Notre Dame Journal of Formal Logic
Volume XI, Number 2, April 1970

AFFINE GEOMETRY WITH S. DOWDY'S "TRAPEZOID" AS PRIMITIVE

ROBERT E. CLAY

In [1] S. Dowdy introduces an axiom system for affine geometry based on the primitive $\mathbf{t}(ABCD)$ which intuitively means that A, B, C, D are the vertices of a trapezoid. In this paper the system is first simplified and then altered slightly so that the defined terms which appear in the axioms can be eliminated and still produce a "reasonable" looking system. A system in which $\mathbf{c}(ABC)$, A, B, C are collinear, is the only relation which appears is then given.

The system $\mathbf{T}$ which appears in this paper is Dowdy's $\mathbf{A}*$ in [1]. System $\mathbf{T}'$ is derived from $\mathbf{T}$ by the following simplifications: Two disjuncts are removed from *D2*, one conjunct is removed from the last disjunct of *D3*, *A3* is eliminated, the equivalence in *A4* is replaced by an implication, *A5* is replaced by a shorter simpler axiom, and a conjunct is removed from the antecedent of *A8*. System $\mathbf{T}''$ is obtained from $\mathbf{T}'$ by shortening and at the same time strengthening the definition of collinearity so that *A5'*, the transitivity of collinearity, follows from *A6*, the transitivity of parallelism. The theses prefixed with an L are to be found in [1], pp. 245-255.

1. SYSTEM T

D1 $[A]:A\varepsilon\alpha\,.\equiv.\,[\exists BCD]\,.\,\mathbf{t}(ABCD)$

D2 $[AB]\,.\because.\,\mathbf{r}(AB)\,.\equiv:\,[\exists CD]:\mathbf{t}(ABCD)\,.\mathsf{v}.\,\mathbf{t}(ACBD)\,.\mathsf{v}.\,\mathbf{t}(CBAD)$

D3 $[ABC]\,.\because.\,\mathbf{c}(ABC)\,.\equiv:\,\mathbf{r}(BC):A=B\,.\mathsf{v}.\,A=C\,.\mathsf{v}.\,[\exists XY]\,.\,\mathbf{t}(BCXY)\,.\,\mathbf{t}(BAXY)\,.\,\mathbf{t}(CAXY)$

A1 $[\exists ABCD]\,.\,\mathbf{t}(ABCD)$

A2 $[ABCD]:\mathbf{t}(ABCD)\,.\supset.\,A\neq B$

A3a $[ABCD]:\mathbf{t}(ABCD)\,.\supset.\,\mathbf{t}(DCAB)$

A3b $[ABCD]:\mathbf{t}(ABCD)\,.\supset.\,\mathbf{t}(ABDC)$

A4 $[ABC]::A\,\varepsilon\alpha\,.\,B\varepsilon\alpha\,.\,C\varepsilon\alpha\,.\supset\because\sim\mathbf{c}(CAB):\equiv:[\exists D]\,.\,\mathbf{t}(ABCD)\,.\mathsf{v}.\,A=B$

A5 $[ABCMN]:A\neq B\,.\,\mathbf{c}(AMN)\,.\,\mathbf{c}(BMN)\,.\,\mathbf{c}(CMN)\,.\supset.\,\mathbf{c}(CAB)$

A6 $[ABCDEFG]:\mathbf{t}(ABCD)\,.\,\mathbf{t}(ABEF)\,.\,\mathbf{t}(CDEG)\,.\supset.\,\mathbf{t}(CDEF)$

Received December 2, 1967

A7 $[ABCD]\therefore\mathsf{t}(ABCD).\equiv:\mathsf{r}(AB).\mathsf{r}(CD):[M]:\mathsf{c}(MAB).\supset.\sim\mathsf{c}(MCD)$

A8 $[ABCDEFGHX]:\mathsf{t}(ABCD).\mathsf{t}(BEDF).\mathsf{t}(AECG).\mathsf{t}(AEFH).\mathsf{c}(XAC).$
$\mathsf{c}(XBD).\mathsf{c}(XEF).\supset.\mathsf{t}(AECF)$

2. SYSTEM T′

D1 $[A]:A\varepsilon\alpha.\equiv.[\exists BCD].\mathsf{t}(ABCD)$

D2′ $[AB]:\mathsf{r}'(AB).\equiv.[\exists CD].\mathsf{t}(ABCD)$

D3′ $[ABC]\therefore\mathsf{c}'(ABC).\equiv:\mathsf{r}'(BC):A=B.\mathrm{v}.A=C.\mathrm{v}.[\exists XY].\mathsf{t}(BCXY).$
$\mathsf{t}(BAXY)$

A1 $[\exists ABCD].\mathsf{t}(ABCD)$

A2 $[ABCD]:\mathsf{t}(ABCD).\supset.A\neq B$

A3 $[ABCD]:\mathsf{t}(ABCD).\supset.\mathsf{t}(ABDC)$

A4′ $[ABC]:A\varepsilon\alpha.B\varepsilon\alpha.C\varepsilon\alpha.\sim\mathsf{c}'(CAB).A\neq B.\supset.[\exists D].\mathsf{t}(ABCD)$

A5′ $[ABMN]:A\neq B.\mathsf{c}'(AMN).\mathsf{c}'(BMN).\supset.\mathsf{c}'(MAB)$

A6 $[ABCDEFG]:\mathsf{t}(ABCD).\mathsf{t}(ABEF).\mathsf{t}(CDEG).\supset.\mathsf{t}(CDEF)$

A7′ $[ABCD]\therefore\mathsf{t}(ABCD).\equiv:\mathsf{r}'(AB).\mathsf{r}'(CD):[M]:\mathsf{c}'(MAB).\supset.\sim\mathsf{c}'(MCD)$

A8′ $[ABCDEFGX]:\mathsf{t}(ABCD).\mathsf{t}(BEDF).\mathsf{t}(AECG).\mathsf{c}'(XAC).\mathsf{c}'(XBD).$
$\mathsf{c}'(XEF).\supset.\mathsf{t}(AECF)$

2.1 Equivalence of systems T and T′

2.1.1 System **T** implies system **T′**

Z0 $[ABCD]:\mathsf{t}(ABCD).\supset.A\varepsilon\alpha.B\varepsilon\alpha.C\varepsilon\alpha.D\varepsilon\alpha$ [*L1, L2, L3, D1*]

Z1 $[AB]:\mathsf{r}'(AB).\supset.\mathsf{r}(AB)$ [*D2, D2′*]

L22a $[ABC]:A\neq B.\mathsf{c}(ACB).\supset.\mathsf{c}(CAB)$ [*L22*]

Z2 $[ABCD]:\mathsf{t}(ACBD).\supset.[\exists E].\mathsf{t}(ABCE)$

PF $[ABCD]:\mathrm{Hyp}(1).\supset.$

2) $\sim\mathsf{c}(BAC).$ [*A4, Z0*, 1]

3) $\sim\mathsf{c}(BCA).$ [*L23*, 2]

4) $A\neq C.$ [*A2*, 1]

5) $\sim\mathsf{c}(CBA).$ [*L22a*, 3, 4]

6) $\sim\mathsf{c}(CAB).$ [*L23*, 5]

7) $A\neq B.$ [*L14*, 1]

$[\exists E].\mathsf{t}(ABCE)$ [*A4, Z0*, 1, 6, 7]

Z3 $[AB]:\mathsf{r}(AB).\supset.\mathsf{r}'(AB)$

PF $[AB]\therefore\mathrm{Hyp}(1).\supset:$

2) $[\exists CD].\mathsf{t}(ABCD).\mathrm{v}.\mathsf{t}(ACBD).\mathrm{v}.\mathsf{t}(CBAD):$ [*D2*, 1]

3) $[\exists CD].\mathsf{t}(ABCD).\mathrm{v}.\mathsf{t}(ACBD).\mathrm{v}.\mathsf{t}(ADBC):$ [*L5*, 2]

4) $[\exists CDEF].\mathsf{t}(ABCD).\mathrm{v}.\mathsf{t}(ABCE).\mathrm{v}.\mathsf{t}(ABDF):$ [3, *Z2*]

$\mathsf{r}'(AB)$ [*D2′*, 4]

Z4 $[AB]:\mathsf{r}(AB).\equiv.\mathsf{r}'(AB)$ [*Z1, Z3*]

Z5 $[ABC]:\mathsf{c}(ABC).\supset.\mathsf{c}'(ABC)$ [*D3, D3′, Z4*]

Z6 $[BC]:\mathsf{c}(BBC).\equiv.\mathsf{c}'(BBC)$ [*D3, D3′, Z4*]

Z7 $[BC]:\mathsf{c}(CBC).\equiv.\mathsf{c}'(CBC)$ [*D3, D3′, Z4*]

Z8 $[ABC]:\mathsf{c}'(ABC).A\neq B.A\neq C.\supset.\mathsf{c}(ABC)$

PF $[ABC]\therefore\mathrm{Hyp}(3).\supset:$

$[\exists XY]:$

4)	$\mathsf{t}(BCXY)$.	[$D3'$, 1, 2, 3]
5)	$\mathsf{t}(BAXY)$.	
6)	$\mathsf{t}(XYBC)$.	[$L5$, 4]
7)	$\mathsf{t}(XYAB)$:	[$L2$, 5]
8)	$\mathsf{t}(BCAB) . \mathrm{v} . [D] . \sim\mathsf{t}(BCAD)$:	[$A6$, 6, 7]
9)	$[D] . \sim\mathsf{t}(BCAD)$.	[$L16$, 8]
10)	$B \neq C$.	[$A2$, 4]
	$\mathsf{c}(ABC)$	[$A4$, $Z0$, 4, 5, 10, 9]

Z9 $[ABC] : \mathsf{c}'(ABC) . \supset . \mathsf{c}(ABC)$ [$Z8$, $Z6$, $Z7$]

Z10 $[ABC] : \mathsf{c}'(ABC) . \equiv . \mathsf{c}(ABC)$ [$Z5$, $Z9$]

Z11 $[ABMN] : A \neq B . \mathsf{c}(AMN) . \mathsf{c}(BMN) . \supset . \mathsf{c}(MAB)$

PF $[ABMN]$: Hyp (3) $. \supset .$

4)	$\mathsf{r}(MN)$.	[$D3$, 3]
5)	$\mathsf{c}(MMN)$.	[$D3$, 4]
	$\mathsf{c}(MAB)$	[$A5$, 1, 2, 3, 5]

Z12 $[ABCDEFGX] : \mathsf{t}(ABCD) . \mathsf{t}(BEDF) . \mathsf{t}(AECG) . \mathsf{c}(XAC) . \mathsf{c}(XBD) .$
$\mathsf{c}(XEF) . [H] . \sim\mathsf{t}(AEFH) . \supset . [\exists H] . \mathsf{t}(AEFH)$

PF $[ABCDEFGX]$: Hyp(7) $. \supset .$

8)	$A \neq E$.	[$A2$, 2]
9)	$\mathsf{c}(FAE)$.	[$A4$, $Z0$, 1, 2, 7, 8]
10)	$E \neq F$.	[$L17$, 2]
11)	$\mathsf{c}(AFE)$.	[$L22a$, 9, 10]
12)	$\mathsf{c}(AEF)$.	[$L23$, 11]
13)	$\mathsf{t}(ABDC)$.	[$A3b$, 1]
14)	$\sim\mathsf{c}(DAB)$.	[$A4$, $Z0$, 1, 13]
15)	$A \neq B$.	[$A2$, 1]
16)	$\sim\mathsf{c}(ADB)$.	[$L22a$, 15]
17)	$\sim\mathsf{c}(ABD)$.	[$L23$, 16]
18)	$X \neq A$.	[17, 5]
19)	$\mathsf{c}(EAX)$.	[$Z11$, 18, 12, 6]
20)	$\mathsf{c}(XCA)$.	[$L23$, 4]
21)	$\mathsf{c}(CXA)$.	[$L22$, 20, 18]
22)	$\mathsf{c}(CAX)$.	[$L23$, 21]
23)	$E \neq C$.	[$L15$, 3]
24)	$\mathsf{c}(ACE)$.	[$Z11$, 23, 22, 19]
25)	$\mathsf{c}(CAE)$.	[$L22a$, 24, 8]
26)	$\sim\mathsf{c}(CAE)$.	[$A4$, $Z0$, 3]
	$[\exists H] . \mathsf{t}(AEFH)$	[25, 26]

Z13 $[ABCDEFGX] : \mathsf{t}(ABCD) . \mathsf{t}(BEDF) . \mathsf{t}(AECG) . \mathsf{c}(XAC) . \mathsf{c}(XBD) .$
$\mathsf{c}(XEF) . \supset . \mathsf{t}(AECF)$ [$A8$, $Z12$]

A4′ $[ABC] : A \varepsilon \alpha . B \varepsilon \alpha . C \varepsilon \alpha . \sim\mathsf{c}'(CAB) . A \neq B . \supset . [\exists D] . \mathsf{t}(ABCD)$ [$A4$, $Z10$]

A5′ $[ABMN] : A \neq B . \mathsf{c}'(AMN) . \mathsf{c}'(BMN) . \supset . \mathsf{c}'(MAB)$ [$Z11$, $Z10$]

A7′ $[ABCD] \therefore \mathsf{t}(ABCD) . \equiv : \mathsf{r}'(AB) . \mathsf{r}'(CD) : [M] : \mathsf{c}'(MAB) . \supset . \sim\mathsf{c}'(MCD)$
[$A7$, $Z4$, $Z10$]

A8′ $[ABCDEFGX] : \mathsf{t}(ABCD) . \mathsf{t}(BEDF) . \mathsf{t}(AECG) . \mathsf{c}'(XAC) . \mathsf{c}'(XBD) .$
$\mathsf{c}'(XEF) . \supset . \mathsf{t}(AECF)$ [$Z13$, $Z10$]

Therefore **T** implies **T′**.

2.1.2 System $\mathbf{T}'$ implies system $\mathbf{T}$

L14 $[ABCD]:\mathsf{t}(ABCD).\supset.A\neq C.$ (New proof)
PF $[ABCD].\because.\mathrm{Hyp}(1).\supset:$
2) $\mathsf{r}(AB).$
3) $\mathsf{r}(CD):$ [*A7*, 1]
4) $[M]:\mathsf{c}(MAB).\supset.\sim\mathsf{c}(MCD):$
5) $\mathsf{c}(AAB).$ [*D3*, 2]
6) $\mathsf{c}(CCD).$ [*D3*, 3]
7) $\sim\mathsf{c}(ACD).$ [4, 5]
$A\neq C$ [6, 7]

L20 $[ABC]:\mathsf{c}(ABC).\supset.A\varepsilon\alpha.B\varepsilon\alpha.C\varepsilon\alpha$ (New proof)
PF $[ABC].\because.\mathrm{Hyp}(1).\supset:$
2) $\mathsf{r}(BC):$
3) $A=B.\mathrm{v}.A=C.\mathrm{v}.[\exists XY].\mathsf{t}(BAXY):$ [*D3*, 1]
4) $A=B.\mathrm{v}.A=C.\mathrm{v}.[\exists XY].\mathsf{t}(ABXY):$ [*L4*, 3]
5) $B\varepsilon\alpha.$
6) $C\varepsilon\alpha.$ [*L19*, 2]
7) $A\varepsilon\alpha.$ [4, 5, 6, *D1*]
$A\varepsilon\alpha.B\varepsilon\alpha.C\varepsilon\alpha$ [5, 6, 7]

Substituting the two above proofs for the originals, *L1* through *L20* are now valid in $\mathbf{T}'$, if c and r are replaced by c' and r' respectively in both statements and proofs and any axiom used as a reason is replaced by the corresponding primed axiom.

Z'2 $[ABCD]:\mathsf{t}(ABCD).\supset.\sim\mathsf{c}'(CAB)$
PF $[ABCD].\because.\mathrm{Hyp}.\supset:$
2) $\mathsf{t}(CDAB).$ [*L5*, 1]
3) $\mathsf{r}'(CD):$
4) $[M]:\mathsf{c}'(MCD).\supset.\sim\mathsf{c}'(MAB):$ [*A7'*, 2]
5) $\mathsf{c}'(CCD).$ [*D3'*, 3]
$\sim\mathsf{c}'(CAB)$ [4, 5]

Z'3 $[ABC]:\mathsf{c}'(ABC).\supset.\mathsf{c}'(ACB)$
PF $[ABC]:\mathrm{Hyp}(1).\supset.$
2) $\mathsf{r}'(BC).$ [*D3'*, 1]
3) $B\neq C.$ [*L19*, 2]
4) $[D].\sim\mathsf{t}(BCAD).$ [*Z'2*, 1]
5) $[D].\sim\mathsf{t}(CBAD).$ [*L4*, 4]
$\mathsf{c}'(ACB)$ [*A4'*, *L20*, 1, 3, 5]

Z'4 $[ABC]:\mathsf{c}'(ABC).A\neq B.A\neq C.\supset.\mathsf{c}'(CBA)$ [*D3'*]
Z'5 $[ABC]:\mathsf{c}'(ABC).A\neq B.\supset.\mathsf{c}'(CBA)$ [*Z'4*]
Z'6 $[ABC]:\mathsf{c}'(ABC).A\neq C.\supset.\mathsf{c}'(BAC)$ [*Z'3*, *Z'5*, *Z'3*]

Z0 through *Z7* are now valid in $\mathbf{T}'$ if c is replaced by c' in the proof of *Z2*, and the theses given as reasons are replaced by their analogs in $\mathbf{T}'$, e.g. *A4* is replaced by *Z'2* in the proof of *Z2*.

Z'7 $[ABCDMN]:\mathsf{t}(ABCD).\mathsf{c}'(ABM).\mathsf{c}'(NBM).A\neq N.\supset.\sim\mathsf{c}'(NCD)$
PF $[ABCDMN].\because.\mathrm{Hyp}(4).\supset.$

	5)	$\mathbf{c}'(BAN)$.	[$A5'$, 4, 2, 3]
	6)	$A \neq B$.	[$A2$, 1]
	7)	$\mathbf{c}'(NAB)$:	[$Z'5$, 5, 6]
	8)	$[P]:\mathbf{c}'(PAB).\supset.\sim\mathbf{c}'(PCD)$:	[$A7'$, 1]
		$\sim\mathbf{c}'(NCD)$	[8, 7]
$Z'8$		$[ABCD]:\mathbf{t}(ABCD).\supset.\sim\mathbf{c}'(ACD)$	
PF		$[ABCD]$.˙. Hyp(1)$.\supset$:	
	2)	$\mathbf{r}'(AB)$:	[$A7'$, 1]
	3)	$[M]:\mathbf{c}'(MAB).\supset.\sim\mathbf{c}'(MCD)$:	
	4)	$\mathbf{c}'(AAB)$.	[$D3'$, 2]
		$\sim\mathbf{c}'(ACD)$	[3, 4]
$Z'9$		$[ABCDMN]:\mathbf{t}(ABCD).\mathbf{c}'(ABM).\supset.\mathbf{t}(BMCD)$	
PF		$[ABCDMN]$.˙. Hyp. (2)$.\supset$:	
	3)	$\mathbf{r}'(CD)$.	[$A7'$, 1]
	4)	$\mathbf{r}'(BM)$:	[$D3'$, 2]
	5)	$[N]:\mathbf{c}'(NBM).\supset.\sim\mathbf{c}'(NCD)$:	[$Z'7$, 1, 2, $Z'8$, 1]
		$\mathbf{t}(BMCD)$	[$A7'$, 4, 3, 5]
$Z'10$		$[ABC]:\mathbf{c}'(ABC).A\neq C.\supset.\mathbf{c}(ABC)$	
PF		$[ABC]$: Hyp. (2)$.\supset$.	
	3)	$\mathbf{c}'(BAC)$.	[$Z'6$, 1, 2]
	4)	$\mathbf{c}'(BCA)$.	[$Z'3$, 3]
	5)	$\mathbf{r}'(BC)$.	[$D3'$, 1]
		$[\exists XY]$.	
	6)	$\mathbf{t}(BCXY)$.	[$D3'$, 1]
	7)	$\mathbf{t}(BAXY)$.	
	8)	$\mathbf{t}(CAXY)$.	[$Z'9$, 6, 4]
		$\mathbf{c}(ABC)$	[$D3$, $Z4$, 5, 6, 7, 8]
$Z'11$		$[ABC]:\mathbf{c}'(ABC).\equiv.\mathbf{c}(ABC)$	[$Z'10$, $Z7$, $Z5$]
$Z'12$		$[ABC]:A\varepsilon\alpha.B\varepsilon\alpha.C\varepsilon\alpha.A=B.\supset.\sim\mathbf{c}'(CAB)$	[$L19$, $D3'$]
$A4$		$[ABC]::A\varepsilon\alpha.B\varepsilon\alpha.C\varepsilon\alpha.\supset$.˙. $\sim\mathbf{c}(CAB).\equiv:[\exists D].\mathbf{t}(ABCD).\mathrm{v}.A=B$	[$Z'11$, $A4'$; $Z'11$, $Z'12$, $Z'2$]
$Z'13$		$[ABMN]:A\neq B.\mathbf{c}'(AMN).\mathbf{c}'(BMN).\supset.\mathbf{c}'(AAB).\mathbf{c}'(BAB)$	
PF		$[ABMN]$: Hyp(3)$.\supset$.	
	4)	$A\varepsilon\alpha$.	[$L20$, 2]
	5)	$B\varepsilon\alpha$.	[$L20$, 3]
	6)	$\mathbf{r}'(AB)$.	[$L19$, 4, 5, 1]
	7)	$\mathbf{r}'(BA)$.	
	8)	$\mathbf{c}'(AAB)$.	[$D3'$, 6]
	9)	$\mathbf{c}'(BBA)$.	[$D3'$, 7]
	10)	$\mathbf{c}'(BAB)$.	[$L23$, 9]
		$\mathbf{c}'(AAB).\mathbf{c}'(BAB)$	[8, 10]
$Z'14$		$[ABCN]:A\neq B.B\neq C.\mathbf{c}'(BAN).\mathbf{c}'(CAN).\supset.\mathbf{c}'(CAB)$	
PF		$[ABCN]$: Hyp(4)$.\supset$.	
	5)	$\mathbf{c}'(ACB)$.	[$A5'$, 2, 4, 3]
		$\mathbf{c}'(CAB)$	[$L22a$, 5, 1]
$Z'15$		$[ABCMN]:A\neq B.A\neq C.B\neq C.A\neq M.\mathbf{c}'(AMN).\mathbf{c}'(BMN).\mathbf{c}'(CMN)$ $.\supset.\mathbf{c}'(CAB)$	

PF $[ABCMN]:\mathrm{Hyp}(7).\supset.$

8) $\mathsf{c}'(MBA).$ [$A5'$, 1, 6, 5]

9) $\mathsf{c}'(MCA).$ [$A5'$, 2, 7, 5]

10) $\mathsf{c}'(BMA).$ [$L22a$, 8, 4]

11) $\mathsf{c}'(CMA).$ [$L22a$, 9, 4]

12) $\mathsf{c}'(BAM).$ [$L23$, 10]

13) $\mathsf{c}'(CAM).$ [$L23$, 11]

14) $\mathsf{c}'(ACB).$ [$A5'$, 3, 13, 12]

$\mathsf{c}'(CAB)$ [$L22a$, 14, 1]

A5 $[ABCMN]:A\neq B.\mathsf{c}(AMN).\mathsf{c}(BMN).\mathsf{c}(CMN).\supset.\mathsf{c}(CAB)$ [$Z'11$, $Z'15$, $Z'13$, $Z'14$]

A7 $[ABCD].\because.\mathsf{t}(ABCD).\equiv:\mathsf{r}(AB).\mathsf{r}(CD):[M]:\mathsf{c}(MAB).\supset.\sim\mathsf{c}(MCD)$ [$A7'$, $Z4$, $Z'11$]

A8 $[ABCDEFGHX]:\mathsf{t}(ABCD).\mathsf{t}(BEDF).\mathsf{t}(AECG).\mathsf{t}(AEFH).\mathsf{c}(XAC).$
$\mathsf{c}(XBD).\mathsf{c}(XEF).\supset.\mathsf{t}(AECF)$ [$A8'$, $Z'11$]

Therefore $\mathbf{T}'$ implies $\mathbf{T}$. So the two systems are equivalent.

In $\mathbf{T}'$, *D1* and *D2'* may easily be eliminated, thus leaving *D3'* as the only important definition. We can strengthen *D3'* so that transitivity is part of the definition. This will enable us to eliminate *A5'* entirely. The new definition of collinearity is also one atom shorter than *D3'*. These simplifications will make it possible to introduce a fairly natural system in which only the primitive '**t**' appears.

3. SYSTEM $\mathbf{T}''$

D1 $[A]:A\,\varepsilon\,\alpha.\equiv.[\exists BCD].\mathsf{t}(ABCD)$

D2' $[AB]:\mathsf{r}'(AB).\equiv.[\exists CD].\mathsf{t}(ABCD)$

D3'' $[ABC]::\mathsf{c}''(ABC).\equiv.\because.\mathsf{r}'(BC).\because.A=B.\mathsf{v}:[XY]:\mathsf{t}(BCXY).\supset.\mathsf{t}(BAXY)$

A1 $[\exists ABCD].\mathsf{t}(ABCD)$

A2 $[ABCD]:\mathsf{t}(ABCD].\supset.A\neq B$

A3 $[ABCD]:\mathsf{t}(ABCD].\supset.\mathsf{t}(ABDC)$

A4'' $[ABC]:A\,\varepsilon\,\alpha.B\,\varepsilon\,\alpha.C\,\varepsilon\,\alpha.\sim\mathsf{c}''(CAB).A\neq B.\supset.[\exists D].\mathsf{t}(ABCD)$

A6 $[ABCDEFG]:\mathsf{t}(ABCD].\mathsf{t}(ABEF).\mathsf{t}(CDEG).\supset.\mathsf{t}(CDEF)$

A7'' $[ABCD].\because.\mathsf{t}(ABCD).\equiv:\mathsf{r}'(AB).\mathsf{r}'(CD):[M]:\mathsf{c}''(MAB).\supset.\mathsf{c}''(MCD)$

A8'' $[ABCDEFGX]:\mathsf{t}(ABCD).\mathsf{t}(BEDF).\mathsf{t}(AECG).\mathsf{c}''(XAC).\mathsf{c}''(XBD).$
$\mathsf{c}''(XEF).\supset.\mathsf{t}(AECF)$

3.1 Equivalence of systems $\mathbf{T}'$ and system $\mathbf{T}''$

3.1.1 System $\mathbf{T}'$ implies system $\mathbf{T}''$

Z'16 $[BC]:\mathsf{c}''(BBC).\equiv.\mathsf{c}'(BBC)$ [$D3'$, $D3''$]

Z'17 $[BC]:\mathsf{c}''(CBC).\equiv.\mathsf{c}'(CBC)$ [$D3'$, $D3''$]

Z'18 $[ABC]:\mathsf{c}''(ABC).A\neq B.\supset.\mathsf{c}'(ABC)$

PF $[ABC].\because.\mathrm{Hyp}(2).\supset:$

3) $\mathsf{r}'(BC):$

4) $[XY]:\mathsf{t}(BCXY).\supset.\mathsf{t}(BAXY):$ [$D3''$, 1, 2]

$[\exists DE].$

	5)	$\mathsf{t}(BCDE)$.	[$D2'$, 3]
	6)	$\mathsf{t}(BADE)$.	[4, 5]
		$\mathsf{c}'(ABC)$	[$D3'$, 3, 5, 6]
$Z'19$		$[ABC]:\mathsf{c}''(ABC).\supset.\mathsf{c}'(ABC)$	[$Z'16$, $Z'17$, $Z'18$]
$Z'20$		$[ABC]:\mathsf{c}'(ABC).A\neq B.\mathsf{t}(BCXY).\supset.\mathsf{t}(BAXY)$	
PF		$[ABC]:\mathrm{Hyp}(3).\supset.$	
	4)	$\mathsf{t}(CBXY)$.	[$L4$, 3]
	5)	$\mathsf{c}'(ACB)$.	[$Z'3$, 1]
	6)	$\mathsf{c}'(CAB)$.	[$Z'6$, 5, 2]
	7)	$\mathsf{c}'(CBA)$.	[$Z'3$, 6]
		$\mathsf{t}(BAXY)$	[$Z'9$, 4, 7]
$Z'21$		$[ABC]\therefore\mathsf{c}'(ABC).A\neq B.\supset:[XY]:\mathsf{t}(BCXY).\supset.\mathsf{t}(BAXY)$	[$Z'20$]
$Z'22$		$[ABC]:\mathsf{c}'(ABC).A\neq B.\supset.\mathsf{c}''(ABC)$	[$D3''$, $D3'$, $Z'21$]
$Z'23$		$[ABC]:\mathsf{c}'(ABC).\supset.\mathsf{c}''(ABC)$	[$Z'22$, $Z'16$]
$Z'24$		$[ABC]:\mathsf{c}'(ABC).\equiv.\mathsf{c}''(ABC)$	[$Z'19$, $Z'23$]

$A4''$, $A7''$, and $A8''$ now follow immediately from $Z'24$ and $A4'$, $A7'$, and $A8'$. Therefore $\mathbf{T}'$ implies $\mathbf{T}''$.

3.1.2 System $\mathbf{T}''$ implies sytem $\mathbf{T}'$

In $\mathbf{T}''$, $L1$ through $L19$ hold with c' replaced by c'' in both theorems and proofs and with $D3'$ replaced by $D3''$ in the reasons; $Z''16$ through $Z'19$ and $Z0$ hold as they stand. $L20$ holds with c replaced by c''.

$Z''1$	$[ABC]:\mathsf{c}''(ABC).\supset.A\varepsilon\alpha.B\varepsilon\alpha.C\varepsilon\alpha$	[$L20$, $Z'19$]

$Z'2$ and $Z'3$ are now valid for c'' (with the obvious replacements).

$Z''2$		$[ABC]:\mathsf{c}''(ABC).A\neq B.\supset.\mathsf{c}''(CBA)$	
PF		$[ABC]\therefore\mathrm{Hyp}(2).\supset:$	
	3)	$\mathsf{r}'(BC):$	[$D3''$, 1, 2]
	4)	$[XY]:\mathsf{t}(BCXY).\supset.\mathsf{t}(BAXY):$	[$D3''$, 1, 2]
		$[\exists DE]:$	
	5)	$\mathsf{t}(BCDE)$.	[$D2'$, 4]
	6)	$\mathsf{t}(BADE)$.	[5, 6]
	7)	$\mathsf{t}(DECB)$.	[$L2$, 5]
	8)	$\mathsf{t}(DEBA):$	[$L5$, 6]
	9)	$\mathsf{t}(BACB).\mathrm{v}.[F].\sim\mathsf{t}(BACF):$	[$A6$, 7, 8]
	10)	$[F].\sim\mathsf{t}(BACF)$.	[$L16$, 9]
		$\mathsf{c}''(CBA)$	[$A4''$, $Z0$, 5, 6, 2, 10]
$Z''3$		$[ABMN]:A\neq B.A\neq M.B\neq M.\mathsf{c}''(AMN).\mathsf{c}''(BMN).\supset.\mathsf{c}''(MAB)$	
PF		$[ABMN]\therefore\mathrm{Hyp}(5).\supset:$	
	6)	$\mathsf{r}'(MN):$	[$D3''$, 4]
	7)	$[XY]:\mathsf{t}(MNXY).\supset.\mathsf{t}(MAXY):$	[$D3''$, 4, 2]
	8)	$[XY]:\mathsf{t}(MNXY).\supset.\mathsf{t}(MBXY):$	[$D3''$, 5, 3]
		$[\exists DE]$.	
	9)	$\mathsf{t}(MNDE)$.	[$D2'$, 6]
	10)	$\mathsf{t}(MADE)$.	[7, 9]
	11)	$\mathsf{t}(MBDE)$.	[8, 9]

12) $\mathbf{t}(DEAM)$. [$L2$, 10]
13) $\mathbf{t}(DEBM)$: [$L2$, 11]
14) $\mathbf{t}(AMBM).\mathrm{v}.[F].\sim\mathbf{t}(AMBF)$: [$A6$, 12, 13]
15) $[F].\sim\mathbf{t}(AMBF)$. [$L17$, 14]
16) $\mathbf{c}''(BAM)$. [$A4''$, $Z0$, 10, 11, 1, 15]
17) $\mathbf{c}''(MAB)$ [$Z''2$, 16, 1]

$Z''4$ $[ABMN]:A\neq B.\mathbf{c}''(AMN).\mathbf{c}''(BMN).\supset.\mathbf{c}''(AAB).\mathbf{c}''(BAB)$ [$Z''1$, $L19$, $D3''$]

$Z''5$ $[ABMN]:A\neq B.\mathbf{c}''(AMN).\mathbf{c}''(BMN).\supset.\mathbf{c}''(MAB)$ [$Z''3$, $Z''4$]

$Z''6$ $[ABC]:\mathbf{c}'(ABC).A\neq B.A\neq C.\supset.\mathbf{c}''(ABC)$

PF $[ABC]\therefore\mathrm{Hyp}(3).\supset$:
4) $\mathbf{r}'(BC)$. [$D3'$, 1]
5) $B\neq C$: [$D2'$, 4, $A2$]
$[\exists XY]$.
6) $\mathbf{t}(BCXY)$.
7) $\mathbf{t}(BAXY)$. [$D3'$, 1, 2, 3]
8) $\mathbf{t}(XYBC)$. [$L5$, 6]
9) $\mathbf{t}(XYAB)$: [$L2$, 7]
10) $\mathbf{t}(BCAB).\mathrm{v}.[D].\sim\mathbf{t}(BCAD)$ [$A6$, 8, 9]
11) $[D].\sim\mathbf{t}(BCAD)$ [$L16$, 10]
12) $\mathbf{c}''(ABC)$ [$A4''$, $L20$, 1, 5, 11]

$Z''7$ $[ABC]:\mathbf{c}'(ABC).\supset.\mathbf{c}''(ABC)$ [$Z''6$, $Z'16$, $Z'17$]

$Z''8$ $[ABC]:\mathbf{c}'(ABC).\equiv.\mathbf{c}''(ABC)$ [$Z''7$, $Z'19$]

$A4'$, $A4'$, $A7'$, and $A8'$ now follow immediately from $Z''8$ and $A4''$, $Z''5$, $A7''$, and $A8''$. Therefore $\mathbf{T}''$ implies $\mathbf{T}'$. So $\mathbf{T}''$ is equivalent to $\mathbf{T}'$.

4. SYSTEM $\mathbf{T}^+$

We now list a system in which $\mathbf{t}$ is the only relation which appears.

T^+1 $[\exists ABCD].\mathbf{t}(ABCD)$

T^+2 $[ABCD]:\mathbf{t}(ABCD).\supset.A\neq B$

T^+3 $[ABCD]:\mathbf{t}(ABCD).\supset.\mathbf{t}(ABDC)$

T^+4 $[ABCEFGHIJKLM]\therefore\mathbf{t}(AEFG).\mathbf{t}(BHIJ).\mathbf{t}(CKLM).[D].\sim\mathbf{t}(ABCD).$
$A\neq B.\supset:[\exists NO].\mathbf{t}(ABNO):A=C.\mathrm{v}:[XY]:\mathbf{t}(ABXY).\supset.\mathbf{t}(ACXY)$

T^+6 $[ABCDEFG]:\mathbf{t}(ABCD).\mathbf{t}(ABEF).\mathbf{t}(CDEG).\supset.\mathbf{t}(CDEF)$

T^+7 $[ABCD]::\mathbf{t}(ABCD).\equiv\therefore[\exists EFGH].\mathbf{t}(ABEF).\mathbf{t}(CDGH)\therefore[M]\therefore$
$A=M.\mathrm{v}:[XY]:\mathbf{t}(ABXY).\supset.\mathbf{t}(AMXY):\supset.M\neq C.[\exists IJ].\mathbf{t}(CDIJ).$
$\sim\mathbf{t}(CMIJ)$

T^+8 $[ABCDEFGX]::\mathbf{t}(ABCD).\mathbf{t}(BEDF).\mathbf{t}(AECG)\therefore[HI]\therefore\mathbf{t}(ACHI).\supset.$
$\mathbf{t}(AXHI):\mathbf{t}(BDHI).\supset.\mathbf{t}(BXHI):\mathbf{t}(EFHI).\supset.\mathbf{t}(EXHI)\therefore\supset.\mathbf{t}(AECF)$

T^+1, T^+2, T^+3, T^+6 are identical with axioms of $\mathbf{T}''$. T^+4 is merely $A4''$ with substitutions from $D1$, $D2'$, and $D3''$. The first two conditions to the right of the equivalence sign in T^+7 are substitutions from $D2'$. They eliminate the necessity of using $\mathbf{r}'(AB)$ and $\mathbf{r}'(CD)$ when the substitutions from $D3''$ are made in the third condition to the right of the equivalence sign. The proof which follows shows that T^+8 is equivalent to $A8''$ in the system $\mathbf{T}''$ from which $A8''$ has been deleted.

4.1 Equivalence of systems $\mathbf{T}''$ and $\mathbf{T}^+$

Z″9 $[ABCDX]::\mathbf{t}(ABCD).\mathbf{c}''(XBD).\supset.\because.\mathbf{c}''(XAC).\equiv:[FG]:\mathbf{t}(ACFG).\supset.\mathbf{t}(AXFG)$

PF $[ABCDX]::\mathrm{Hyp}(2).\supset.\because.$

3)	$\mathbf{t}(ABDC)$.	$[A3,\ 1]$
4)	$\sim\mathbf{c}''(DAB)$.	$[Z'2,\ 3]$
5)	$A\neq B$.	$[A2,\ 1]$
6)	$\sim\mathbf{c}''(ADB)$.	$[Z'6,\ 4,\ 5]$
7)	$\sim\mathbf{c}''(ABD)$.	$[L23,\ 6]$
8)	$X\neq A$.	$[2,\ 7]$
9)	$A\neq C$.	$[L14,\ 1]$
10)	$\mathbf{r}'(AC).\because.$	$[L19,\ Z0,\ 1,\ 9]$
	$\mathbf{c}''(XAC).\equiv:[FG]:\mathbf{t}(ACFG).\supset.\mathbf{t}(AXFG)$	$[D3'',\ 10,\ 5]$

Z″10 $[BDEFX]::\mathbf{t}(BEDF):[HI]:\mathbf{t}(BDHI).\supset.\mathbf{t}(BXHI):\supset.\because.\mathbf{c}''(XEF).\equiv:[JK]:\mathbf{t}(EFJK).\supset.\mathbf{t}(EXJK)$

PF $[BDEFX]::\mathrm{Hyp}\,(2).\supset.\because.$

3)	$B\neq D$.	$[L14,\ 1]$
4)	$\mathbf{r}'(BD)$.	$[L19,\ Z0,\ 1,\ 3]$
5)	$\mathbf{c}''(XBD)$.	$[D3'',\ 4,\ 2]$
6)	$\mathbf{t}(EBFD).\because.$	$[L7,\ 1]$
	$\mathbf{c}''(XEF).\equiv:[JK]:\mathbf{t}(EFJK).\supset.\mathbf{t}(EXJK)$	$[Z''9,\ 6,\ 5]$

Z″11 $A8''.\equiv.T^{+}8$ $[Z''9,\ Z''9,\ Z''10]$

Therefore $\mathbf{T}''$ is equivalent to $\mathbf{T}^+$.

A slight variation of $\mathbf{T}^+$ can be made by replacing $T^{+}4$ with

$T^{+}4a$ $[ABDEFGHI]:\mathbf{t}(ADEF).\mathbf{t}(BGHI).A\neq B.\supset.[\exists JK].\mathbf{t}(ABJK)$

and

$T^{+}4b$ $[ABCEFXY]:\mathbf{t}(ACEF).\mathbf{t}(ABXY).\mathbf{t}(ACXY).\supset.[\exists D].\mathbf{t}(ABCD).$

The equivalence is easily proved.

5. SYSTEM C

We now give an axiom system in which collinearity is the sole primitive.

C1a $[\exists AB].\mathbf{c}(AAB)$

C1b $[ABC]:\mathbf{c}(ABC).\supset.[\exists DE].\mathbf{c}(EED).\sim\mathbf{c}(DBC)$

C2 $[ABC]:\mathbf{c}(ABC).\supset.B\neq C.$

C3 $[ABC]:\mathbf{c}(ABC).\supset.\mathbf{c}(ACB)$

C4 $[ABCEFGHIJ]::\mathbf{c}(EFA).\mathbf{c}(GHB).\mathbf{c}(IJC).\sim\mathbf{c}(CAB).A\neq B.\supset.\because.\mathbf{c}(AAB).\because.[\exists D].\because.\mathbf{c}(CCD):\mathbf{c}(MAB).\supset.\sim\mathbf{c}(MCD)$

C5 $[ABMN]:A\neq B.\mathbf{c}(AMN).\mathbf{c}(BMN).\supset.\mathbf{c}(MAB)$

C6 $[ABCDEFN].\because.\mathbf{c}(AAB).\mathbf{c}(CCD).\mathbf{c}(EEF):[M]:\mathbf{c}(MAB).\supset.\sim\mathbf{c}(MCD).\sim\mathbf{c}(MEF):\sim\mathbf{c}(ECD).\mathbf{c}(NCD).\supset.\sim\mathbf{c}(NEF)$

C8 $[ABCDEFNX]::\mathbf{c}(AAB).\mathbf{c}(CCD).\mathbf{c}(BBE).\mathbf{c}(DDF).\because.[M].\because.\mathbf{c}(MAB).\supset.\sim\mathbf{c}(MCD):\mathbf{c}(MBE).\supset.\sim\mathbf{c}(MDF).\because.\sim\mathbf{c}(CAE).\mathbf{c}(XAC).\mathbf{c}(XBD).\mathbf{c}(XEF).\mathbf{c}(NAE).\because.\supset.\sim\mathbf{c}(NCF)$

5.1 Equivalence of systems $\mathbf{T}''$ and $\mathbf{C}$

To prove the equivalence of $\mathbf{C}$ to $\mathbf{T}''$ we introduce the following definitions.

DC1	$[A]:A\,\varepsilon\,\alpha\,.\equiv.\,[\exists BC].\mathsf{c}(BCA)$	
DC2	$[AB]:\mathsf{r}(AB).\equiv.\mathsf{c}(AAB)$	
DC3	$[ABCD].\because.\mathsf{t}(ABCD).\equiv:\mathsf{r}(AB).\mathsf{r}(CD):[M]:\mathsf{c}(MAB).\supset.\sim\mathsf{c}(MCD)$	
C9	$[ABC]:\mathsf{c}(CAB).\supset.[\exists DE].\mathsf{t}(ABDE)$	
PF	$[ABC].\because.\mathrm{Hyp}(1).\supset:$	
	2) $\mathsf{c}(CBA)$.	[*C3*, 1]
	3) $A \neq B:$	[*C2*, 1]
	$[\exists DE]:$	
	4) $\mathsf{c}(EED)$.	[*C1b*, 1]
	5) $\sim\mathsf{c}(DAB)$.	
	6) $[\exists F].\mathsf{t}(ABDF)$	[*C4*, 2, 1, 4, 5, 3, *DC3*, *DC2*]
	$[\exists DE].\mathsf{t}(ABDE)$	[6]
C10	$[A]:A\,\varepsilon\alpha\,.\supset.[\exists BCD].\mathsf{t}(ABCD)$	
PF	$[A].\because.\mathrm{Hyp}(1).\supset:$	
	$[\exists BC]:$	
	2) $\mathsf{c}(BCA)$.	[*DC1*, 1]
	3) $\mathsf{c}(BAC)$.	[*C3*, 1]
	4) $[\exists DE].\mathsf{t}(ACDE)$.	[*C9*, 3]
	$[\exists BCD].\mathsf{t}(ABCD)$	[4]
C11	$[ABCD]:\mathsf{t}(ABCD).\supset.A\,\varepsilon\,\alpha$	
PF	$[ABCD]:\mathrm{Hyp}(1).\supset.$	
	2) $\mathsf{r}(AB)$.	[*DC3*, 1]
	3) $\mathsf{c}(AAB)$.	[*DC2*, 2]
	4) $\mathsf{c}(ABA)$.	[*C3*, 3]
	$A\,\varepsilon\,\alpha$	[*DC1*, 4]
C12	$[A]:A\varepsilon\,\alpha\,.\equiv.[\exists BCD].\mathsf{t}(ABCD)$	[*C10*, *C11*]
C13	$[AB]:\mathsf{r}(AB).\equiv.[\exists CD].\mathsf{t}(ABCD)$	[*DC2*, *C9*, *DC3*]
C14	$[AB]:A\,\varepsilon\alpha\,.\,B\varepsilon\,\alpha\,.\,A\neq B\,.\sim\mathsf{c}(AAB).\supset.\mathsf{c}(AAB)$	[*C4*, *C/A*, *DC1*]
C15	$[AB]:A\varepsilon\,\alpha\,.\,B\varepsilon\,\alpha\,.\,A\neq B\,.\supset.\mathsf{c}(AAB)$	[*C14*]
C16	$[ABC]:\mathsf{c}(CAB).\supset.\mathsf{c}(AAB)$	
PF	$[ABC]:\mathrm{Hyp}(1).\supset.$	
	2) $\mathsf{c}(CBA)$.	[*C3*, 1]
	3) $A\neq B$.	[*C2*, 1]
	$\mathsf{c}(AAB)$	[*C15*, *DC1*, 1, 2, 3]
C17	$[AB]:\mathsf{c}(AAB).\supset.A\varepsilon\,\alpha\,.\,B\varepsilon\,\alpha\,.\,A\neq B$	[*DC1*, *C3*, *C2*]
C18	$[AB]:\mathsf{r}(AB).\equiv.A\varepsilon\,\alpha\,.\,B\varepsilon\,\alpha\,.\,A\neq B$	[*C15*, *C17*, *DC2*]
C19	$[AB]:\mathsf{c}(AAB).\equiv.\mathsf{c}(BBA)$	[*C18*, *DC2*]
C20	$[AB]:\mathsf{r}(AB).\equiv.\mathsf{r}(BA)$	[*C19*, *DC2*]
C21	$[ABC]:\mathsf{c}(ABC).A\neq C\,.\supset.\mathsf{c}(BAC)$	
PF	$[ABC]:\mathrm{Hyp}(1).\supset.$	
	3) $\mathsf{c}(ACB)$.	[*C3*, 1]
	4) $\mathsf{c}(CCB)$.	[*C16*, 3]

5) $\mathrm{c}(CBC)$. [$C3$, 4]
$\mathrm{c}(BAC)$ [$C5$, 2, 1, 5]

C22 $[ABC]:\mathrm{c}(ABC).\supset.B\varepsilon\alpha.C\varepsilon\alpha.A\varepsilon\alpha$

PF $[ABC]\therefore\mathrm{Hyp}(1).\supset:$
2) $\mathrm{c}(ACB):$ [$C3$, 1]
3) $A\neq C.\supset.\mathrm{c}(BAC):$ [$C21$, 1]
4) $A\neq C.\supset.\mathrm{c}(BCA):$ [$C3$, 3]
$B\varepsilon\alpha.C\varepsilon\alpha.A\varepsilon\alpha$ [$DC1$, 2, 1, 4]

C23 $[ABCXY]:\mathrm{c}(ABC).A\neq B.\mathrm{t}(BCXY).\mathrm{c}(MBA).M\neq C.\supset.\sim\mathrm{c}(MXY)$

PF $[ABCXY]\therefore\mathrm{Hyp}(5).\supset:$
6) $[N]:\mathrm{c}(NBC).\supset.\sim\mathrm{c}(NXY):$ [$DC3$, 3]
7) $\mathrm{c}(ACB)$. [$C3$, 1]
8) $\mathrm{c}(CAB)$. [$C21$, 7, 2]
9) $\mathrm{c}(CBA)$. [$C3$, 8]
10) $\mathrm{c}(BMC)$. [$C5$, 5, 4, 9]
11) $B\neq C$. [$C2$, 1]
12) $\mathrm{c}(MBC)$. [$C21$, 10, 11]
$\sim\mathrm{c}(MXY)$ [6, 12]

C24 $[BCXY]:\mathrm{t}(BCXY).\supset.\sim\mathrm{c}(CXY)$

PF $[BCXY]\therefore\mathrm{Hyp}(1).\supset:$
2) $\mathrm{c}(BBC):$
3) $[M]:\mathrm{c}(MBC).\supset.\sim\mathrm{c}(MXY):$ [$DC3$, $DC2$, 1]
4) $\mathrm{c}(CCB)$. [$C19$, 2]
5) $\mathrm{c}(CBC)$. [$C3$, 4]
$\sim\mathrm{c}(CXY)$ [3, 5]

C25 $[ABC]\therefore\mathrm{c}(ABC).A\neq B.\mathrm{t}(BCXY).\supset:[M]:\mathrm{c}(MBA).\supset.\sim\mathrm{c}(MXY)$
[$C23$, $C24$]

C26 $[ABC]:\mathrm{c}(ABC).A\neq B.\mathrm{t}(BCXY).\supset.\mathrm{t}(BAXY)$

PF $[ABC]\therefore\mathrm{Hyp}(3).\supset:$
4) $A\varepsilon\alpha$.
5) $B\varepsilon\alpha$. [$C22$, 1]
6) $\mathrm{r}(BA)$. [$C18$, 5, 4, 2]
7) $\mathrm{r}(XY):$ [$DC3$, 3]
8) $[M]:\mathrm{c}(MBA).\supset.\sim\mathrm{c}(MXY):$ [$C25$, 1, 2, 3]
$\mathrm{t}(BAXY)$ [$DC3$, 6, 7, 8]

C27 $[ABC]::\mathrm{c}(ABC).\supset\therefore\mathrm{r}(BC)\therefore A=B.\vee:[XY]:\mathrm{t}(BCXY).\supset.\mathrm{t}(BAXY)$

PF $[ABC]::\mathrm{Hyp}(1).\supset\therefore$
2) $\mathrm{r}(BC)\therefore$ [$DC2$, $C16$, 1]
3) $A=B.\vee:[XY]:\mathrm{t}(BCXY).\supset.\mathrm{t}(BAXY)\therefore$ [$C26$, 1]
$\mathrm{r}(BC)\therefore A=B.\vee:[XY]:\mathrm{t}(BCXY).\supset.\mathrm{t}(BAXY)$ [2, 3]

C28 $[ABCD]:\mathrm{t}(ABCD).\supset.\sim\mathrm{c}(CAB)$

PF $[ABCD]\therefore\mathrm{Hyp}(1).\supset:$
2) $\mathrm{r}(CD):$
3) $[M]:\mathrm{c}(MAB).\supset.\sim\mathrm{c}(MCD):$ [$CD3$, 1]
4) $[M]:\mathrm{c}(MCD).\supset.\sim\mathrm{c}(MAB):$ [3]
5) $\mathrm{c}(CCD)$. [$DC2$, 2]
$\sim\mathrm{c}(CAB)$ [4, 5]

C29 $[ABC]:A\varepsilon\alpha.B\varepsilon\alpha.C\varepsilon\alpha.\sim c(CAB).A\neq B.\supset.[\exists D].\mathsf{t}(ABCD)$ [*C4, DC1, DC3, DC2*]

C30 $[ABC]\therefore\mathsf{r}(BC):[XY]:\mathsf{t}(BCXY).\supset.\mathsf{t}(BAXY):\sim\mathsf{c}(ABC):\supset.\wedge$

PF $[ABC]\therefore\text{Hyp}(3).\supset:$

$[\exists DE].$

4) $\mathsf{t}(BCDE).$ [*C13*, 1]

5) $\mathsf{t}(BADE).$ [4, 2]

6) $\mathsf{r}(BA).$ [*DC3*, 5]

7) $A\varepsilon\alpha.$ [*C18*, 6]

8) $\mathsf{r}(AB).$ [*C20*, 6]

9) $\mathsf{c}(AAB).$ [*DC2*, 8]

10) $\mathsf{c}(ABA).$ [*C3*, 9]

$[\exists F].$

11) $\mathsf{t}(BCAF).$ [*C29, C18*, 1, 7, 3]

12) $\mathsf{t}(BAAF).$ [2, 11]

13) $\sim\mathsf{c}(ABA).$ [*C28*, 12]

$\wedge$ [10, 13]

C31 $[ABC]::\mathsf{r}(BC)\therefore A=B.\mathsf{v}:[XY]:\mathsf{t}(BCXY).\supset.\mathsf{t}(BAXY)\therefore\supset.\mathsf{c}(ABC)$ [*DC2, C30*]

C32 $[ABC]::\mathsf{c}(ABC).\equiv\therefore\mathsf{r}(BC)\therefore A=B.\mathsf{v}:[XY]:\mathsf{t}(BCXY).\supset.\mathsf{t}(BAXY)$ [*C27, C31*]

C33 $[\exists ABCD].\mathsf{t}(ABCD)$

PF $[\exists AB]\therefore$

1) $\mathsf{c}(AAB):$ [*C1a*]

$[\exists CD]:$

2) $\mathsf{c}(CCD).$

3) $\sim\mathsf{c}(DAB).$ [*C1b*, 1]

4) $D\varepsilon\alpha.$ [*C22*, 2]

5) $[\exists E].\mathsf{t}(ABDE)\therefore$ [*C29, C17*, 1, 4, 3]

$[\exists ABCD].\mathsf{t}(ABCD)$ [5]

C34 $[ABCD]:\mathsf{t}(ABCD).\supset.A\neq B$ [*DC3, C18*]

C35 $[ABCD]:\mathsf{t}(ABCD).\supset.\mathsf{t}(ABDC)$ [*DC3, C20, C3*]

C36 $[ABCD]:\mathsf{t}(ABCD).\mathsf{t}(ABEF).\mathsf{t}(CDEG).\supset.\mathsf{t}(CDEF)$

PF $[ABCD]\therefore\text{Hyp}(3).\supset:$

4) $\mathsf{r}(AB).$

5) $\mathsf{r}(CD):$ [*DC3*, 1]

6) $[M]:\mathsf{c}(MAB).\supset.\sim\mathsf{c}(MCD):$

7) $\mathsf{r}(EF):$

8) $[M]:\mathsf{c}(MAB).\supset.\sim\mathsf{c}(MEF):$ [*DC3*, 2]

9) $[M]:\mathsf{c}(MAB).\supset.\sim\mathsf{c}(MCD).\sim\mathsf{c}(MEF):$ [6, 8]

10) $\sim\mathsf{c}(ECD):$ [*C28*, 3]

11) $[N]:\mathsf{c}(NCD).\supset.\sim\mathsf{c}(NEF):$ [*C6, DC2*, 4, 5, 7, 9, 10]

$\mathsf{t}(CDEF)$ [*DC3*, 5, 7, 11]

C37 $[ABCD]:\mathsf{t}(ABCD).\supset.\mathsf{t}(CDAB)$ [*DC3*]

C38 $[ABCDEFGX]:\mathsf{t}(ABCD).\mathsf{t}(BEDF).\mathsf{t}(AECG).\mathsf{c}(XAC).\mathsf{c}(XBD).$
$\mathsf{c}(XEF).\supset.\mathsf{t}(AECF)$

PF $[ABCDEFGX]\therefore \text{Hyp}(6).\supset:$

7) $\mathsf{r}(AB).$ [DC3, 1]
8) $\mathsf{r}(CD):$
9) $[M]:\mathsf{c}(MAB).\supset.\sim\mathsf{c}(MCD):$
10) $\mathsf{r}(BE).$ [DC3, 2]
11) $\mathsf{r}(DF):$
12) $[M]:\mathsf{c}(MBE).\supset.\sim\mathsf{c}(MDF):$
13) $\sim\mathsf{c}(CAE):$ [C28, 3]
14) $[N]:\mathsf{c}(NAE).\supset.\sim\mathsf{c}(NCF):$ [C8, DC2, 7, 8, 10, 11, 9, 12, 13, 4, 5, 6]
15) $A\neq E.$ [C34, 3]
16) $\mathsf{r}(AE).$ [C18, C18, 7, 10, 15]
17) $C\neq D.$ [C34, 1]
18) $\mathsf{c}(BCD).$ [C24, 1]
19) $\mathsf{c}(CBD).$ [C21, 17, 18]
20) $X\neq C.$ [19, 5]
21) $\mathsf{c}(AXC).$ [C21, 4, 20]
22) $\mathsf{c}(ACX):$ [C3, 21]
23) $C=F.\supset.\mathsf{c}(EXC):$ [C21, 6, 20]
24) $C=F.\supset.\mathsf{c}(ECX):$ [C3, 23]
25) $C=F.\supset.\mathsf{c}(CAE):$ [C5, 15, 22, 24]
26) $C\neq F.$ [25, 13]
27) $\mathsf{r}(CF).$ [C18, C18, 8, 11, 26]

$\mathsf{t}(AECF)$ [DC3, 16, 27, 14]

If one now corresponds c to c'' and r to r', then *C12, C13, C32, C33, C34, C29, C36, DC3, C38* correspond to the definitions and axioms of $\mathbf{T}''$. Therefore $\mathbf{C}$ implies $\mathbf{T}''$. To prove that $\mathbf{T}''$ implies $\mathbf{C}$ we have all the theorems of $\mathbf{T}$, $\mathbf{T}'$, and $\mathbf{T}''$ at our disposal. Since in the $\mathbf{T}$ systems r and r', and c, c', c'' are equivalent we shall use r and c in what follows instead of r' and c''.

T1 $[A]:A\varepsilon\alpha.\supset.[\exists BC].\mathsf{c}(BCA)$

PF $[A]:\text{Hyp}(1).\supset.$

$[\exists B].$

2) $\mathsf{r}(AB).$ [D1, 1, D2′]
3) $\mathsf{r}(BA).$ [L19, 2]
4) $\mathsf{c}(BBA).$ [D3″, 3]

$[\exists BC].\mathsf{c}(BCA)$ [4]

T2 $[ABC]:\mathsf{c}(BCA).\supset.A\varepsilon\alpha$

PF $[ABC]:\text{Hyp}(1).\supset.$

2) $\mathsf{r}(CA).$ [D3″, 1]
3) $\mathsf{r}(AC).$ [L19, 2]

$A\varepsilon\alpha$ [D2′, 3, D1]

T3 $[A]:A\varepsilon\alpha.\equiv.[\exists BC].\mathsf{c}(BCA)$ [T1, T2]

T4 $[AB]:\mathsf{r}(AB).\equiv.\mathsf{c}(AAB)$ [D3″]

T5 $[\exists AB].\mathsf{c}(AAB)$ [A1, A7″, D3″]

T6 $[ABC]:\mathsf{c}(ABC).\supset.[\exists DE].\mathsf{c}(EED).\sim\mathsf{c}(DBC)$

PF $[ABC]:\mathrm{Hyp}(1).\supset.$

2) $\mathsf{r}(BC).$ [*D3''*, 1]

$[\exists DE].$

3) $\mathsf{t}(BCDE)$ [*D2'*, 2]

4) $\mathsf{r}(DE).$ [*A7''*, 3]

5) $\mathsf{r}(ED).$ [*L19*, 4]

6) $\mathsf{c}(EED).$ [*D3''*, 5]

7) $\sim\mathsf{c}(DBC).$ [*Z'2*, 3]

$[\exists DE].\mathsf{c}(EED).\sim\mathsf{c}(DBC)$ [6, 7]

T7 $[ABC]:\mathsf{c}(ABC).\supset.B\neq C$ [*D3''*, *L19*]

T8 $[ABCEFGHIJ]::\mathsf{c}(EFA).\mathsf{c}(GHB).\mathsf{c}(IJC).\sim\mathsf{c}(CAB).A\neq B.\supset\therefore$ $\mathsf{c}(AAB)\therefore[\exists D]\therefore\mathsf{c}(CCD):[M]:\mathsf{c}(MAB).\supset.\sim\mathsf{c}(MCD)$

PF $[ABCEFGHIJ]::\mathrm{Hyp}(5).\supset\therefore$

6) $A\varepsilon\alpha.B\varepsilon\alpha.C\varepsilon\alpha.$ [*T3*, 1, 2, 3]

7) $\mathsf{c}(AAB)\therefore$ [*L19*, 6, 5, *T4*]

$[\exists D]\therefore$

8) $\mathsf{t}(ABCD).$ [*A4''*, 6, 4, 5]

9) $\mathsf{r}(CD):$
10) $[M]:\mathsf{c}(MAB).\supset.\sim\mathsf{c}(MCD):$ [*A7''*, 8]

11) $\mathsf{c}(CCD).$ [*T4*, 9]

$\mathsf{c}(AAB)\therefore[\exists D]\therefore\mathsf{c}(CCD):[M]:\mathsf{c}(MAB).\supset.\sim\mathsf{c}(MCD)$ [7, 11, 10]

T9 $[ABCDEFN]\therefore\mathsf{c}(AAB).\mathsf{c}(CCD).\mathsf{c}(EEF):[M]:\mathsf{c}(MAB).\supset.\sim\mathsf{c}(MCD).$ $\sim\mathsf{c}(MEF):\sim\mathsf{c}(ECD).\mathsf{c}(NCD):\supset.\sim\mathsf{c}(NEF)$

PF $[ABCDEFN]\therefore\mathrm{Hyp}(6).\supset:$

7) $\mathsf{t}(ABCD).$ [*A7''*, *T4*, 1, 2, 4]

8) $\mathsf{t}(ABEF).$ [*A7''*, *T4*, 1, 3, 4]

9) $[\exists G].\mathsf{t}(CDEG).$ [*A4''*, *T4*, *L19*, 2, 3, 5]

10) $\mathsf{t}(CDEF):$ [*A6*, 7, 8, 9]

11) $[M]:\mathsf{c}(MCD).\supset.\sim\mathsf{c}(MEF):$ [*A7''*, 10]

$\sim\mathsf{c}(NEF)$ [11, 6]

T10 $[ABCDEFNX]::\mathsf{c}(AAB).\mathsf{c}(CCD).\mathsf{c}(BBE).\mathsf{c}(DDF)\therefore[M]\therefore$ $\mathsf{c}(MAB).\supset.\sim\mathsf{c}(MCD):\mathsf{c}(MBE).\supset.\sim\mathsf{c}(MDF)\therefore\sim\mathsf{c}(CAE).\mathsf{c}(XAC).$ $\mathsf{c}(XBD).\mathsf{c}(XEF).\mathsf{c}(NAE)\therefore\supset.\sim\mathsf{c}(NCF)$

PF $[ABCDEFNX]::\mathrm{Hyp}(10).\supset:$

11) $\mathsf{t}(ABCD).$ [*A7''*, *T4*, 1, 2, 5]

12) $\mathsf{t}(BEDF).$ [*A7''*, *T4*, 3, 4, 5]

13) $A\neq E.$ [*T7*, 10]

14) $[\exists G].\mathsf{t}(AECG).$ [*A4''*, *L20*, 1, 2, 3, 6, 13]

15) $\mathsf{t}(AECF):$ [*A4''*, 11, 12, 14, 7, 8, 9]

16) $[M]:\mathsf{c}(MAE).\supset.\sim\mathsf{c}(MCF):$ [*A7''*, 15]

$\sim\mathsf{c}(NCF)$ [16, 10]

T5, *T6*, *T7*, *Z'3*, *T8*, *Z''5*, *T9*, *T10*, *T3*, *T4*, *D3''* are just the axioms and auxilliary definitions of **C**. Therefore **T''** implies **C**. Therefore **C** is equivalent to **T''**.

REFERENCES

[1] Dowdy, S.: A quaternary relation as the primitive notion in several geometries, *Notre Dame Journal of Formal Logic,* Vol. VI (1965), pp. 241-295.

University of Notre Dame
Notre Dame, Indiana