Notre Dame Journal of Formal Logic
Volume XIX, Number 2, April 1978
NDJFAM

A NEW FAMILY OF MODAL SYSTEMS

G. N. GEORGACARAKOS

1 It is well-known that a Lemmon-style axiomatization of modal system S3 is provided in the following fashion:

Formation Rules:

(1) Any propositional variable is a wff.
(2) If x is a wff, then so is Nx.
(3) If x is a wff, then so is Lx.
(4) If both x and y are wffs, then so is Cxy.
(5) Nothing else is a wff.

Axioms:

A1 $CLpp$
A2 $CLCpqLCLpLq$

Rules of Inference:

(a) Uniform Substitution for Variables.
(b) Detachment (**D**): If both Cxy and x are theses, then y is a thesis.
(c) Restricted Necessitation (**RN**): If x is a **PC**-tautology or a thesis of the form $CLyz$, then Lx is a thesis. (This version of the rule of necessitation is taken from Zeman in [8], p. 105.)
(d) Tautology Rule (**PCR**): If x is a **PC**-tautology, then x is a thesis.

Definitions:

B1 $Kxy =_{Df} NCxNy$
B2 $Axy =_{Df} NKNxNy$
B3 $Exy =_{Df} KCxyCyx$
B4 $Mx =_{Df} NLNx$
B5 $Fxy =_{Df} LCxy$
B6 $Hxy =_{Df} KFxyFyx$

Now if we append to the above basis for S3 the additional axiom

M1 $CNLpLNLp$

Received March 3, 1975

we obtain a modal system called S3.5 (*cf.* [1], p. 82). Quite obviously this system is properly contained in S5; and since S5 is properly contained in **PC**, it must be the case that S3.5 is also properly contained in **PC**. However, there does exist a well-known class of irregular extensions of S3 which are not contained in **PC**. They are axiomatized in the following way:

S7 = {S3; *NLLp*}
S8 = {S3; *NMLLp*}
S9 = {S3.5; *NLLp*}

What we are calling S9 has been called S7.5 by Åqvist in [1]. However, Cresswell has shown in [2] that Åqvist's S7.5 contains S8 and so it seems more appropriate to follow Cresswell in calling it S9 (also see relevant footnote in [4], p. 272).

The containment relations existing among these irregular systems and some other well-known systems are exhibited in the following diagram:

$$\begin{tikzcd}
\mathrm{S9} \arrow[d] \arrow[dr] & & \mathbf{PC} \arrow[d] \\
\mathrm{S8} \arrow[d] & \mathrm{S3.5} \arrow[d] & \mathrm{S5} \arrow[l] \arrow[d] \\
\mathrm{S7} \arrow[r] & \mathrm{S3} & \mathrm{S4} \arrow[l]
\end{tikzcd}$$

Not only are these irregular systems not contained in **PC**, but they are inconsistent with it. After all, they are known to be inconsistent with S4, and S4 is contained in **PC**.

Now it is known that appending

N1 *LCpLp*

as an axiom to the axiomatic basis of a modal system at least as weak as S3 will collapse it into **PC**. Consequently, we might think of **PC** as being axiomatized by appending N1 to S3. Quite obviously this would be a very inelegant axiomatization of **PC**, but nonetheless one that will serve our purposes.

Keeping in mind this consideration, we might speculate about what would happen if we were to append

P2 *ANLLpLCqLq*

to the axiomatic basis of S3? In the primitive notation of S3, the formula in question would be

P1 *CLLpLCqLq*

Note that the resulting modal system would apparently commit us to the view that either S7 or **PC** gives us the truths of logic. Appending P1 to S3.5, on the other hand, would apparently yield a modal system telling us that either S9 or **PC** is the correct system capturing the laws of logic. Analogously, appending

Q1 *CMLLpLCqLq*

to the basis of S3 would apparently yield a modal system telling us that either S8 or **PC** is the correct system concerning the laws of logic.

Using formulae P1 and Q1, it appears possible to construct an entirely new family of modal systems which we shall call Modal Family $\mathcal{J}$. We shall label the resulting systems in the following way:

$$J1 = \{S3;\ P1\}$$
$$J2 = \{S3;\ Q1\}$$
$$J3 = \{S3.5;\ P1\}$$

The purpose of this paper then shall be to define the relationships existing among these systems and to show that Modal Family $\mathcal{J}$ is comprised of intersection systems possessing the following properties:

$$J1 = S7 \cap \mathbf{PC}$$
$$J2 = S8 \cap \mathbf{PC}$$
$$J3 = S9 \cap \mathbf{PC}$$

An acquaintance with 4, 8, and 16 valued ordinary Boolean matrices for functors C and N is assumed. In this paper I shall use the following matrices which are presented here only for functors M and L:

01

	*			
p	1	2	3	4
Mp	1	1	1	4
Lp	1	4	4	4

02

	*	*						
p	1	2	3	4	5	6	7	8
Mp	1	1	1	3	5	5	5	7
Lp	2	4	4	4	6	8	8	8

03

	*	*						
p	1	2	3	4	5	6	7	8
Mp	1	1	3	3	1	1	3	7
Lp	2	6	8	8	6	6	8	8

04

	*	*	*					
p	1	2	3	4	5	6	7	8
Mp	1	1	3	3	1	1	3	7
Lp	2	6	8	8	6	6	8	8

05

	*	*														
p	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Mp	1	1	1	1	1	5	1	7	9	9	9	9	9	13	9	15
Lp	2	8	4	8	8	8	8	8	10	16	12	16	16	16	16	16

The designated values for each of these matrices are indicated by the asterisks. Moreover, all of these matrices verify S3. Matrix 01 is the familiar Group III of Lewis-Langford (*cf.* [5], p. 493). Matrix 03 is taken from Parry in [6]. The rest of them are mine.

2 Let us begin with modal system J1. Quite obviously, this system contains S3; that it contains S3 properly is established by the consideration that matrix 01 which is known to validate the entire axiomatic basis of S5, and hence S3, falsifies formula P1 for $p/1$ and $q/2$: $CLL1LC2L2 = C1LC2L2 = C1LC24 = C1L3 = C14 = 4$. It is an easy matter to show that J1 is a subsystem of S7; we need only demonstrate that $NLLp$ entails P1 in the field of S3!

G1	$NLLp$	Axiom of S7
G2	$CNpCpq$	**PCR**
G3	$CNLLpCLLpLCqLq$	G2, p/LLp; $q/LCqLq$
P1	$CLLpLCqLq$	G1, G3, **D**

To demonstrate that J1 is a *proper* subsystem of S7, we make use of matrix 03. This matrix verifies the entire basis of J1 but invalidates $NLLp$ whenever p takes on any one of the values 1, 2, 5, or 6. Clearly then, J1 is a proper extension of S3 and a proper subsystem of S7.

We now wish to show that modal system J2 contains J1. To accomplish this, we need only demonstrate that Q1 entails P1 in the field of S3:

G4	$CMLLpLCqLq$	Q1
G5	$CLCLpLqCLpLq$	A1, $p/CLpLq$
G6	$CCpqCNqNp$	**PCR**
G7	$CCLppCNpNLp$	G6, p/Lp; q/p
G8	$CNpNLp$	A1, G7, **D**
G9	$CNNpNLNp$	G8, p/Np
G10	$CpNNp$	**PCR**
G11	$CCpqCCqrCpr$	**PCR**
G12	$CCpNNpCCNNpNLNpCpNLNp$	G11, q/NNp; $r/NLNp$
G13	$CCNNpNLNpCpNLNp$	G10, G12, **D**
G14	$CpNLNp$	G9, G13, **D**
G15	$CpMp$	G14, Def[M]
G16	$CLLpMLLp$	G15, p/LLp
G17	$CCLLpMLLpCCMLLpLCqLqCLLpLCqLq$	G11, p/LLp; $q/MLLp$; $r/LCqLq$
G18	$CCMLLpLCqLqCLLpLCqLq$	G16, G17, **D**
P1	$CLLpLCqLq$	G4, G18, **D**

Not only does J2 contain J1, but it contains J1 properly. To see this, consider matrix 03 again which, as we have already pointed out, verifies the entire basis of J1. We observe that this matrix rejects formula Q1 for $p/1$ and $q/1$: $CMLL1LC1L1 = CML2LC12 = CM6L2 = C16 = 6$.

Modal system J2, on the other hand, is contained in system S8 since the characteristic axiom of S8 entails Q1 in the field of S3:

G19	$NMLLp$	Axiom of S8
G20	$CNMLLpCMLLpLCqLq$	G2, $p/MLLp$; $q/LCqLq$
Q1	$CMLLpLCqLq$	G19, G20, **D**

That J2 is a proper subsystem of S8 is indicated by the consideration that matrix 05, which validates the axiomatic basis for J2, falsifies $NMLLp$ whenever p has any of the values 1, 2, 3, 4, 5, 6, 7, or 8.

At this point we might wonder about the relationship between systems S7 and J2? Now matrix 05, again a matrix which verifies J2, falsifies *NLLp*, the proper axiom of S7, whenever *p* has any of the values 1, 2, 3, 4, 5, 6, 7, or 8. Matrix 04, on the other hand, verifies the basis for S7, but rejects Q1 when *p*/2 and *q*/4: *CMLL*2*LC*4*L*4 = *CML*6*LC*48 = *CM*6*L*5 = *C*16 = 6. Clearly then, systems S7 and J2 are independent of one another.

As we might expect, J3 contains J2. This is demonstrated by simply showing that M1 and P1 jointly entail Q1 in the field of S3:

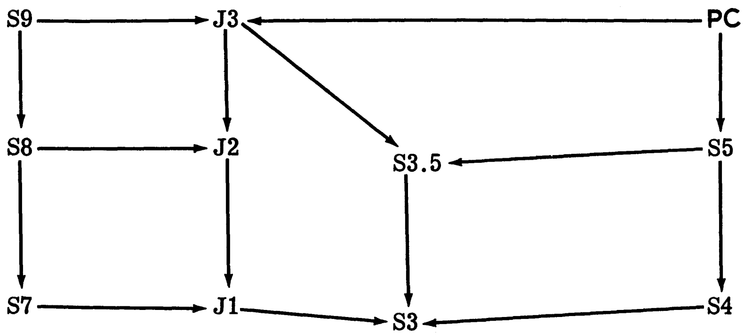
G21	*CLLpLCqLq*	P1
G22	*CNLpLNLp*	M1
G23	*CCNpqCNqp*	**PCR**
G24	*CCNLpLNLpCNLNLpLp*	G23, *p*/*Lp*; *q*/*LNLp*
G25	*CNLNLpLp*	G22, G24, **D**
G26	*CMLpLp*	G25, Def[*M*]
G27	*CMLLpLLp*	G26, *p*/*Lp*
G28	*CCMLLpLLpCCLLpLCqLqCMLLpLCqLq*	G11, *p*/*MLLp*; *q*/*LLp*; *r*/*LCqLq*
G29	*CCLLpLCqLqCMLLpLCqLq*	G27, G28, **D**
Q1	*CMLLpLCqLq*	G21, G30, **D**

In order to show that J2 is properly contained in J3, we direct our attention to matrix 05 which verifies J2 but rejects M1 for *p*/3: *CNL*3*LNL*3 = *CN*4*LN*4 = *C*13*L*13 = *C*1316 = 4. Now S9 obviously contains J3; that S9 contains J3 properly is confirmed by matrix 02 which verifies J3 but rejects *NLLp* whenever *p* has any of the values 1, 2, or 3. Again it is obvious that J3 is an extension of S3.5; that it is a proper extension of S3.5 is equally obvious since matrix 01 verifies the entire basis of S3.5 but rejects, as we have seen above, formula P1. In fact this consideration also establishes that S3.5 contains none of the systems of Modal Family $\mathcal{J}$.

We noted above that J3 does not contain *NLLp* as a thesis; consequently, in order to establish that J3 and S8 are independent of one another, it will suffice to point out, as it is well-known, that S8 does not contain M1 as a thesis.

We have yet to consider the relationship of Modal Family $\mathcal{J}$ to **PC**. Quite obviously **PC** contains every one of these systems since *LCqLq* is a thesis of **PC**. However, none of the systems of Family $\mathcal{J}$ contain **PC**, except of course as a fragment. (Perhaps it would be better to say that none of the systems of Family $\mathcal{J}$ collapse into **PC**.) This is evidenced by the consideration that matrix 02 which verifies all of the systems of Family $\mathcal{J}$ rejects *LCqLq* when *q* has any of the values 1, 2, 3, 5, 6, or 7. Clearly this consideration, along with the fact that none of the systems of Modal Family $\mathcal{J}$ contain any of the irregular systems, also establishes that all of the systems of the new family are unreasonable in the sense of Halldén (*cf.* [3]).

We are now in a position to visualize the containment relations which the systems of Family $\mathcal{J}$ bear to one other and other well-known systems.



3 What I would now like to show is that Modal Family $\mathcal{J}$ is comprised of intersection systems having the following properties:

$$J1 = S7 \cap \mathbf{PC}$$
$$J2 = S8 \cap \mathbf{PC}$$
$$J3 = S9 \cap \mathbf{PC}$$

Let us begin with modal system J1. Clearly, if we are to show that J1 is the intersection of both S7 and **PC**, we must prove the following metalogical result:

(A) *A formula, x, is a thesis of* J1 *if and only if x is a thesis of both* S7 *and* **PC**.

In the last section, it was shown that J1 is contained properly in both S7 and **PC**; hence if a formula is a thesis of J1 it must also be a thesis of both S7 and **PC**. Quite obviously then, the proof of (A) reduces to a demonstration that the following holds:

(A′) *If x is a thesis of both* S7 *and* **PC**, *then x is a thesis of* J1.

In order to accomplish this task, we shall first show that each of the following formulae are theses of J1:

R1 *CNLLCpLpNLLp*
R2 *CNLLCpLpNLLCqLq*
R3 *CLLCpLpLCpLp*
R4 *CLLCpLpLLCqLq*

Then we shall prove two lemmata, to be stated below, from which (A′) will follow.

Assume the field of J1; i.e., {S3; P1}:

H1	*LCpCqp*	**PCR**, **RN**
H2	*CCLCpqLCLpLqCCLCLpLqCLpLqCLCpqCLpLq*	
		G11, *p/LCpq*; *q/LCLpLq*; *r/CLpLq*
H3	*CCLCLpLqCLpLqCLCpqCLpLq*	A2, H2, **D**
H4	*CLCpqCLpLq*	G5, H3, **D**
H5	*CLCpCqpCLpLCqp*	H4, *q/Cqp*
H6	*CLpLCqp*	H1, H5, **D**

H7	$CLCqpLCLqLp$	A2, p/q; q/p
H8	$CCLpLCqpCCLCpqLCLqLpCLpLCLqLp$	G11, p/Lp; $q/LCqp$; $r/LCLqLp$
H9	$CCLCqpLCLqLpCLpLCLqLp$	H6, H8, D
H10	$CLpLCLqLp$	H7, H9, D
H11	$CLCLqLpCLqLp$	A1, $p/CLqLp$
H12	$CCLCpqLCLqLpCCLCLqLpCLqLpCLCqpCLqLp$	G11, $p/LCqp$; $q/LCLqLp$; $r/CLqLp$
H13	$CCLCLqLpCLqLpCLCqpCLqLp$	H7, H12, D
H14	$CLCqpCLqLp$	H11, H13, D
H15	$CLCLqLpCLLqLLp$	H14, p/Lp; q/Lq
H16	$CCLpLCLqLpCCLCLqLpCLLqLLpCLpCLLqLLp$	G11, p/Lp; $q/LCLqLp$; $r/CLLqLLp$
H17	$CCLCLqLpCLLqLLpCLpCLLqLLp$	H10, H16, D
H18	$CLpCLLqLLp$	H15, H17, D
H19	$CCpCqrCqCpr$	PCR
H20	$CCLpCLLqLLpCLLqCLpLLp$	H19, p/Lp; q/LLq; r/LLp
H21	$CLLqCLpLLp$	H18, H20, D
H22	$CLLpCLLpLLLp$	H21, p/Lp; q/p
H23	$CCpCpqCpq$	PCR
H24	$CCLLpCLLpLLLpCLLpLLLp$	H23, p/LLp; $q/LLLp$
H25	$CLLpLLLp$	H22, H24, D
H26	$LCLLpLCqLq$	P1, RN
H27	$CLCLLpLCqLqCLLLpLLCqLq$	H4, p/LLp; $q/LCqLq$
H28	$CLLLpLLCqLq$	H26, H27, D
H29	$CCLLpLLLpCCLLLpLLCqLqCLLpLLCqLq$	G11, p/LLp; $q/LLLp$; $r/LLCqLq$
H30	$CCLLLpLLCqLqCLLpLLCqLq$	H25, H29, D
H31	$CLLpLLCqLq$	H28, H30, D
H32	$CCLLpLLCqLqCNLLCqLqNLLp$	G6, p/LLp; $q/LLCqLq$
H33	$CNLLCqLqNLLp$	H31, H32, D
*R1	$CNLLCpLpNLLp$	H33, q/p
*R2	$CNLLCpLpNLLCqLq$	H33, $p/CqLq$; q/p
*R3	$CLLCpLpLCpLp$	A1, $p/LCpLp$
*R4	$CLLCpLpLLCqLq$	H31, $p/CpLp$

Before stating and proving the two lemmata to which we alluded above, we ask that the reader imagine that we are working with a different axiomatization for S3 rather than the Lemmon-style axiomatization which was presented in section 1; say, an axiomatization due to Leo Simons in [7]. It looks like this:

Formation Rules:

(1) Any propositional variable is a wff.
(2) If x is a wff, then so is Nx.
(3) If x and y are wffs, then so is Kxy.
(4) If x is a wff, then so is Mx.
(5) Nothing else is a wff.

Axioms:

K1 $FpKpp$
K2 $FKpqp$
K3 $FKKrpNKqrKpNq$
K4 $CNMpNp$
K5 $FpMp$
K6 $FFpqFNMqNMp$

Rules of Inference:

(a) Uniform Substitution for Variables.
(b) Material Detachment: If both Cxy and x are theses, then y is also a thesis.

Definitions:

L1 $Axy =_{Df} NKNxNy$
L2 $Cxy =_{Df} NKxNy$
L3 $Exy =_{Df} KCxyCyx$
L4 $Lx =_{Df} NMNx$
L5 $Fxy =_{Df} NMKxNy$
L6 $Hxy =_{Df} KFxyFyx$

Now we may think of J1, J2, J3, S7, S8, S9, and **PC** as being axiomatized respectively as follows:

$$\begin{aligned} \text{J1} &= \{\text{S3(Simons)} + \text{P1}\} \\ \text{J2} &= \{\text{S3(Simons)} + \text{Q1}\} \\ \text{J3} &= \{\text{S3(Simons)} + \text{M1, P1}\} \\ \text{S7} &= \{\text{S3(Simons)} + NLLp\} \\ \text{S8} &= \{\text{S3(Simons)} + NMLLp\} \\ \text{S9} &= \{\text{S3(Simons)} + \text{M1}, NLLp\} \\ \textbf{PC} &= \{\text{S3(Simons)} + \text{N1}\} \end{aligned}$$

The advantage of this is twofold. First, uniform substitution and material detachment will be our only primitive rules of inference and thus we will not have to bother with the restricted rule of necessitation. Second, we will be able to follow the same pattern outlined by Hughes and Cresswell in [4], pp. 271-272, for their proof that S3 is the intersection of S7 and S4.

Lemma 1 *If x is a thesis of* S7, *then $CNLLCpLpx$ is a thesis of* J1.

The proof of this lemma proceeds by induction on the various ways of proving theorems in S7.

(1) If x is a J1 axiom then by $CpCqp$ and detachment we have $CNLLCpLpx$ as a thesis of J1.
(2) If x is the characteristic S7 axiom, $NLLp$, then $CNLLCpLpx$ is simply R1 and hence is a thesis of J1.
(3) Concerning Material Detachment: If x is obtained from the two proved formulae y and Cyx, then it follows by the induction hypothesis that both

CNLLCpLpy and *CNLLCpLpCyx* are theses of J1. But *CCrpCCrCpqCrq* is a tautology and hence a thesis of J1. Putting *NLLCpLp* for *r*, *y* for *p*, and *x* for *q*, and employing material detachment twice, we obtain *CNLLCpLpx* as a thesis of J1.
(4) Concerning Uniform Substitution: Let us assume that *x* is the result of substituting a formula, *y*, for some propositional variable in an S7 theorem, *z*, then clearly by the induction hypothesis we have *CNLLCpLpz* as a thesis of J1.

(a) Now let us further assume that the variable for which the substitution is made is any variable other than *p*. Obviously then, the same substitution in *CNLLCpLpz* will give us *CNLLCpLpx* as a thesis of J1. (b) Let us now assume instead that the substitution is made for *p*, then the same substitution in *CNLLCpLpz* will give us *CNLLCyLyx* as a thesis of J1. But putting *y* for *q* in R2 will give us *CNLLCpLpNLLCyLy* as a thesis of J1; hence by syllogism we now also have *CNLLCpLpx* as a thesis of J1. We may therefore conclude that the lemma holds for all theses of J1.

Lemma 2 *If x is a thesis of* **PC**, *then CLLCpLpx is a thesis of* J1.

The proof of this lemma proceeds by induction on the various ways of proving theorems in **PC** = {S3(Simons) + N1}.

(1) If *x* is a J1 axiom then by *CpCqp* and detachment we have *CLLCpLpx* as a thesis of J1.
(2) If *x* is the characteristic **PC** axiom, *LCpLp*, then *CLLCpLpx* is simply R3 and hence is a thesis of J1.
(3) Concerning Material Detachment: If *x* is obtained from the two proved formulae *y* and *Cyx*, then it follows by the induction hypothesis that both *CLLCpLpy* and *CLLCpLpCyx* are theses of J1. But *CCrpCCrCpq* is a tautology and hence a thesis of J1. Putting *LLCpLp* for *r*, *y* for *p*, and *x* for *q*, and employing material detachment twice, we obtain *CLLCpLpx* as a thesis of J1.
(4) Concerning Uniform Substitution: Let us assume that *x* is the result of substituting a formula, *y*, for some propositional variable in a **PC** theorem, *z*. Clearly, by the induction hypothesis we have *CLLCpLpz* as a thesis of J1. (a) Now let us assume that the variable for which the substitution is made is any variable other than *p*. Then it follows that the same substitution in *CLLCpLpz* will give us *CLLCpLpx* as a thesis of J1. (b) Suppose this time that the substitution is made for *p*, then the same substitution in *CLLCpLpz* would give us *CLLCyLyx* as a thesis of J1. But putting *y* for *q* in R4 will give us *CLLCpLpLLCyLy* as a thesis of J1; hence by syllogism we now also have *CLLCpLpx* as a thesis of J1. We may therefore conclude that the lemma holds for all theses of **PC**.

Having proved these two lemmata, we are now in a position to prove (A′). Suppose that *x* is a thesis of both S7 and **PC**, then by Lemma 1 we have *CNLLCpLpx* as a thesis of J1, and by Lemma 2 that *CLLCpLpx* as a thesis of J1 too. But *CCNpqCCpqq* is a tautology and hence a thesis of J1.

Putting $LLCpLp$ for p and x for q, we obtain $CCNLLCpLpxCCLLCpLpxx$ as a thesis of J1. Thus two uses of material detachment gives us x as a thesis of J1. Clearly then, J1 is the intersection of S7 and **PC**. Put differently, we may think of J1 as containing all those and only those theses provable in both S7 and **PC**.

Undoubtedly, the proof that J3 is the intersection of S9 and **PC** will proceed similarly with obvious modifications. Consequently, we assert the following:

(B) *A formula, x, is a thesis of* J3 *if and only if x is a thesis of both* S9 *and* **PC**.

Actually the proof that J2 is the intersection of both S8 and **PC** also proceeds similarly; the only difference is that it is necessary to use

T1 $CNLLCpLpNMLLp$

in place of R1 in the above proof. Consequently, all that needs to be done now is to show that T1 is a thesis of J2. This is accomplished in the following way:

I1	$CMLLpLCqLq$	Q1
I2	$CLCqLqCqLq$	A1, $p/CqLq$
I3	$CMLLpCqLq$	I1, I2, Syllogism
I4	$CMLLpCMLLpLMLLp$	I3, $q/MLLp$
I5	$CCMLLpCMLLpLMLLpCMLLpLMLLp$	H23, $p/MLLp$; $q/LMLLp$
I6	$CMLLpLMLLp$	I4, I5, **D**
I7	$CLMLLpMLLp$	A1, $p/MLLp$
I8	$CLMLLpLCqLq$	I1, I7, Syllogism
I9	$LCLMLLpLCqLq$	I8, **RN**
I10	$CLCLMLLpLCqLqCLLMLLpLLCqLq$	H4, $p/LMLLq$; $q/LCqLq$
I11	$CLLMLLpLLCqLq$	I9, I10, Syllogism
I12	$CMLLpLCMLLpLMLLp$	I1, $q/MLLp$
I13	$CLCMLLpLMLLpCLMLLpLLMLLp$	H4, $p/MLLp$; $q/LMLLp$
I14	$CMLLpCLMLLpLLMLLp$	I12, I13, Syllogism
I15	$CLMLLpCLMLLpLLMLLp$	I7, I14, Syllogism
I16	$CLMLLpCLMLLpLLMLLpCLMMLpLLMLLp$	H23, $p/LMLLp$; $q/LLMLLp$
I17	$CLMLLpLLMLLp$	I15, I16, **D**
I18	$CMLLpLLMLLp$	I6, I17, Syllogism
I19	$CMLLpLLCqLq$	I11, I18, Syllogism
I20	$CCMLLpLLCqLqCNLLCqLqNMLLp$	G6, $p/MLLp$; $q/LLCqLq$
I21	$CNLLCqLqNMLLp$	I19, I20, **D**
T1	$CNLLCpLpNMLLp$	I21, q/p

We are now justified in asserting:

(C) *A formula, x, is a thesis of* J2 *if and only if x is a thesis of both* S8 *and* **PC**.

Before concluding this paper we note that since **PC**, S9, S8, and S7 are known to be decidable, it follows, in light of the considerations established above, that the systems of Family $\mathcal{J}$ are also decidable.

REFERENCES

[1] Åqvist, Lennart, "Results concerning some modal systems that contain S2," *The Journal of Symbolic Logic*, vol. 29 (1964), pp. 79-87.

[2] Cresswell, M. J., "Note of a system of Åqvist," *The Journal of Symbolic Logic*, vol. 32 (1967), pp. 58-60.

[3] Halldén, S., "On the semantic non-completeness of certain Lewis calculi," *The Journal of Symbolic Logic*, vol. 16 (1951), pp. 127-129.

[4] Hughes, G. E., and M. J. Cresswell, *An Introduction to Modal Logic*, Methuen and Co., Ltd., London (1968).

[5] Lewis, C. I., and C. H. Langford, *Symbolic Logic*, Dover Publications Inc., New York (1959).

[6] Parry, W. T., "Modalities in the *Survey System* of strict implication," *The Journal of Symbolic Logic*, vol. 4 (1939), pp. 137-154.

[7] Simons, L., "New axiomatizations of S3 and S4," *The Journal of Symbolic Logic*, vol. 18 (1953), pp. 309-316.

[8] Zeman, J. J., *Modal Logic: The Lewis-Modal Systems*, Oxford University Press, London (1973).

Kansas State University
Manhattan, Kansas