. For any maximal pure chain C, the set $\{t_x \in Y : t_x \in C\}$ is a chain in Y, and so, by Lemma 2.1, has size at most c_T. Thus $|X_1| \leq c_T \cdot \mu_T$.

Finally, consider X_2. If $x \in X_2$, then (by assumption) for every $u_x > t_x$, u_x is comparable to s for some $s \in F_x$. Suppose u_x is comparable to distinct $s, s' \in F_x$. Then necessarily $u_x \leq s, s'$, since $s \perp s'$; but this violates our assumption that t_x is the greatest element of T that is less than both s and s'. Thus for any $u_x > t_x$, u_x is comparable to a unique $s \in F_x$; as $2 \leq |F_x| < \omega$, this means that t_x is a fan element. Therefore $|X_2| \leq \varphi_T$, since if $x, y \in X_2$ and $x \neq y$ then $t_x \neq t_y$.

Therefore, recalling that at least one of the four cardinals is infinite since $|T| \geq \omega$, we have

$$\begin{aligned} |X| &= |X_0 \cup X_1 \cup X_2| \\ &\leq |X_0| + |X_1| + |X_2| \\ &\leq (\iota_T) + (c_T \cdot \mu_T) + (\varphi_T) \\ &= \max\{c_T, \iota_T, \varphi_T, \mu_T\}. \end{aligned}$$

□

Thus we have a characterization of the cellularity of a pseudo-tree algebra.

Theorem 3.6 *For any infinite pseudo-tree T,*

$$c(\text{Treealg}(T)) = \max\{c_T, \iota_T, \varphi_T, \mu_T\}.$$

4 The Four Cardinals Are All Necessary

The following examples demonstrate that the four cardinals c_T, ι_T, φ_T, and μ_T are nonredundant. (Here the reader may wish to consult Chapter II, §5 of Kunen [6] for basic facts concerning trees.)

Example 4.1 Let S be a Suslin tree such that for every $s \in S$, s has ω-many immediate successors. (If there is a Suslin tree, then there is a Suslin tree in which every element has infinitely many immediate successors.) Let T be obtained from S as follows: for $r, s \in S$ with s the immediate successor of r, insert a "link" $C_{r,s} = \mathbb{Q}$ with $r < C_{r,s} < s$. Let T be the resulting pseudo-tree, with the induced order. Then

$$\iota_T = \omega, \quad c_T = \omega, \quad \varphi_T = 0, \quad \text{and} \quad \mu_T = \omega_1.$$

Example 4.2 Let $T = \mathbb{R}$ with the usual order. Then

$$\iota_T = 1, \quad c_T = \omega, \quad \varphi_T = 0, \quad \text{and} \quad \mu_T = 1.$$

Example 4.3 Let T be the ω_1-branching tree of height ω. (That is, every element of T has exactly ω_1-many immediate successors, and height$(T) = \omega$.) Then

$$\imath_T = \omega_1, \quad c_T = \omega, \quad \varphi_T = 0, \quad \text{and} \quad \mu_T = 0.$$

Example 4.4 Let T be a Suslin tree in which every element has exactly two immediate successors. (If there is a Suslin tree, then there is a Suslin tree in which every element has exactly two immediate successors.) Then

$$\imath_T = \omega, \quad c_T = \omega, \quad \varphi_T = \omega_1, \quad \text{and} \quad \mu_T = 0.$$

Noting that Examples 4.1 and 4.4 used a Suslin tree, one might ask whether there are examples in ZFC of pseudo-trees T for which either $\mu_T > \max\{c_T, \imath_T\}$ or $\varphi_T > \max\{c_T, \imath_T\}$. The short answer is no; this and related questions are addressed in Brown [1] or [3].

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