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ON THE NUMBER OF VARIABLES IN THE AXIOMS

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§1. *Introduction.* There are three principal results.
(1) A new proof is given of a theorem already obtained by Wajsberg (see [7]), and by Diamond and McKinsey (see [1]), to the effect that if a propositional calculus is complete then at least one of its axioms must contain ≥ 3 distinct variables. (For precise definitions of *propositional calculus*, *complete*, etc., see §2.)
(2) The incomplete calculus whose axioms consist of all tautologies having ≤ 2 distinct variables, is described and shown to be not finitely axiomatizable.
(3) Finally, it is shown that, for a complete propositional calculus to be axiomatizable using only n distinct variables, where $n \geq 3$, it is necessary and sufficient that every logical connective of the system has $\leq n$ argument-places. The axiomatization in the sufficiency case is an adaption of that of Henkin in [3].

§2. *Terminology. Connective.* Short for "logical connective". We regard this as a primitive symbol to which is attached a classical truth-function. Sometimes we use "connective" loosely, meaning just the truth-function.

Variable. This always means one of the *propositional* variables, $p, p_1, p_2, \ldots, q, q_1, q_2, \ldots, r, r_1, r_2, \ldots$.

The wff A is of the form $\phi(p,q)$. This is typical of a whole class of assertions. It implies that no variable distinct from p, q appears in the wff A (*wff* has the usual meaning). The expression, "$\phi(X,Y)$", where X, Y are wff, strands for the wff obtained from A by substituting X, Y for p, q, respectively.

Propositional calculus; *complete*; P_n; $\supset$; $\Gamma \vdash_{\mathsf{P}} A$. Let S be a set of connectives, adequate to express implication; let Γ be a set of tautologies involving no connectives other than those of S; and let $\supset$ be an expression of implication in terms of the connectives of S. Then $\mathsf{P}(S, \Gamma, \supset)$ denotes the propositional calculus with connectives S, axioms Γ, and rules of inference (i) substitution, and (ii) A, $(A \supset B) \vdash B$. For the purposes of this paper,

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every propositional calculus is of this form for some S, Γ, $\supset$. A propositional calculus is *complete* if and only if all its tautologies are theorems. For the remainder of the paper, P_n is to stand for a given propositional calculus whose axioms are precisely those of its tautologies which contain $\leq n$ distinct variables; otherwise, we assume nothing about P_n. We denote the specified expression of implication in P_n by $\supset$. Let Γ be a set of wff, and A a single wff, belonging to a propositional calculus P; then, by

$$\Gamma \vdash_{\mathsf{P}} A,$$

we denote the assertion that A is deducible from Γ and the closure under substitution of the axioms of P, using modus ponens as the sole rule of inference.

Model; $\mathfrak{K}$; *valuation*. A *model* $\mathfrak{M}$ for a propositional calculus P consists of

(i) a set U of elements (often called *truth-values*), some of which are said to be *designated*,
(ii) an assignment to every connective of P of a function, with the appropriate number of argument-places, from and into U.

The classical model, for which $U = \{$t (true, designated), f (false)$\}$, and each connective is assigned its own turth-function, shall be known as $\mathfrak{K}$. An $\mathcal{M}$*-valuation* consists of all that is implied in the model $\mathfrak{M}$, together with an assignment to every variable of a value on U. Our practice will be to denote the value assigned to a symbol under a valuation by printing that symbol boldface, e.g. $\boldsymbol{p}$ for the value of the variable p, $\boldsymbol{\phi}$ for the function assigned to the connective ϕ.

We define

$$(p \vee q) = ((p \supset q) \supset q).$$

(For $\supset$, see earlier.) Intuitively, $(p \vee q)$ stands for "p or q", also "not-p implies q".

We regard $\neg$ as standing for a particular expression of negation in terms of the connectives of whatever propositional calculus is being discussed, *if* negation is expressible; if not, then $\neg$ is undefined.

§3. *Programme*. In §4 the class of theorems of P_1 is described (Theorem 1).

In §5, P_2 is proved to be incomplete (Theorem 2). Its class of theorems is described (Theorem 3), and it is shown to be not finitely axiomatizable (Theorem 4).

In §6, it is shown that for P_n to be complete, where $n \geq 3$, it is sufficient (Theorem 5) and necessary (Theorem 6) that P_n contain no connective with $> n$ argument-places.

§4. *The Propositional Calculus* P_1. THEOREM 1. *The class of theorems of* P_1 *coincides with the closure of the axioms under substitution.*

Proof. Let T be the closure of the axioms of P_1 under substitution. There is only one thing to prove, namely that T is closed under modus ponens. Let A, $(A \supset B)$ both belong to T. Then there exist two tautologies, one of each of the forms

$$\phi(p),$$
$$\psi(p) \supset \chi(p),$$

and wff C, D, such that

$$\phi(C) = A,$$
$$\psi(D) \supset \chi(D) = A \supset B.$$

We make the following assumptions:

(i) p does not appear in C, D (obviously, there is no loss of generality);
(ii) each of C, D contains at least one variable-occurrence;
(iii) p does actually appear at least once in each of $\phi(p)$, $\psi(p)$.

The only point of assumptions (ii), (iii) is to enable us to omit the more trivial cases. From the two facts, $A = \phi(C)$ and $A = \psi(D)$, we deduce that

Every variable-occurrence in A falls within a C-occurrence, (1)
Every variable-occurrence in A falls within a D-occurrence. (2)

There is at least one variable-occurrence in A (assumptions (ii), (iii)); hence we have occurrences of C, D overlapping, and therefore

One of C, D must contain the other. (3)

Case (*a*). C contains D (this includes the case $C = D$). From each wff X let a wff X^* be obtained by replacing every occurrence of D in X by p. Note that no such replacement can bring about a new occurrence of D (see assumption (i)), and that it is clearly impossible for two distinct occurrences of D to overlap; hence the order of the replacements does not matter, and X determines X^* uniquely. Clearly

$$A^* = \psi(p). \tag{4a}$$

Now, it follows from result (2) that every variable-occurrence in C falls within a D-occurrence. Hence

C^* is of the form $\gamma(p)$. (5a)

By combining result (1) with the fact that every D-occurrence contains at least one variable-occurrence, we see that every D-occurrence falls within a C-occurrence. Hence

$$A^* = \phi(C^*). \tag{6a}$$

(The only C-occurrences in A are those arising through substitution for p in $\phi(p)$; this is because every C-occurrence contains at least one variable-occurrence.) Combining results (4a), (5a), (6a), we have

$$\psi(p) = \phi(\gamma(p)).$$

It follows that $\psi(p)$ is a tautology, since it is a substitution instance of the tautology $\phi(p)$. But $(\psi(p) \supset \chi(p))$ is a tautology. Hence so is $\chi(p)$. But B is a substitution instance of $\chi(p)$. Therefore $B \varepsilon T$.

Case (*b*). D properly contains C. From each wff X let a wff X^* be obtained by replacing every occurrence of C in X by p. By arguments similar to those used in case (a), we can show that

$$A^* = \phi(p)\ , \tag{4b}$$

$$D^* \text{ is of the form } \delta(p)\ , \tag{5b}$$

$$A^* = \psi(B^*)\ . \tag{6b}$$

Combining these three results, we have

$$\phi(p) = \psi(\delta(p))\ .$$

The L.H. side of this equation is a tautology; hence so is the R.H. side. Now $(\psi(\delta(p)) \supset \chi(\delta(p)))$ is also a tautology, being a substitution instance of $(\psi(p) \supset \chi(p))$. Hence so is $\chi(\delta(p))$. But if we substitute C for p in this last wff, we obtain B (Clearly, $\delta(C) = D$). Hence $B \varepsilon T$.

This completes the proof of the theorem. We shall not bother to show here that P_1 is incomplete, as this will follow anyway from the incompleteness of P_2, to be proved in the next section.

§5. *The Propositional Calculus* P_2. We shall construct a model $\mathfrak{M}$, so contrived that the class of uniformly designated wff contains all theorems of P_2 but not all classical tautologies.

$\mathfrak{M}$ has 8 truth-values, namely

$$\{0,\ 1,\ a_1,\ a_2,\ a_3,\ b_1,\ b_2,\ b_3\},$$

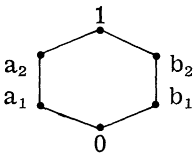
of which the only designated value is 0. An $\mathcal{M}$-function is assigned to each connective ϕ as follows. First we construct an expression which is truth-table equivalent to ϕ under $\mathfrak{K}$ and has N (negation) and & (conjunction) as its only connectives. Then, by re-interpretation of N, &, as below, our expression gives the $\mathcal{M}$-function assigned to ϕ.

TABLE FOR ***N***.

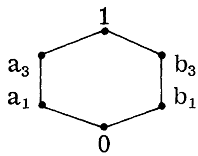
p	$N(p)$
0	1
1	0
a_i	b_i
b_i	a_i

TABLE FOR **&**.

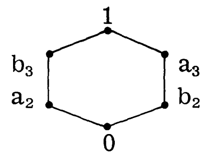
We define $(p \mathbin{\&} q)$ to be the least upper bound of p, q in those of the 3 lattices shown below which display both p and q. The reader will find that this definition is unique, despite the fact that the 3 lattices cannot be combined consistently into a single lattice.



LATTICE 1



LATTICE 2



LATTICE 3

I give two illustrations: $(a_2 \mathbin{\&} b_3) = b_3$ (lattice 1, only); $(a_2 \mathbin{\&} b_2) = 1$ (lattices 1 and 3).

LEMMA 1. *If the variables of any classical tautology A are assigned values from just one of the above 3 lattices, then* $\boldsymbol{A} = 0$ *under* $\mathfrak{M}$.

Proof. It is sufficient to take the case of Lattice 3. First we note that the set of values $\{0, 1, a_2, a_3, b_2, b_3\}$ is closed under $\boldsymbol{N}$, **&**. Hence, if we delete all mention of other truth-values from $\mathfrak{M}$, what is left forms a new, barer model which we shall call $\mathfrak{M}'$.

We have to show that if $\boldsymbol{A} \neq 0$ under some $\mathcal{M}'$-valuation then A can take the value f in some $\mathcal{K}$-valuation, and hence is not a classical tautology. Suppose $\boldsymbol{A} = a_2$, b_3 or 1. Then, under the identification,

$$\begin{cases} 0 = a_3 = b_2 = t, \\ 1 = a_2 = b_3 = f, \end{cases}$$

$\mathfrak{M}'$ becomes $\mathfrak{K}$, i.e. every row of an $\mathfrak{M}'$-table becomes a row of the corresponding classical truth-table. Hence we see that A can take the value f, classically. Now suppose that $\boldsymbol{A} = a_3$ or b_2. Under the identification,

$$\begin{cases} 0 = a_2 = b_3 = t, \\ 1 = a_3 = b_2 = f, \end{cases}$$

$\mathfrak{M}'$ again becomes $\mathfrak{K}$. Hence again A can take the value f, classically.

LEMMA 2. *Modus ponens preserves* $\mathcal{M}$*-designation.*

Proof. It is sufficient to show that $(0 \supset \boldsymbol{p}) = \boldsymbol{p}$ for every value of p. We take just the case $\boldsymbol{p} = a_1$ by way of example. First we note that the set of truth-values

$$\{0, 1, a_1, b_1\}$$

is closed under $\boldsymbol{N}$, **&**, and hence that part of $\mathfrak{M}$ dealing with only these values itself forms a model, which we shall call $\mathfrak{M}'$.

Now, under the identification,

$$\begin{cases} 0 = a_1 = t, \\ 1 = b_1 = f, \end{cases}$$

$\mathfrak{M}'$ translates into $\mathfrak{K}$, and we have

$$(0 \supset a_1) = (t \supset t) = t.$$

Hence, under $\mathfrak{M}'$, $(0 \supset a_1)$ must have been 0 or a_1.

Next we consider the identification,

$$\begin{cases} 0 = b_1 = t, \\ 1 = a_1 = f, \end{cases}$$

under which $\mathfrak{M}'$ again translates into $\mathfrak{K}$. This time we have

$$(0 \supset a_1) = (t \supset f) = f.$$

Hence, under $\mathfrak{M}'$, $(0 \supset a_1)$ must have been 1 or a_1. Combining this with the previous result, we see that, under $\mathfrak{M}'$, and so under $\mathfrak{M}$,

$$(0 \supset a_1) = a_1.$$

LEMMA 3. *All theorems of* P_2 *are uniformly designated under* $\mathfrak{M}$.

Proof. For any 2 given truth-values of $\mathfrak{M}$, at least one of Lattices 1, 2, 3 contains both of them. Hence, by Lemma 1, every axiom of P_2 is uniformly designated. Obviously substitution preserves uniform designation; and so does modus ponens, by Lemma 2.

LEMMA 4. *The classical tautology,*

$$A = ((p \supset q) \supset ((q \supset r) \supset (p \supset r))),$$

is not uniformly designated under $\mathfrak{M}$.

Proof. We need the following preliminary results:

$$\begin{aligned} (0 \supset a_3) &= a_3, \\ (0 \supset b_2) &= b_2, \\ (a_2 \supset a_1) &= 0, \\ (a_1 \supset a_3) &= 0, \\ (a_2 \supset a_3) &= a_3 \text{ or } b_2. \end{aligned}$$

The first two have already been proved during Lemma 2. The other results can be argued out in a similar kind of way. The ambiguity in the last result arises because we have not said precisely which expression in terms of N, &, is to represent $\supset$.

Assuming the preliminary results, if we set

$$\begin{aligned} \boldsymbol{p} &= a_2, \\ \boldsymbol{q} &= a_1, \\ \boldsymbol{r} &= a_3, \end{aligned}$$

we have

$$\boldsymbol{A} = a_3 \text{ or } b_2.$$

Lemmas 3 and 4 give us our theorem at once.

THEOREM 2. P_2 *is incomplete*.

The proofs of Theorems 1 and 2 illustrate two different methods of demonstrating incompleteness:

(1) The *direct*, as in Theorem 1. This involves describing the class of theorems of the system we hope to prove incomplete.

(2) The *indirect*, as in Theorem 2, by means of a specially contrived model.

The direct method provides more information, but tends to be much longer. I do give a direct proof of Theorem 2 in my Thesis, [2], but it is too long to inflict on the reader here (about 6 times as long as the indirect proof). However, for interest's sake, I will quote without proof the extra information it gives about P_2.

THEOREM 3. *The class of theorems of* P_2 *coincides with the closure under substitution of the set of those tautologies which belong to the set T of wff, defined as follows:*

Every wff with ≤ 2 *distinct variables belong to T;*

If A, B both belong to T and have no variable in common, then any wff of the form $\phi(A,B)$ *belongs to T.*

Let me mention that I could have given a shorter, indirect proof of Theorem 1, using a model with 4 truth-values, of which two are designated; however, I wished to illustrate the direct method.

In [1], Diamond and McKinsey use the indirect method. They show the inadequacy of 2-variable axioms for Boolean algebra, and hence deduce a similar result for the propositional calculus.

In [7], an earlier result by Wajsberg is quoted without proof, to the effect that $(p \supset (q \supset (r \supset p)))$ is not deducible from 2-variable tautologies. This result is a corollary of our Theorem 3 (above). Alternatively, an indirect proof can be given using the model $\mathfrak{M}(0,1)$, defined below.

The rest of this section is devoted to showing that P_2 is not finitely axiomatizable. Let Q be any propositional calculus whose axioms consist of a finite set of tautologies having ≤ 2 distinct variables; and let $\supset$ be the specified expression of implication for Q. We shall describe a model $\mathfrak{M}(r,s)$, defined for any $r \geq 0$, $s \geq 1$, with the property that, for sufficiently large r, s, the class of wff uniformly designated under $\mathfrak{M}(r,s)$ contains all theorems of Q, but not all 2-variable tautologies.

THE MODEL $\mathfrak{M}(r,s)$.

The model has the following truth-values:

$$\begin{array}{ll} \text{designated:} & t_1, \ldots, t_{n+2}; \\ \text{non-designated:} & \begin{cases} f_1, \ldots, f_{n+2}, \\ a_1, \ldots, a_{n+2}, \\ b_1, \ldots, b_{n+2}, \end{cases} \end{array}$$

where $n = r + s$. The function assigned to any connective ϕ of Q is obtained by expressing ϕ in terms of N, &, as for the earlier model $\mathfrak{M}$. The functions ***N***, **&** are determined by the following tables, together with the stipulations that (i) there is duality between the symbols a, b, and (ii) the function **&** is symmetric.

TABLE FOR *N*.

p	$N(p)$
t_i	f_i
f_i	t_i
a_i	b_i

TABLE FOR **&**.

p	q	$(p \,\&\, q)$
t_i	t_j	t_k
t_i	f_j	f_k
f_i	f_j	f_k
t_i	a_j	b_{n+2}, if $i, j = n+2, n+1$, respectively, a_k, otherwise.
f_i	a_j	f_k
a_i	a_j	f_{n+2}, if $i, j = n+1, n+2$ or $n+2, n+1$, a_k, otherwise.
a_i	b_j	a_{n+2}, if $i, j = n+2, n+1$ b_{n+2}, if $i, j = n+1, n+2$, f_k, otherwise.

The suffix k in the final column is given by,

$$k = \begin{cases} \max\{i,j\} + 1, & \text{if both of } i, j \leq n, \text{ and at least one of } i, j \leq r, \\ \max\{i,j\}, & \text{otherwise.} \end{cases}$$

I give one illustration: suppose $r = 1$; then, by the table $f_2 \,\&\, a_1 = f_3$; hence, by symmetry of **&**, $a_1 \,\&\, f_2 = f_3$; hence, by duality between a and b, $b_1 \,\&\, f_2 = f_3$.

One further definition before the proof beings: we define the *rank* of a wff to be the total number of occurrences of N, & in the expression representing it.

LEMMA 5. *Any wff A of the form $\phi(p,q)$ with rank $<s$ is uniformly designated under $\mathfrak{M}(r,s)$.*

Proof. We show that $\boldsymbol{A}$ is designated whenever the suffixes of $\boldsymbol{p}$, $\boldsymbol{q}$ both fall into any one of the following sets. (Meaning of "suffix": if $\boldsymbol{p} = t_i$, for instance, then i is the suffix of $\boldsymbol{p}$.)

(i) The set $\{1, \ldots, n+1\}$. Under the identification,

$$\begin{cases} t_1 = \ldots = t_{n+1} = a_1 = \ldots = a_{n+1} = t, \\ f_1 = \ldots = f_{n+1} = b_1 = \ldots = b_{n+1} = f, \end{cases}$$

that part of $\mathfrak{M}(r,s)$ which deals with these values translates into $\mathfrak{K}$. Similarly for the dual of the above identification. Hence $\boldsymbol{A}$ is designated in this case, by the same sort of argument as used in Lemma 1.

(ii) The set $\{r+1, \ldots, n, n+2\}$. It is sufficient to note that under both the identification,

$$\begin{cases} t_{r+1} = \ldots = t_n = t_{n+2} = a_{r+1} = \ldots = a_n = a_{n+2} = t, \\ f_{r+1} = \ldots = f_n = f_{n+2} = b_{r+1} = \ldots = b_n = b_{n+2} = f, \end{cases}$$

and its dual, the relevant part of $\mathfrak{M}(r,s)$ translates into $\mathfrak{K}$.
(iii) The set $\{n+1, n+2\}$. It is sufficient to note that under both the identification,

$$\begin{cases} t_{n+1} = t_{n+2} = a_{n+1} = b_{n+2} = t, \\ f_{n+1} = f_{n+2} = b_{n+1} = a_{n+2} = f, \end{cases}$$

and its dual, the relevant part of $\mathfrak{M}(r,s)$ translates into $\mathfrak{K}$.
(iv) The set $\{1, \ldots, r, n+2\}$. Under the identification,

$$\begin{cases} t_1 = \ldots = t_{n+2} = a_1 = \ldots = a_{n+2} = t, \\ f_1 = \ldots = f_{n+2} = b_1 = \ldots = b_{n+2} = f, \end{cases}$$

and its dual, $\mathfrak{M}(r,s)$ translates into $\mathfrak{K}$, *with the exception of* those rows of the **&**-table in which one argument has suffix $n+1$, and the other has suffix $n+2$. If we can show that the exceptional cases do not arise for rank of $A < s$, it will follow that $\boldsymbol{A}$ is designated. An exceptional case can arise only if the expression for A in terms of N, &, p, q contains a sub-expression B, such that $\boldsymbol{B}$ has suffix $n+1$. But this would require B to contain at least s occurrences of &, which is impossible because the rank of A is too small.

Whatever the suffixes of $\boldsymbol{p}$, $\boldsymbol{q}$, there is always one of the sets (i)-(iv), above, into which they both fall. So we have shown that A is uniformly designated.

LEMMA 6. *If* $(p \supset q)$ *has rank* $< s$, *then modus ponens preserves designation under* $\mathfrak{M}(r,s)$.

Proof. It is sufficient to show that if $\boldsymbol{p}$ is designated but $\boldsymbol{q}$ is not, then $(\boldsymbol{p} \supset \boldsymbol{q})$ is non-designated. The various cases can be classified according to the suffixes of $\boldsymbol{p}$, $\boldsymbol{q}$, and argued out as in the previous lemma. (Compare, too, Lemma 2.)

LEMMA 7. *For sufficiently large* s, *all theorems of* Q *are uniformly designated under* $\mathfrak{M}(r,s)$.

Proof. Choose s larger than the largest rank of axiom in Q, and larger than the rank of $(p \supset q)$. The result then follows from Lemmas 5 and 6.

LEMMA 8. *Given any* $s \geq 1$, *we can choose an* $r \geq 0$ *and construct a* 2-*variable tautology of Q which is not uniformly designated under* $\mathfrak{M}(r,s)$.

Proof. We choose any $r \geq$ rank of $(p \supset q)$.
Consider the 2-variable tautology,

$$A = ((B \supset q) \supset ((Q \supset p) \supset (B \supset p))),$$

where

$$B = (C \supset q),$$

and

$$C = (q \supset (q \supset \ldots (q \supset q) \ldots)),$$

the number of explicit appearances of q being $\geq n$.

Let us set $\boldsymbol{p} = \mathrm{a}_{n+2}$, $\boldsymbol{q} = \mathrm{a}_1$. I give only the outline of the calculation. We have

$$\boldsymbol{C} = \mathrm{t}_i, \text{ where } i \geq n.$$

(It was in order to be able to make statements like the one about the suffix of $\boldsymbol{C}$, that we chose r as large as we did.) Hence

$$\boldsymbol{B} = \mathrm{a}_{n+1}.$$

So

$$\begin{aligned} \boldsymbol{A} &= ((\mathrm{a}_{n+1} \supset \mathrm{a}_1) \supset ((\mathrm{a}_1 \supset \mathrm{a}_{n+2}) \supset (\mathrm{a}_{n+1} \supset \mathrm{a}_{n+2}))), \\ &= \mathrm{a}_{n+2}. \end{aligned}$$

(Compare Lemma 4.)

Combining Lemmas 7 and 8, we see that the class of theorems of Q cannot contain all 2-variable tautologies. This gives us the required theorem.

THEOREM 4. P_2 *is not finitely axiomatizable.*

§6. *The Propositional Calculus* P_n, *where* $n \geq 3$. For the duration of the proof of Theorem 5, let ϕ stand for a typical m-place connective of P_n, and let $\mathcal{V}_{\mathfrak{K}}$ stand for a typical $\mathcal{K}$-valuation. Let

$$A = \phi(p_1, \ldots, p_m).$$

We define

$$\begin{aligned} A^* = {} & (A \vee q), \text{ if } \boldsymbol{A} = \mathrm{t} \text{ under } \mathcal{V}_{\mathfrak{K}}, \\ & (A \supset q), \text{ otherwise}, \end{aligned}$$

and for $1 \leq i \leq m$, we define

$$\begin{aligned} p_i^* = {} & (p_i \vee q), \text{ if } \boldsymbol{p}_i = \mathrm{t} \text{ under } \mathcal{V}_{\mathfrak{K}}, \\ & (p_i \supset q), \text{ otherwise}. \end{aligned}$$

Finally, we define

$$B = (p_1^* \supset (p_2^* \supset \ldots (p_m^* \supset A^*) \ldots)).$$

Intuitively, B represents the assertion that either q is true or A has the appropriate truth-value under $\mathcal{V}_{\mathfrak{K}}$. Clearly B is a tautology.

We can save ourselves some work by borrowing the following result from Henkin (for proof, see [3]):

LEMMA 1. *For a propositional calculus* Q *to be complete, it is sufficient that its axioms include*

(i) *a certain finite set of tautologies, each having* ≤ 3 *distinct variables,*

(ii) *all wff B (above) given by letting ϕ range over all connectives of* Q, *and $\mathcal{V}_{\mathfrak{K}}$ range over all $\mathcal{K}$-valuations.*

Note that, if the set of connectives of Q is finite, then so is the set (ii) of wff, since, so far as any particular connective ϕ is concerned, there are effectively only 2^m distinct $\mathcal{K}$-valuations.

All the wff listed in Lemma 1 are axioms of P_n, except those wff B for which $m \geq n$. Our main object now is to show that, for $m = n$, B is a theorem of P_n, and so deduce that, if no case $m > n$ arises, then P_n must be complete.

Let $\{r_1, \ldots, r_m\}$ be any permutation of $\{p_1, \ldots, p_m\}$, satisfying the condition that, *if* at least one of $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_m$ is f under $\mathcal{V}_{\mathfrak{K}}$, *then* so is $\boldsymbol{r}_m$.

For $1 \leq i \leq m$, we now define a series of 2-variable wff, $p\Delta_i q$, in Tables I and II, below. The definitions are dependent upon the values taken by $r_1, \ldots, r_m, A$, under $\mathcal{V}_{\mathfrak{K}}$.

TABLE I. $1 \leq i < m$.

$\boldsymbol{r}_i$	$p\Delta_i q$
t	$p \supset q$
f	$p \vee q$

TABLE II. $i = m$.

$\boldsymbol{r}_m$	$\boldsymbol{A}$	$p\Delta_m q$
t	t	$p \supset q$
t	f	$p \supset \neg q$
f	t	$p \vee q$
f	f	$q \supset p$

It is necessary to justify the expressibility of $\neg$ in the 2nd row of Table II. Now, $\boldsymbol{r}_m = \mathrm{t}$ implies $\boldsymbol{r}_1 = \boldsymbol{r}_2 = \ldots = \boldsymbol{r}_{m-1} = \mathrm{t}$. Hence, if $\boldsymbol{r}_m = \mathrm{t}$, $\boldsymbol{A} = \mathrm{f}$, then A takes the value f when all its variables take the value t. Thus a wff truth-table equivalent to not-p is given, for instance, by

$$(p \supset \phi(p, \ldots, p)).$$

We define

$$C = (r_1\Delta_1(r_2\Delta_2 \ldots (r_m\Delta_m A) \ldots)).$$

Intuitively, C represents the assertion that A has the appropriate truth-value under $\mathcal{V}_{\mathfrak{K}}$. For instance,

$$(r_1 \vee (r_2 \supset (A \supset r_3)))$$

may be read as

"not-r_1 implies (r_2 implies (not-r_3 implies not-A))".

Clearly C is a tautology.

LEMMA 2. *If $m = n$, then $\vdash_{\mathsf{P}_n} B$.*

Proof. Since C (above) is an n-variable tautology, we have

$$\vdash_{\mathsf{P}_n} C \tag{1}$$

Now $(C \supset ((C \supset q) \supset q))$ is an instance (i.e., substitution instance) of a 2-variable tautology and hence is a theorem of P_n. Applying modus ponens to it and to result (1), we have

$$\vdash_{\mathsf{P}_n} ((C \supset q) \supset q). \tag{2}$$

We now wish to show that, for any wff X, and set Γ of wff,

$$\text{for } 1 \leq i < m, \text{ if } \Gamma \vdash_{\mathsf{P}_n} (((r_i \Delta_i X) \supset q) \supset q) \text{ then } \Gamma, r_i^* \vdash_{\mathsf{P}_n} ((X \supset q) \supset q). \tag{3}$$

This will follow by two applications of modus ponens, if we can show

$$\vdash_{\mathsf{P}_n} [((r_i \Delta_i X) \supset q) \supset q] \supset [r_i^* \supset [(X \supset q) \supset q]].$$

But this last line is an instance of a 3-variable tautology. So result (3) is established.

Next we require the result, for any set Γ of wff,

$$\text{if } \Gamma \vdash_{\mathsf{P}_n} (((r_m \Delta_m A) \supset q) \supset q) \text{ then } \Gamma, r_m^* \vdash_{\mathsf{P}_n} A^*. \tag{4}$$

This can be proved in the same sort of way as result (3), by 2 applications of modus ponens to a suitable instance of a 3-variable tautology.

From results (2) and (4), and from $(m - 1)$ applications of result (3), it follows that

$$r_1^*, \ldots, r_m^* \vdash_{\mathsf{P}_n} A^*.$$

Hence, by the deduction theorem (which depends only upon axioms with ≤ 3 distinct variables), we have

$$\vdash_{\mathsf{P}_n} B.$$

THEOREM 5. *For* P_n *to be complete, where* $n \geq 3$, *it is sufficient that it contain no connective with* $> n$ *argument-places. Furthermore, if this condition is satisfied and* P_n *has only finitely many connectives, then* P_n *is finitely axiomatisable.*

Proof. At once from Lemmas 1 and 2.

The rest of this section is devoted to showing that the sufficiency condition of Theorem 5 is also a necessary one. The definitions of $\mathcal{V}_{\mathfrak{K}}$, A, B, etc., specially introduced for Theorem 5, are now discarded.

A model $\mathfrak{M}$ is now to be described, whose class of uniformly designated wff includes all theorems of P_n but not all classical tautologies involving connectives with $>n$ argument-places. $\mathfrak{M}$ has $n + 3$ truth-values, namely

$$\{\mathrm{t}, \mathrm{f}, \mathrm{a}_1, \ldots, \mathrm{a}_{n+1}\},$$

with t as the designated value.

Before assigning an $\mathcal{M}$-function to each connective, we need a preliminary definition. For each truth-value $\boldsymbol{p}$, we define

$$\boldsymbol{p}^* = \mathrm{t}, \text{ if } \boldsymbol{p} = \mathrm{t}, \quad \mathrm{f}, \text{ otherwise}.$$

Note that the range of $\boldsymbol{p}^*$ consists of those truth-values common to both $\mathfrak{M}$ and $\mathfrak{K}$. Now let ϕ be any m-place connective. We complete the description of $\mathfrak{M}$ by defining

$$\phi(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_m) = \text{the value of } \phi(\boldsymbol{p}_1^*, \ldots, \boldsymbol{p}_m^*) \text{ under } \mathfrak{K},$$

all of $\mathrm{a}_1, \ldots, \mathrm{a}_{n+1}$ appear among $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_m$, in which case we have

$$\phi(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_m) = \text{whichever of t, f is } \textit{not} \text{ the value of } \phi(\boldsymbol{p}_1^*, \ldots, \boldsymbol{p}_m^*) \text{ under } \mathfrak{K}.$$

LEMMA 3. *For each $\mathcal{M}$-valuation, $\mathcal{V}_{\mathfrak{M}}$, let $\mathcal{V}_{\mathfrak{K}}$ be the corresponding $\mathcal{K}$-valuation such that, for each variable x,*

$$\text{value of } x \text{ under } \mathcal{V}_{\mathfrak{K}} = (\text{value of } x \text{ under } \mathcal{V}_{\mathfrak{M}})^*.$$

Let A be any wff with $\leq n$ distinct variables. Then

$$\text{value of } A \text{ under } \mathcal{V}_{\mathfrak{K}} = (\text{value of } A \text{ under } \mathcal{V}_{\mathfrak{M}})^*.$$

Proof. The last part of the description of $\mathfrak{M}$, namely the proviso, "unless . . . under $\mathfrak{K}$", is obviously never applied when calculating the value of a wff with $\leq n$ distinct variables. So we may delete this proviso when dealing solely with A.

Once the proviso is deleted, we can transform $\mathfrak{M}$ into $\mathfrak{K}$, and every $\mathcal{M}$-valuation $\mathcal{V}_{\mathfrak{M}}$ into the corresponding $\mathcal{K}$-valuation $\mathcal{V}_{\mathfrak{K}}$, by the identification

$$\mathrm{a}_1 = \ldots = \mathrm{a}_{n+1} = \mathrm{f}.$$

The lemma follows at once.

LEMMA 4. *All theorems of P_n are uniformly designated under $\mathfrak{M}$.*

Proof. It follows immediately from Lemma 3 that the axioms of P_n take the value t uniformly under $\mathfrak{M}$. Obviously substitution preserves uniform designation. There remains modus ponens. By applying Lemma 3 to the case when $A = (p \supset q)$, we have that, in any $\mathcal{M}$-valuation, if $\boldsymbol{q} \neq \mathrm{t}$ then $(\boldsymbol{p} \supset \boldsymbol{q}) \neq \mathrm{t}$ (otherwise, in the corresponding $\mathcal{K}$-valuation, we would have $(\mathrm{t} \supset \mathrm{f}) = \mathrm{t}$). In other words, modus ponens preserves designation.

LEMMA 5. *If P_n contains a connective with $>n$ argument-places, then the wff of P_n include a classical tautology which is not uniformly designated under $\mathfrak{M}$.*

Proof. Let θ be a connective of P_n with m argument-places, where $m > n$. As our counter-example we present the wff,

$$A = (p_1 \vee (p_2 \vee \ldots (p_m \vee B) \ldots)),$$

where

$$B = \begin{cases} (p_m \vee \theta(p_1, \ldots, p_m), & \text{if } \boldsymbol{\theta}(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_m) = \text{t under the } \mathcal{K}\text{-valuation in which } \boldsymbol{p}_1 = \ldots = \boldsymbol{p}_m = \text{f}, \\ (\theta(p_1, \ldots, p_m) \supset p_m), & \text{otherwise,} \end{cases}$$

Intuitively, A represents the assertion that B has the appropriate truth-value in the $\mathcal{K}$-valuation in which all variables take the value f. Clearly A is a tautology.

Consider the $\mathcal{M}$-valuation,

$$\boldsymbol{p}_i = \begin{cases} \text{a}_i, & \text{if } 1 \leq i \leq n+1, \\ \text{a}_{n+1}, & \text{otherwise.} \end{cases}$$

Using the results that

(i) if $\boldsymbol{q} \neq \text{t}$, then $(\text{t} \supset \boldsymbol{q}) \neq \text{t}$ (already proved during Lemma 4),
(ii) if $\boldsymbol{p} \neq \text{t}$, $\boldsymbol{q} \neq \text{t}$, then $(\boldsymbol{p} \vee \boldsymbol{q}) \neq \text{t}$ (proof similar to that of (i), taking $A = (p \vee q)$ in Lemma 3),

we easily calculate that, under our $\mathcal{M}$-valuation,

$$\boldsymbol{A} \neq \text{t}.$$

Combining Lemmas 4 and 5, we have the required theorem.

THEOREM 6. *For P_n to be complete, where $n \geq 3$, it is necessary that it contain no connective with $> n$ argument-places.*

§7. *Final Discussion.* I list some lines of enquiry in this field.

(1) Restrictions may be placed upon the number of distinct variables appearing in each axiom (as here), the number of occurrences of each axiom, the number of occurrences of all variables in each axiom, the total number of occurrences of variables and connectives in each axiom, the number of axioms, and the particular connectives available. By making one or more of these restrictions simultaneously, various minimum conditions compatible with completeness may be obtained. I am aware of the following results of this kind.

(i) Let n = the greatest number of variable-occurrences in any one axiom. Then, for the C - N propositional calculus, the least value of n compatible with completeness is 5. (Sobociński, [8].)

(ii) The shortest single axiom adequate for the implicational calculus contains 7 variable-occurrences. (Łukasiewicz, [6].)

(iii) Let P be a propositional calculus whose specified expression for implication is a single 2-place connective. Then, for P to be complete, it is necessary that its axioms include at least one wff eleven symbols long, or at least 2 wff each 9 symbols long (no. of symbols = no. of variables + no. of connectives). (Jaśkowski, [4].)

(iv) Let n = greatest number of occurrences of any one variable in any one axiom. Then the least value of n compatible with completeness is 3. (Jaśkowski, [5].)

(2) The fact that a lower bound of 3 crops up both in the present paper and in Jaśkowski's paper, [5] (just mentioned), suggests that there may be a deeper link between the two results, but I have not discovered any yet.

(3) Each result of type (1) presents us with a set of incomplete systems to be investigated, as P_1, P_2 have been, in the present paper. However, not even P_1, P_2 have been investigated to exhaustion. One may ask:

(i) Is P_1 finitely axiomatizable? The answer No is obtained, fairly easily.

(ii) Does either of P_1, P_2 have a finitely axiomatizable, undecidable sub-system? Open.

(4) Do the results listed above extend to the Intuitionistic Propositional Calculus?

(5) It is natural to ask: Which, if any, of the above lines of enquiry are important enough to be worth pursuing? But I do not have any answer to offer myself.

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