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ON YABLONSKII THEOREM CONCERNING FUNCTIONALLY COMPLETENESS OF k-VALUED LOGIC

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In his paper, S. B. Yablonskii [1] proved a theorem concerning the functional completeness in k-valued logic (see [1], p. 64). The theorem asserts that the system of functions consisting of constant $k - 2$, $\sim x$, and $x_1 \supset x_2$ is functionally complete in this logic. His proof is incomplete. In this paper, we shall give a simple proof of this theorem.

Let P_k be the set of all functions that are defined on the set $\{0, 1, \ldots, k - 1\}$ and take their values on the same set. First, we shall give a lemma needed for the proof of the theorem.

Lemma *The system consisting of functions* $0, 1, \ldots, k - 1$, $\max(x_1, x_2)$, $\min(x_1, x_2)$ *and* $j_i(x) (0 \leq i \leq k - 1)$ *defined by*

$$j_i(x) = \begin{cases} k - 1, & \text{if } x = i, \\ 0, & \text{if } x \neq i, \end{cases}$$

is functionally complete in P_k.

Proof: We use the induction. All the constants are already given. If we put

$$\max(y_1, y_2, \ldots, y_n) = \max[\max\{\ldots \max(\max(y_1, y_2), y_3) \ldots\}, y_n],$$

then

$$f(x_1, \ldots, x_n, x_{n+1}) = \max[\min\{f(x_1, \ldots, x_n, 0), j_0(x_{n+1})\},$$
$$\min\{f(x_1, \ldots, x_n, 1), j_1(x_{n+1})\}, \ldots, \min\{f(x_1, \ldots, x_n, k - 1), j_{k-1}(x_{n+1})\}].$$

Therefore, from the induction hypothesis we can construct every $n + 1$-variable function in P_k by superposition. The lemma is proved.

Now we shall prove the following theorem:

Theorem *The system of functions consisting of the constant* $k - 2$, $\sim x$, *and* $x_1 \supset x_2$, *where* $x_1 \supset x_2 = \min(k - 1, x_2 - x_1 + k - 1)$, *is functionally complete in* P_k.

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Proof: It is easy to see that

$$\begin{aligned}(x_1 \supset x_2) \supset x_2 &= \min[k-1,\ x_2 - (x_1 \supset x_2) + k - 1] \\ &= \min[k-1,\ x_2 - \min(k-1,\ x_2 - x_1 + k - 1) + k - 1] \\ &= \max(x_1, x_2).\end{aligned}$$

By superposition of functions $\sim x$ and $\max(x_1, x_2)$, we can define $\min(x_1, x_2)$ as follows:

$$\min(x_1, x_2) = \sim \max(\sim x_1, \sim x_2).$$

Let us consider the function $h_i(x)$ defined by means of the following way:

$$h_1(x) = \sim x \text{ and } h_{i+1}(x) = x \supset h_i(x) \ (i = 1, 2, \ldots, k-2).$$

Then

$$h_1(x) = k - 1 - x,$$

and

$$\begin{aligned}h_2(x) &= x \supset \sim x \\ &= \min[k-1,\ (k-1-x) - x + k - 1)] \\ &= \min[k-1,\ 2(k-1-x)].\end{aligned}$$

From the assumption

$$h_m(x) = \min[k-1,\ m(k-1-x)],$$

it follows that

$$\begin{aligned}h_{m+1}(x) &= x \supset h_m(x) \\ &= \min[k-1,\ h_m(x) - x + k - 1] \\ &= \min[k-1,\ \min\{k-1,\ m(k-1-x)\} - x + k - 1] \\ &= \min[k-1,\ (m+1)(k-1-x)].\end{aligned}$$

Hence,

$$h_n(x) = \min[k-1,\ n(k-1-x)]$$

for any positive integer n. Hence

$$h_{k-1}(x) = \min[k-1,\ (k-1)(k-1-x)] = \begin{cases} 0, & \text{if } x = k-1, \\ k-1, & \text{if } x \neq k-1. \end{cases}$$

From this function, we can obtain

$$\mathrm{j}_{k-1}(x) = \sim h_{k-1}(x),$$

and

$$\mathrm{j}_0(x) = \mathrm{j}_{k-1}(\sim x).$$

Let

$$f_1(x) = \max(h_{k-2}(x), x) \text{ and } f_2(x) = \min(h_{k-2}(x), x).$$

Then we consider the function

$$j_{k-1}(f_1(x) \supset f_2(x)).$$

In order to calculate the values of the function $f_1(x) \supset f_2(x)$ we shall consider the function $h_{k-2}(x)$. Since

$$2(k - 2) \geqslant k - 1, \text{ for } k \geqslant 3,$$

$$h_{k-2}(x) = \begin{cases} 0, & \text{if } x = k - 1, \\ k - 2, & \text{if } x = k - 2, \\ k - 1, & \text{otherwise.} \end{cases}$$

Therefore,

$$f_1(x) = \max(h_{k-2}(x), x) = \begin{cases} k - 1, & \text{if } x = k - 1, \\ k - 2, & \text{if } x = k - 2, \\ k - 1, & \text{otherwise,} \end{cases}$$

and

$$f_2(x) = \min(h_{k-2}(x), x) = \begin{cases} 0, & \text{if } x = k - 1, \\ k - 2, & \text{if } x = k - 2, \\ x, & \text{otherwise.} \end{cases}$$

Thus it follows that the function $f_1(x) \supset f_2(x)$ takes the value $k - 1$ if and only if $x = k - 2$. The results above show

$$j_{k-1}(f_1(x) \supset f_2(x)) = j_{k-2}(x),$$

and

$$j_1(x) = j_{k-2}(\sim x).$$

Every constant is constructed as follows:

$$\begin{gathered} \sim(k - 2) = 1, \\ k - 2 \supset 1 = 2, \\ k - 2 \supset 2 = 3, \\ \dots\dots\dots\dots \\ k - 2 \supset k - 2 = k - 1, \\ \sim(k - 1) = 0. \end{gathered}$$

Hence

$$\sim(x_1 \supset x_2) = \begin{cases} x_1 - x_2, & \text{if } x_1 \geqslant x_2, \\ \\ 0, & \text{if } x_1 < x_2, \end{cases}$$

implies

$$j_2(x) = j_1(\sim(x \supset 1)).$$

Similarly

$$\begin{gathered} j_3(x) = j_2(\sim(x \supset 1)), \\ j_4(x) = j_3(\sim(x \supset 1)), \\ \dots\dots\dots\dots \\ \dots\dots\dots\dots \\ j_{k-3}(x) = j_{k-4}(\sim(x \supset 1)). \end{gathered}$$

Thus we can obtain every constant, functions $\max(x_1, x_2)$, $\min(x_1, x_2)$ and $j_i(x)$ $(i = 0, 1, \ldots, k - 1)$. The theorem follows from the lemma.

Remark: *We can construct the functions* $j_i(x)$ *as follows:*

$$
\begin{aligned}
j_2(x) &= j_1(\sim(x \supset 1)),\\
j_3(x) &= j_1(\sim(x \supset 2)),\\
&\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\\
j_{k-3}(x) &= j_1(\sim(x \supset k - 4)).
\end{aligned}
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REFERENCE

[1] С. В. Яблонский, "Функциональные построения в к-значной логике," Труды Математического Института Имени В. А. Стеклова, Том 51 (1958), страны 5-142.

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