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SOME METHODS OF FORMAL PROOFS. III

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In [1] I gave a characterization of the first order functional calculus by means of a truncated definition of satisfiability; in [2] I gave a proof of completeness of the functional calculus of an arbitrary order (without the extensionality axiom and the definition axiom) in the non-standard sense. So there appears a question about an analogical characterization—by means of the truncated satisfiability definition—of the last calculus. This paper gives an answer to the question, that is, it gives a characterization of theses of the calculus of an arbitrary order by means of a truncated satisfiability definition.

We use the notation of [1] with the following:

$Q(k)$ - a set of tables of rank k,
$\{i_{w(F)}\}$ - indices of free variables occurring in F,
$w(F)$ - the number of free variables occurring in F,
$p(F)$ - the number of apparent variables occurring in F,
$n(F) = \max\{w(F) + p(F), \{i_{w(F)}\}\}$.

A non-standard model is defined by means of a sequence $\mathbf{M} = \langle B_1, \{F_q^t\}, B_2, \{G_q^t\}, B_3, \{H_q^t\}, \ldots\rangle$ where B_1 is a non-empty domain of arbitrary elements, B_2 is an arbitrary non-empty domain of relations on B_1, B_3 is a non-empty domain of relations of a given type and order which contains previously used relations of the type; and for each type there exists a domain B_i such that B_i is a non-empty domain of relations of the given type including previously used relations of the type; and $\{F_q^t\}$ is a sequence of relations from the domain B_2, $\{G_q^t\}$ is a sequence of relations from the domain B_3, and analogously for each sequence of relations of a given type. A model with finite domains is called a table; if domains of the table have k elements then it is called a table of rank k.

f - a function on domains of a table T with values in domains of a model $\mathbf{M}$.

A_k - a sample of k individuals and k relations of each type.

V_k - the set of all elements of each domain of a table in natural order.

$P(B, V_k)$ means: B is a permutation of V_k.

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$\mathsf{M}f/A_k$ means a non-standard table $T = \langle B_1, \{\phi_q^t\}, B_2, \{\psi_q^t\}, \ldots \rangle$ of the rank k such that the set of arguments of the function f is A_k, values of f belong to suitable domains of M and for each relation F^m of the model M: $F^m(f(s_1), \ldots, f(s_m)). \equiv . \phi^m(s_1, \ldots, s_m)$, for each $s_1, \ldots, s_m \epsilon A_k$ and $f(\phi^m) = F^m$ (i.e., F^m corresponds to ϕ^m). For brevity we omit f and write simply $T = \mathsf{M}/A_k$.

D.1. $TT_1/A . \equiv . (T/A = T_1/A)$.

In [2] we proved:

T.1 A formula E is a thesis, iff it is true in the non-standard sense.

We give more details about the truncated characterization.

D.2. $m(Q,k) . \equiv . Q(k) \wedge (B)(T)\{P(B, V_k) \wedge (T \epsilon Q) \rightarrow (T/B \epsilon Q)\}$

D.3. $N(Q, k) . \equiv . m(Q, k) \wedge (B_1)(B_2)(B_3)(T_1)(T_2)(\exists T_3)\{(B_1 \cup B_2 \cup B_3 \subset V_k) \wedge (T_1, T_2 \epsilon Q) \wedge T_1 T_2/B_1 \rightarrow (T_3 \epsilon Q) \wedge T_1 T_3/B_1 \cup B_2 \wedge (T_2 T_3/B_1 \cup B_3)\}$

D.4. $Q = \mathsf{M}[k] . \equiv . (T)(\exists A_k)\{(T \epsilon Q) . \equiv . T = \mathsf{M}/A_k\}$

D.5. $R(\mathsf{M}) . \equiv . (s_1)(s_2)\{\mathsf{M}/s_1/ = \mathsf{M}/s_2/ \rightarrow (s_1 = s_2)\}$

D.6. $T \epsilon Q|k| . \equiv . (\exists n)(\exists T_1)(\exists A_k)\{(n \geq k) \wedge Q(n) \wedge (T_1 \epsilon Q) \wedge (T = T_1/A_k)\}$

L.1. *If* $Q = \mathsf{M}[k]$, *then* $m(Q, k)$.

L.2. *If* $R(\mathsf{M})$ *and* $Q = \mathsf{M}[k]$, *then* $N(Q, k)$.

L.3. *If* $N(Q^0, k)$, *then* Q^0 *can be extended to such minimal* Q *that* $Q^0 \subset Q|k|$ *and* $N(Q, k+1)$.

L.4. *If* $N(Q, n)$, $k \leq n$ *and* $Q^0 = Q|k|$, *then* $N(Q^0, k)$.

Assuming $Q(n)$ we give the finite interpretation of the general quantifier:

(4d) $V\{Q, T, \Pi aF\} = 1 . \equiv . (i)(T_1)\{(i \leq n) \wedge (T_1 \epsilon Q) \wedge TT_1/\{i_{w(F)}\} \rightarrow V\{Q, T_1 F(x_i/a)\} = 1\}$.

D.7. $E \epsilon P\{Q, n\} . \equiv . (T)\{(T \epsilon Q) \wedge Q(n) \rightarrow V\{Q, T, E\} = 1\}$.

D.8. $E \epsilon P\{n\} . \equiv . (Q)\{N(Q, n) \rightarrow E \epsilon P\{Q, n\}\}$.

D.9. $E \epsilon P . \equiv . E \epsilon P\{n(E)\}$.

It can be proved:

L.5. Let E^0 *result from* E *by replacing free variables with indices* $\{i_{w(E)}\}$ *corresponding to free variables with indices* $\{j_{w(E^0)}\}$ *(then* E *results from* E^0 *by an inverse substitution). Let* $T, T^0 \epsilon Q$, $m(Q, k)$ *and* $T^0/\{j_{w(E^0)}\} = T/\{i_{w(E)}\}$. *Then*

$$V\{Q, T, E\} = 1 . \equiv . V\{Q, T^0, E^0\} = 1.$$

L.6'. Let $m(Q, k+1)$, $m(Q^0, k)$, $k \geq n(E)$, $Q^0 = Q|k|$, $T \epsilon Q$, $T^0 \epsilon Q^0$ *and* $T^0 = T/V_k$. *Then*

$$V\{Q, T, E\} = 1 . \equiv . V\{Q^0, T^0, E\} = 1$$

L.6. Let $N(Q^0, k)$ *and let* Q *be of the rank* $k+1$ *and let* Q *be the minimal extension of* Q^0 *respectively to the property* $N(Q^0, k)$ *(then according to L.3. also* $N(Q, k+1)$*). Let* $k \geq n(E)$, $T \epsilon Q$, $T^0 = T/V_k$. *Then,*

$$V\{Q, T, E\} = 1 \,.\equiv.\, V\{Q^0, T^0, E\} = 1.$$

T.2. If E is a thesis, then $E \epsilon P$.

T.3. If E has Skolem's normal form for theses, $F \epsilon C\{E\}$, $\mathbf{M}\{E\} = 0$, $n \geqslant n(E)$, $Q = \mathbf{M}[n]$, $T \epsilon Q$. Then for a given function f:

If $\mathbf{M}\{F(f(s_1), \ldots, f(s_{u(F)}))\} = 0$, $\mathbf{M}/f(s_1), \ldots, f(s_{w(F)})/ = T/s_1, \ldots, s_{w(F)}$ /then $V\{Q, T, F\} = 0$.

T.4. The formula E is a thesis if and only if $E \epsilon P$; in other words: the formula E is true if and only if $E \epsilon P$.

For each i, n, Γ, F:

$1/x_i$ means, the first variable x_i such that $i \leqslant n$ and $F(x_i/a) \notin \Gamma$,

$2/x_i$ means the first variable x_i such that $i \leqslant n$ and x_i does not belong to F.

The truncated satisfiability definition given above creates sequent proof rules analogous to other cases; namely according to the interpretation E as O and according to the truncated satisfiability definition to an arbitrary formula E—called a top formula—and with $n = n(E)$ we apply the following proof rules:

$$(\mathbf{A})\quad \frac{\Gamma, E \div F}{\Gamma, E, F} \qquad (\mathbf{K})\quad \frac{\Gamma, F \div G'}{\Gamma, F' | \Gamma, G'} \qquad (\mathbf{N})\quad \frac{\Gamma, F''}{\Gamma, F}$$

$$(\Pi_1)\quad \frac{\Gamma, (\Pi a F)'}{\Gamma, (\Pi a F)', F'(x_i/a)}, \quad i = 1, 2, \ldots, n$$

$$(\Pi_2)\quad \frac{\Pi a F}{\Gamma | \Gamma_1, F(x_i^0/a)}$$

where $\Gamma(\{i_{w(F)}\}) = \Gamma_1(\{i_{w(F)}\})$ and $\Gamma(\{i_{w(F)}\})$ denotes the set of formulas with variables of indices $\{i_{u(F)}\}$. The rule (**K**) creates two diagrams and we read it: From Γ, $(F + G)$ follows Γ, F' or Γ, G'; the rule (Π_2) determines two last lines.

Applying the above rules to a given formula E we receive a generalized diagram. A generalized diagram is correct if and only if:

1. $(\Pi a F)'$ occurs in some column Γ_1 of the diagram and Γ_2 is another column such that $\Gamma_1(\{i_{u(F)}\}) = \Gamma_2(\{i_{w(F)}\})$, then $(\Pi a F)'$ occurs in Γ_2; if we need we add to Γ_2 the formula $(\Pi a F)'$.
2. The set of last lines of the generalized diagram is closed under permutation of its elements.
3. If $\Gamma_1(\{i_r\}) = \Gamma_2(\{i_r\})$, then there exists a column Γ such that for each i and j: $\Gamma(\{i_r\}, i) = \Gamma_1(\{i_r\}, i)$ and $\Gamma(\{i_r\}, j) = \Gamma_2(\{i_r\}, j)$.

T.5. If for $n = n(E)$ all lines of each column of a certain generalized and correct diagram of a formula E are not fundamental, then E is not a thesis.

T.6. If a line of a certain column of each correct diagram is fundamental for $n = n(E)$, then E is a thesis.

T.7. A formula E is a thesis if and only if for $n = n(E)$ each of its correct diagrams has a fundamental line.

The proofs of these lemmas and theorems are analogous to the proofs given in [3] and [4]; e.g. the proof of *T.5* is inductive on the length of the formula *E*. The infinite character of these theorems lies in the property $N(Q, k)$, for in order to prove the property we must consider in general an infinite number of relations of one argument. Examples are given in [4].

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