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COMBINATORIAL OPERATORS AND THEIR QUASI - INVERSES

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1. *Introduction.* Combinatorial operators were introduced by J. Myhill ([1], [2]) as a fundamental tool in the study of isols. A systematic exposition of those operators is given in the monograph [3] of J. Dekker, to which we refer for the notations. In [3], Dekker proved the following

Theorem 1.1. *Let ϕ be a combinatorial operator and ϕ^{-1} its quasi-inverse. If ϕ is recursive, then $\phi(\varepsilon)$ is a recursively enumerable set, and there is a partial recursive function χ, whose domain is $\phi(\varepsilon)$, such that*

$$\phi^{-1}(x) = \rho_{\chi(x)} \text{ for all } x \in \phi(\varepsilon). \tag{1.1.}$$

In this paper we investigate the measure in which the existence of a p.r. (partial recursive) function χ, such that $\phi^{-1}(x) = \rho_{\chi(x)}$ for all $x \in \phi(\varepsilon)$, determines the recursive character of the operator ϕ.

Besides the notations from [3], we shall use the following ones: $\langle\omega_i\rangle$, $i = 0,1,\ldots\ldots$, is the Post-enumeration of all r.e. (recursively enumerable) sets; F_R denotes the set of all r. (recursive) functions of one variable, and $\widetilde{F}_R$ denotes the set of all p.r. functions of one variable.

2. *The Fundamental Theorem.* Let ϕ be a combinatorial operator and ϕ_0 its dispersive operator. We shall say that ϕ (resp. ϕ_0) is *sub-effective* iff (if and only if) there is a disjoint r.e. sequence $\langle\omega_{\varphi_0(i)}\rangle_{i\in\varepsilon}$, $\phi_0 \in F_R$, of r.e. sets such that

$$\phi_0(\rho_n) \subset \omega_{\varphi_0(n)} \text{ for all } n \in \varepsilon. \tag{2.1.}$$

All theorems of this paper are, essentially, strengthenings of the following fundamental

Theorem 2.1. *A combinatorial operator ϕ is sub-effective iff there is a $\chi \in \widetilde{F}_R$ such that*

$$\phi^{-1}(x) = \rho_{\chi(x)} \text{ for all } x \in \phi(\varepsilon). \tag{2.2.}$$

Proof. Let ϕ be sub-effective and φ_0 as in (2.1). Then, $E = \bigcup_{i=0}^{\infty} \omega_{\varphi_0(i)}$ is a r.e. set and $\phi(\varepsilon) = \bigcup_{n=0}^{\infty} \phi_0(\rho_n) \subset E$ (where ϕ_0 is the dispersive operator of ϕ).

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Define $\chi \epsilon \widetilde{F}_R$ as follows: the domain of χ is E and, for every $x \epsilon E$, $\chi(x)$ is the unique n such that $x \epsilon \omega_{\varphi_0(n)}$. We have the obvious implication

$$(2.3) \qquad x \epsilon \phi(\varepsilon) \rightarrow x \epsilon \omega_{\varphi_0(\chi(x))}.$$

By P.7 of [3], we have

$$(2.4) \qquad x \epsilon \phi(\varepsilon) \rightarrow (x \epsilon \phi_0(\rho_n) \leftrightarrow \phi^{-1}(x) = \rho_n).$$

Thus, by (2.3) and (2.4)

$$x \epsilon \phi(\varepsilon) \rightarrow (x \epsilon \omega_{\varphi_0(\chi(x))} \leftrightarrow \phi^{-1}(x) = \rho_{\chi(x)})$$

which gives

$$\phi^{-1}(x) = \rho_{\chi(x)} \text{ for all } x \epsilon \phi(\varepsilon).$$

Conversely, suppose that ϕ is combinatorial and that there is a $\chi \epsilon \widetilde{F}_R$ such that $\phi^{-1}(x) = \rho_{\chi(x)}$ for all $x \epsilon \phi(\varepsilon)$. Let E be the domain of χ. Define the sets E_i by

$$\begin{aligned} E_i &= \{x \epsilon E \mid \phi^{-1}(x) = \rho_i\} \\ &= \{x \epsilon E \mid \rho_{\chi(x)} = \rho_i\} \\ &= \{x \epsilon E \mid \chi(x) = i\}. \end{aligned}$$

The sequence $\langle E_i \rangle_{i \epsilon \varepsilon}$ is a disjoint r.e. sequence of r.e. sets. Let $\phi_0 \epsilon F_R$ be such that, for all $i \epsilon \varepsilon$, $E_i = \omega_{\varphi_0(i)}$. Then, by (2.4)

$$\phi_0(\rho_n) \subset \omega_{\varphi_0(n)} \text{ for all } n \epsilon \phi(\varepsilon),$$

i.e. ϕ (resp. ϕ_0) is a sub-effective combinatorial operator.

In [4] the author has introduced the notions of an almost r. set and of an almost r.e. set. We show now the relation between almost r.e. sets and sub-effective combinatorial operators.

Theorem 2.2. a) *Let ϕ be a sub-effective combinatorial operator and ϕ_0 its dispersive operator. If, for every $n \epsilon \varepsilon$, $\phi_0(\rho_n)$ is not empty, then $\phi(\varepsilon)$ contains an infinite almost r.e. set.*

b) *For every infinite almost r.e. set A, there is a sub-effective combinatorial operator ϕ, such that, for every $n \epsilon \varepsilon$, $\phi_0(\rho_n)$ is not empty, and such that $A \subset \phi(\varepsilon)$.*

Proof. Define the function α as follows: $\alpha(i)$ is a chosen element of $\phi_0(\rho_i)$. Let χ be as in the first part of the proof of Theorem 2.1. Then $\chi(\alpha(i)) = i$ for all $i \epsilon \varepsilon$. Thus, A = the range of α is an almost r.e. set and $\alpha^{-1} = \chi | A$ ("$\chi | A$" means the restriction of χ to A). To prove b), let A be an almost r.e. set, and let α be a 1-1 function such that $A = \{\alpha(i) | i \epsilon \varepsilon\}$. Let $\chi \epsilon \widetilde{F}_R$ be such that $\alpha^{-1} = \chi | A$, and denote by E the domain of χ. Define the dispersive operator ϕ_0: $Q \rightarrow Q$ by $\phi_0(\rho_n) = \{\alpha(n)\}$, where $\{\alpha(n)\}$ is the singletone whose unique member is $\alpha(n)$. ϕ_0 defines a combinatorial operator ϕ: $V \rightarrow V$ by

$$\phi(\chi) = \bigcup_{\rho_n \subset \chi} \phi_0(\rho_n) = \bigcup_{\rho_n \subset \chi} \{\alpha(n)\}.$$

If $E_i = \{x \epsilon E | \chi(x) = i\}$, then $\alpha(i) \epsilon E_i$ for all $i \epsilon \varepsilon$. Thus, if $\phi_0 \epsilon F_R$ is such that $E_i = \omega_{\varphi_0(i)}$ for all $i \epsilon \varepsilon$, then $\phi_0(\rho_n) \subset \omega_{\varphi_0(n)}$ for all $n \epsilon \varepsilon$, and, as easily checked, $\phi^{-1}(x) = \rho_{\chi(x)}$ for all $x \epsilon \phi(\varepsilon)$.

3. *Strengthening of the Fundamental Theorem.* We impose now more stringent conditions on a combinatorial operator, in order to obtain necessary and sufficient conditions for Theorem 1.1. Let us call a combinatorial operator ϕ *effective* iff there is a disjoint r.e. sequence $\langle\omega_{\varphi_0(i)}\rangle_{i\epsilon N}$ of finite r.e. sets such that

(3.1) $\phi_0(\rho_n) = \omega_{\varphi_0(n)}$ for all $n \epsilon \varepsilon$,

where ϕ_0 is the dispersive operator of ϕ. ($\phi_0 \epsilon F_R$).

Theorem 3.1. *A combinatorial operator ϕ is effective iff there is a $\chi \epsilon \widetilde{F}_R$ such that $\phi(\varepsilon)$ is the domain of χ and*

(3.2) $\phi^{-1}(x) = \rho_{\chi(x)}$ *for all* $x \epsilon \phi(\varepsilon)$.

Proof. Let ϕ be effective and φ_0 as in (3.1). Then, $E = \phi(\varepsilon) = \bigcup_{n=0}^{\infty} \omega_{\varphi_0(n)}$ is a r.e. set. Define $\chi \epsilon \widetilde{F}_R$ by $\chi(x)$ = the only n such that $x \epsilon \omega_{\varphi_0(n)}$, for all $x \epsilon E$. Then, as in the proof of Theorem 2.1, we obtain (3.2). Conversely, if (3.2) holds, define φ_0 by $\omega_{\varphi_0(i)} = \{x \epsilon \text{ domain of } \chi \mid \chi(x) = i\}$. Then, by (2.4), for all $x \epsilon E = \bigcup_{i=0}^{\infty} \omega_{\varphi_0(i)}$,

$$x \epsilon \phi_0(\rho_n) \longleftrightarrow \phi^{-1}(x) = \rho_n \longleftrightarrow \chi(x) = n.$$

Thus, each $\omega_{\varphi_0(i)}$ is finite and $\phi_0(\rho_n) = \omega_{\varphi_0(n)}$ for all $n \epsilon \varepsilon$.

Corollary 3.1.1. a) *Let ϕ be an effective combinatorial operator and ϕ_0 its dispersive operator. If, for every $n \epsilon \varepsilon$, $\phi_0(\rho_n) \neq \emptyset$, then $\phi(\varepsilon)$ contains an infinite r.e. set.*

b). *For every infinite r.e. set A there is an effective combinatorial operator ϕ, such that, for every $n \epsilon \varepsilon$, $\phi_0(\rho_n) = \emptyset$, where ϕ_0 is the dispersive operator of ϕ, and $A \subset \phi(\varepsilon)$.*

Proof. Similar to the proof of Theorem 2.2.

A combinatorial operator ϕ is recursive iff its dispersive operator ϕ_0 is recursive. Thus, ϕ is recursive iff there is a $\varphi_0 \epsilon F_R$ such that $\langle\rho_{\varphi_0(i)}\rangle_{i\epsilon\varepsilon}$ is a disjoint sequence of finite sets satisfying

(3.3) $$\phi_0(\rho_n) = \rho_{\varphi_0(n)} \text{ for all } n \epsilon \varepsilon.$$

The difference between (3.1) and (3.3) is well-known: for every $n \epsilon \varepsilon$, we can find effectively the maximal member and the cardinality of $\rho_{\varphi_0(n)}$, but not of $\omega_{\varphi_0(n)}$ (although each $\omega_{\varphi_0(n)}$ is finite).

Theorem 3.2. *A combinatorial operator ϕ is recursive iff there is a $\chi \epsilon \widetilde{F}_R$ such that $\phi(\varepsilon)$ = the domain of χ,*

(3.4) $$\phi^{-1}(x) = \rho_{\chi(n)} \text{ for all } x \epsilon \phi(\varepsilon)$$

and there is a $\varphi_0 \epsilon F_R$ such that $\langle\rho_{\varphi_0(i)}\rangle_{i\epsilon\varepsilon}$ is a disjoint sequence of finite sets, satisfying $\phi(\varepsilon) = \bigcup_{i=0}^{\infty} \rho_{\varphi_0(i)}$ and

$$\rho_{\varphi_0(i)} = \{x \epsilon \phi(\varepsilon) \mid \chi(x) = i\}.$$

Proof. If the conditions of the theorem are satisfied, then, as easily checked, $\phi_0(\rho_n) = \rho_{\varphi_0(n)}$. Conversely, if $\phi_0(\rho_n) = \rho_{\varphi(n)}$ for some $\varphi \epsilon F_R$, defining χ by

$$\chi(x) = \text{the only } n \text{ such that } x \epsilon \rho_{\varphi(n)}, \text{ for all } x \epsilon \bigcup_{n=0}^{\infty} \rho_{\varphi(n)},$$

we obtain easily (3.4) with $\varphi_0 = \varphi$.

Let ϕ be a combinatorial operator. If there is a $\varphi_0 \epsilon F_R$ such that $\langle \omega_{\varphi_0(i)} \rangle_{i \epsilon \varepsilon}$ is a disjoint r.e. sequence of finite sets and

(3.5) $\phi_0(\rho_n) \subset \omega_{\varphi_0(n)}$ for all $n \epsilon \varepsilon$,

we shall say that ϕ is *finitely sub-effective*.

If there is a $\varphi_0 \epsilon F_R$ such that $\langle \rho_{\varphi_0(i)} \rangle_{i \epsilon \varepsilon}$ is a disjoint r.e. sequence of finite sets, satisfying

(3.6) $\phi_0(\rho_n) \subset \rho_{\varphi_0(n)}$ for all $n \epsilon \varepsilon$,

we shall say that ϕ is *sub-recursive*.

Similarly to previous theorems we can prove

Theorem 3.3. *A combinatorial operator ϕ is finitely sub-effective iff there is a $\chi \epsilon \widetilde{F}_R$ such that each set $E_i = \{x \epsilon$ domain of $\chi | \chi(x) = i\}$ is finite,* $\phi(\varepsilon) \subset \bigcup_{i=0}^{\infty} E_i$ and

$$\phi^{-1}(x) = \rho_{\chi(x)} \text{ for all } x \epsilon \phi(\varepsilon).$$

Theorem 3.4. *A combinatorial operator ϕ is sub-recursive iff there is a $\chi \epsilon F_R$ and a $\varphi_0 \epsilon F_R$* such that $\rho_{\varphi_0(n)} = \{x \epsilon$ *domain* $\chi | \chi(x) = n\}$, $\phi(\varepsilon) \subset$ *domain of χ and*

$$\phi^{-1}(x) = \rho_{\chi(x)} \text{ for all } x \epsilon \phi(\varepsilon).$$

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