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THE STRONG FUTURE TENSE

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1 *Introduction* If the universe is deterministic, to say at time **t** that it will be the case that p is to say that p is true in the only physically possible future relative to **t**. But if the universe is indeterministic, the meaning of "it will be the case that p" becomes more problematic. Relative to **t** there are many alternative possible futures instead of one. In which of these should we require that p be true? The answer given by classical tense logic is that $\mathsf{F}p$ is true iff p is true at some point in at least one such future (see for example [6], p. 38.). But this answer makes it quite possible for $\mathsf{F}p$ to be true while p never is; this happens if p is true in some possible future which turns out not to be actual, i.e., not to be the one that the history of the world follows. This is a defect of F *qua* representative of the future tense of natural languages. If p turned out not to be true we would be justified in accusing the person who previously uttered "$\mathsf{F}p$" of speaking falsely. In what follows we shall examine a different sort of future tense operator which avoids this defect.

The most straightforward way of avoiding the difficulty of having $\mathsf{F}p$ true and p false in an indeterministic future-branching universe is to replace F by a stronger operator. "$\mathsf{F}p$" says in effect that p is true somewhere on some future branch. Let "$\mathsf{S}p$" assert that p is true somewhere on every future branch. Then a situation in which $\mathsf{S}p$ is true and p never is cannot arise. However, the converse situation can arise: it is possible for $\mathsf{S}p$ to fail to be true even though p turns out later to be true. (This can occur when p is true on some future branches but not on all.) Although this might seem to render $\mathsf{S}p$ as deficient as $\mathsf{F}p$, on balance $\mathsf{S}p$ appears to fit the use of the future tense in natural languages better. The man who arrives at the powerhouse during a torrential downpour and asks breathlessly, "Will the dam burst?", is not asking if the dam's bursting is a feature of *some* possible futures, but of *all*.

Against what has just been said it might be objected that what determines the truth of any statement of the form "it will be the case that p" is not whether p is true in some possible futures, or in all, but whether p is ture in *the actual* future. That is, in the branch that becomes history. But

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this presupposes that there *is* such a branch: that of all the possible futures relative to a given moment there is one and only one singled out as *the* future. (We may not know, according to this theory which the branch is, but this is irrelevant; one branch is ontologically distinguished from its neighbors.) We may call such a view *predestinarian*. It represents a philosophical tradition which is not without interest, but the author can at present find no metaphysical (or for that matter theological) justification for it.[1] More interesting is the view that there is only one way to evaluate the truth or falsehood of a future contingent statement, namely by waiting and seeing. This view does not require the existence of an ontologically distinguished future, but to elaborate it formally as a tense logic would require a semantics the model structures of which were dynamic rather than static. (At time $\mathbf{t}$ a given future-branching model structure would have a certain tree-like form; at time $\mathbf{t} + \delta\mathbf{t}$ it would have lost some branches and its main stem would be longer.) In this paper, no attempt will be made to deal with dynamic model structures. Instead we shall confine ourselves to exploring the properties of the strong future tense operator S, which corresponds to what Prior in [6], p. 132, calls the "Peircian" theory of the future tense.

2 *Semantics for the strong future tense* We have, then, the two future tense operators F and S, and defining G as $\sim$F$\sim$ and I as $\sim$S$\sim$ we arrive at the following truth-conditions:

Fp is true iff p is true somewhere on some future branch.
Gp is true iff p is true everywhere on every future branch.
Sp is true iff p is true somewhere on every future branch.
Ip is true iff p is true everywhere on some future branch.

Embedding these truth-conditions in a formal semantics requires that we have some way of quantifying over branches, and branches (i.e., possible "histories" or "scenarios") may be regarded as sets of instantaneous world-states, the latter being 3-dimensional cross-sections of 4-dimensional manifolds of events.[2] These world-states are ordered by the relation "later than". The model structures for our semantics will therefore be ordered pairs $\langle W, L\rangle$, where W is the set of instantaneous world-states and L the relation "later than". Branches are defined as maximal L-chains on W. Let v be an assignment of truth-values to propositional variables over $\langle W, L\rangle$. Then the valuation function $v_{\mathfrak{M}}$ determined by the model $\mathfrak{M} = \langle W, L, v\rangle$ is defined inductively as follows, where $x, z \in W$ and b is any branch on $\langle W, L\rangle$. Note that truth-conditions are given for formulae containing not only the future-tense operators F, G, S, and I, but also the past-tense operators P and H.[3]

1. (Base clause). Where A is any propositional variable,

$$v_{\mathfrak{M}}(A, z) = v(A, z)$$

2. (Induction step).

$$
\begin{aligned}
v_{\mathfrak{M}}(\sim A, z) &= \mathbf{T} \text{ iff } v_{\mathfrak{M}}(A, z) = \mathbf{F} \\
v_{\mathfrak{M}}(A \,\&\, B, z) &= \mathbf{T} \text{ iff } v_{\mathfrak{M}}(A, z) = v_{\mathfrak{M}}(B, z) = \mathbf{T} \\
v_{\mathfrak{M}}(\mathsf{F}A, z) &= \mathbf{T} \text{ iff } (\exists x)(Lxz \,\&\, v_{\mathfrak{M}}(A, x) = \mathbf{T}) \\
v_{\mathfrak{M}}(\mathsf{G}A, z) &= \mathbf{T} \text{ iff } (x)(Lxz \supset v_{\mathfrak{M}}(A, x) = \mathbf{T}) \\
v_{\mathfrak{M}}(\mathsf{S}A, z) &= \mathbf{T} \text{ iff } (b)[z \epsilon b \supset (\exists x)(x \epsilon b \,\&\, Lxz \,\&\, v_{\mathfrak{M}}(A, x) = \mathbf{T})] \\
v_{\mathfrak{M}}(\mathsf{I}A, z) &= \mathbf{T} \text{ iff } (\exists b)[z \epsilon b \,\&\, (x)[(x \epsilon b \,\&\, Lxz) \supset v_{\mathfrak{M}}(A, x) = \mathbf{T}]] \\
v_{\mathfrak{M}}(\mathsf{P}A, z) &= \mathbf{T} \text{ iff } (\exists x)(Lzx \,\&\, v_{\mathfrak{M}}(A, x) = \mathbf{T}) \\
v_{\mathfrak{M}}(\mathsf{H}A, z) &= \mathbf{T} \text{ iff } (x)(Lzx \supset v_{\mathfrak{M}}(A, x) = \mathbf{T})
\end{aligned}
$$

The relation L may be subjected to a number of different constraints, each of which restricts the variety of acceptable model structures. For example, L may be transitive, or be subject to conditions which produce model structures which are non-beginning and/or non-ending, dense, and non-branching toward the past and/or future. In classical tense logic, without the operators S and I, it is not difficult to investigate the set of valid formulae corresponding to each successive restriction on L, and to produce characteristic axioms for the corresponding deductive systems.

But with S and I, the problem becomes more difficult. The author has succeeded in constructing a cumbersome axiomatization of the future-tense fragment of "minimal" tense logic (i.e., operators F, G, S, and I only, and no restrictions on L). But he has so far found no way of adding past-tense operators without imposing restrictions on L and hence abandoning the minimal system. In the next two sections, a system will be presented and proved complete with respect to an L-transitive semantics with non-beginning and non-ending model structures.[4]

3 *The system* $\mathbf{TNK_{ts}}$ The following is the basis of the axiomatic system $\mathbf{TNK_{ts}}$, corresponding to the restrictions $(Lxy \,\&\, Lyz) \supset Lxz$, $(\exists x)Lyx$ and $(\exists x)Lxy$ placed upon the relation L.

Primitive symbols &, $\sim$, G, S, H.

Definitions Usual definitions of $\supset$, $\vee$, $\equiv$, and

$$\mathsf{F} = \sim\mathsf{G}\sim,\ \mathsf{I} = \sim\mathsf{S}\sim,\ \mathsf{P} = \sim\mathsf{H}\sim.$$

Rules of inference

1. Substitution
2. Detachment
3. RG: $\vdash A \rightarrow \vdash \mathsf{G}A$
4. RH: $\vdash A \rightarrow \vdash \mathsf{H}A$

Axioms

1. Any set sufficient for 2-valued logic
2. $\mathsf{G}(p \supset q) \supset (\mathsf{G}p \supset \mathsf{G}q)$
3. $\mathsf{G}(p \supset q) \supset (\mathsf{S}p \supset \mathsf{S}q)$
4. $\mathsf{H}(p \supset q) \supset (\mathsf{H}p \supset \mathsf{H}q)$
5. $p \supset \mathsf{GP}p$
6. $p \supset \mathsf{HF}p$
7. $\mathsf{S}p \supset \mathsf{F}p$

8. $\mathrm{H}p \supset \mathrm{P}p$
9. $\mathrm{G}p \supset \mathrm{S}p$
10. $\mathrm{FF}p \supset \mathrm{F}p$
11. $\mathrm{PP}p \supset \mathrm{P}p$
12. $\mathrm{S}(\mathrm{S}p \vee p) \supset \mathrm{S}p$
13. $\mathrm{PS}p \supset (p \vee \mathrm{S}p \vee \mathrm{P}p)$
14. $(\mathrm{F}p \,\&\, \mathrm{S}q) \supset \mathrm{F}[(p \,\&\, q) \vee (p \,\&\, \mathrm{S}q) \vee (\mathrm{F}p \,\&\, q)]$
15. The sequence TIS2, TIS3, . . ., TISn, . . .

where

TIS2 is: $(\mathrm{I}p \,\&\, \mathrm{S}q \,\&\, \mathrm{S}r) \supset \mathrm{F}[(p \,\&\, q \,\&\, r) \vee (p \,\&\, q \,\&\, \mathrm{F}(p \,\&\, r))$
$\vee (p \,\&\, r \,\&\, \mathrm{F}(p \,\&\, q))]$

TIS3 is: $(\mathrm{I}p \,\&\, \mathrm{S}q \,\&\, \mathrm{S}r \,\&\, \mathrm{S}\mathbf{t}) \supset \mathrm{F}[(p \,\&\, q \,\&\, r \,\&\, \mathbf{t}) \vee (p \,\&\, q \,\&\, r \,\&\, \mathrm{F}(p \,\&\, \mathbf{t}))$
$\vee (p \,\&\, q \,\&\, \mathbf{t} \,\&\, \mathrm{F}(p \,\&\, r)) \vee (p \,\&\, r \,\&\, \mathbf{t} \,\&\, \mathrm{F}(p \,\&\, q))$
$\vee (p \,\&\, q \,\&\, \mathrm{F}(p \,\&\, r \,\&\, \mathbf{t})) \vee (p \,\&\, r \,\&\, \mathrm{F}(p \,\&\, q \,\&\, \mathbf{t}))$
$\vee (p \,\&\, \mathbf{t} \,\&\, \mathrm{F}(p \,\&\, q \,\&\, r)) \vee (p \,\&\, q \,\&\, \mathrm{F}(p \,\&\, r \,\&\, \mathrm{F}(p \,\&\, \mathbf{t})))$
$\vee (p \,\&\, q \,\&\, \mathrm{F}(p \,\&\, \mathbf{t} \,\&\, \mathrm{F}(p \,\&\, r))) \vee (p \,\&\, r \,\&\, \mathrm{F}(p \,\&\, q \,\&\, \mathrm{F}(p \,\&\, \mathbf{t})))$
$\vee (p \,\&\, r \,\&\, \mathrm{F}(p \,\&\, \mathbf{t} \,\&\, \mathrm{F}(p \,\&\, q))) \vee (p \,\&\, \mathbf{t} \,\&\, \mathrm{F}(p \,\&\, q \,\&\, \mathrm{F}(p \,\&\, r)))$
$\vee (p \,\&\, \mathbf{t} \,\&\, \mathrm{F}(p \,\&\, r \,\&\, \mathrm{F}(p \,\&\, q)))]$

and where the number of disjuncts in TISn is the number of different ways in which, allowing multiple occupation and ignoring empty boxes, n distinguishable objects can occupy a row of n boxes.[5]

The fact that 15 consists of a sequence of axioms rather than one indicates that $\mathbf{TNK}_{ts}$ is not finitely axiomatizable. A proof of this will be found in section **6** below. $\mathbf{TNK}_{ts}$ is, however, decidable, and the completeness proof given for it provides a decision procedure. Of the axioms listed above, 8 is falsifiable if any branch of our model structures has a first moment, 9 if any branch has a last moment, and 9-14 if L is not transitive. The remaining axioms hold unrestrictedly.

Certain important features of $\mathbf{TNK}_{ts}$ distinguishing the strong from the weak future tense are the absence of the theses $p \supset \mathrm{HS}p$ and $\mathrm{S}p \vee \mathrm{S}{\sim}p$, and the presence of $\mathrm{PS}p \supset (p \vee \mathrm{S}p \vee \mathrm{P}p)$. By contrast, we have $p \supset \mathrm{HF}p$ and $\mathrm{F}p \vee \mathrm{F}{\sim}p$, but not $\mathrm{PF}p \supset (p \vee \mathrm{F}p \vee \mathrm{P}p)$. The first of these theses has played an important role since the time of Cicero or earlier in discussions of fatalism, God's omniscience and the freedom of the will. Prior devotes quite a lot of space to discussing how to get rid of it in [6], pp. 117-134, and [5], pp. 157-161. Concerning the second thesis, Thomason in [8], p. 267, remarks that "It will or it won't" has the force of tautology, from which it might seem that $\mathrm{S}p \vee \mathrm{S}{\sim}p$ ought to hold. But the reason why "it will or it won't" has the force of tautology is that "it will" and "it won't" are generally thought of as contradictories. $\mathrm{S}p$ and $\mathrm{S}{\sim}p$, on the other hand, are not contradictories; nor are they, unlike $\mathrm{F}p$ and $\mathrm{F}{\sim}p$, sub-contraries, which is the reason why $\mathrm{F}p \vee \mathrm{F}{\sim}p$ holds and $\mathrm{S}p \vee \mathrm{S}{\sim}p$ does not. "It will or it won't" is representable either by $\mathrm{F}p \vee {\sim}\mathrm{F}p$ or $\mathrm{S}p \vee {\sim}\mathrm{S}p$. Finally, although $\mathrm{PF}p \supset (p \vee \mathrm{F}p \vee \mathrm{P}p)$ is falsifiable in future-branching model structures, $\mathrm{PS}p \supset (p \vee \mathrm{S}p \vee \mathrm{P}p)$ is not. Use of the strong future tense, then, avoids the awkwardness of ever being in a position to assert that it was the case that p would be true, while at the same time denying that p is, was, or ever will be.

4 *Completeness of* $\mathbf{TNK_{ts}}$ The completeness proof given here makes use of semantic tableaux, and is patterned on Kripke's proof in [2]. Two different kinds of tableau constructions will be used, in one of which the R-relation between tableaux is transitive, while in the other it is not. These will be known, respectively, as R-constructions and $\mathbf{R}$-constructions. The overall structure of the completeness proof is as follows. Where

(1) denotes "A is a thesis of $\mathbf{TNK_{ts}}$" (abbreviated "$\vdash A$")
(2) denotes "A is valid" (in the semantics of section **2**)
(3) denotes "The R-construction for $\sim A$ closes"
(4) denotes "The $\mathbf{R}$-construction for $\sim A$ closes",

we show, successively, (1) $\supset$ (2), (2) $\supset$ (3), (3) $\supset$ (4), (4) $\supset$ (1).

Theorem 1 *If* $\vdash A$, *then* A *is valid.*

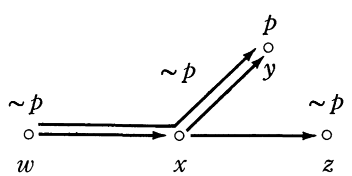
Proof: Detailed verification that each axiom of $\mathbf{TNK_{ts}}$ is valid, and that the rules of inference preserve validity, presents no great difficulty. Assume for example, in the case of Axiom 9, that for some model $\mathfrak{M}$ and some world-state z the following hold:

1. $v_{\mathfrak{M}}(\mathsf{G}p, z) = \mathbf{T}$
2. $v_{\mathfrak{M}}(\mathsf{S}p, z) = \mathbf{F}$

We derive a contradiction as follows:

3. $(x)(Lxz \supset v_{\mathfrak{M}}(p, z) = \mathbf{T})$ [1]
4. $(\exists b)[z \in b \;\&\; (x)((x \in b \;\&\; Lxz) \supset v_{\mathfrak{M}}(p, x) = \mathbf{F})]$ [2]
5. $z \in b' \;\&\; (x)[(x \in b' \;\&\; Lxz) \supset v_{\mathfrak{M}}(p, x) = \mathbf{F}]$ [4, EI]
6. $z \in b' \supset (\exists x)(x \in b' \;\&\; Lxz)$ [Condition of non-endingness, which together with the transitivity of L ensures that every branch is non-ending]
7. $w \in b' \;\&\; Lwz$ [6, 5, MP, EI]
8. $v_{\mathfrak{M}}(p, w) = \mathbf{F}$ [5, UI, 7, MP]
9. $v_{\mathfrak{M}}(p, w) = \mathbf{T}$ [3, UI, 7, MP]

It is worth noting the role played by the transitivity of L in the validity of such axioms as 13. If L were not transitive, the following would be a countermodel for 13:



Branches are $\{x, y\}$, $\{x, z\}$, $\{w, x, y\}$ but not $\{w, x, z\}$ (i.e., we have Lxw and Lzx but not Lzw). $\mathsf{PS}p$ is true at x, but not p, $\mathsf{S}p$, or $\mathsf{P}p$.

Before proceeding to Theorem 2, we shall give semantic tableaux rules for R-constructions. Our tableaux will differ from Kripke's in being one-sided rather than two-sided, but each alternative set T of tableaux will be written Kripke-style on a separate piece of paper.[6] Instead of there being, as in modal logic, only one relation R among tableaux, we shall have for tense-logical tableaux two distinct relations RE, corresponding to "earlier", and RL, corresponding to "later". Note that RE and RL are both transitive.

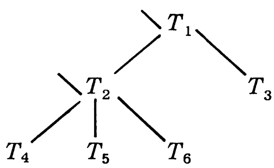
1. Starting rule. To start an R-construction for A, begin a tableau t with A as initial item.
2. $\sim\sim$ If $\sim\sim A$ appears in any tableau t, put A in tableau t.
3. & If A & B appears in tableau t, add A and B separately to t.
4. $\sim$& If $\sim(A$ & $B)$ appears in tableau t, where t belongs to the set T of tableaux, we replace T by two alternative sets T' and T'', formed by copying out T twice. Let t$'$ and t$''$ be the copies of t in T' and T'' respectively. Then add $\sim A$ to t$'$ and $\sim B$ to t$''$.
5. $\sim$ F. If $\sim$ FA appears in t, add G$\sim A$ to t.
6. $\sim$G. If $\sim$GA appears in t, add F $\sim A$ to t.
7. $\sim$ P. If $\sim$ PA appears in t, add H$\sim A$ to t.
8. $\sim$H. If $\sim$HA appears in t, add P$\sim A$ to t.
9. $\sim$S. If $\sim$SA appears in t, add I $\sim A$ to t.
10. $\sim$I. If $\sim$IA appears in t, add S$\sim A$ to t.
11. F. If FA appears in t, begin a new tableau t$'$ such that tREt$'$ with A as initial item.
12. G. If GA appears in t, put A in any tableau t$'$ such that tREt$'$ or t$'RL$t.
13. GN. If GA appears in t, begin a new tableau t$'$ such that tREt$'$ with A as initial item.
14. P. If PA appears in t, begin a new tableau t$'$ such that tRLt$'$ with A as initial item.
15. H. If HA appears in t, put A in any tableau t$'$ such that tRLt$'$ or t$'RE$t.
16. HN. If HA appears in t, begin a new tableau t$'$ such that tRLt$'$ with A as initial item.
17. S. If SA appears in t, begin a new tableau t$'$ such that tREt$'$ with A as initial item.
18. S1. If SA appears in t, where t ϵ T, and if there is a tableau t$'$ "next" to t such that tREt$'$ or t$'RL$t and no further tableau falls between t$'$ and t, then replace T by three alternative sets T', T'', and T''', formed by copying out T three times. In one copy T' add A to t$'$; in T'' add SA to t$'$; and in T''' begin a new tableau t$''$ with initial item A such that, if tREt$'$, then tREt$''$ and t$''RE$t$'$, and if t$'RL$t, then t$'RL$t$''$ and t$''RL$t.
19. I. If IA appears in t, begin a new tableau t$'$ such that tREt$'$ with A as initial item.
20. IS. If items IA and SB appear in t, begin a new tableau t$'$ such that tREt$'$ with A and B as initial items.[7]
21. IS2. If items IA, SB, and SC appear in t, where t ϵ T, replace T by three alternative sets T', T'', and T''', formed by copying out T three

times. In T' add a new tableau $\mathfrak{t}'$ such that $\mathfrak{t}RE\mathfrak{t}'$ with initial items A, B, and C. In T'' add a new tableau $\mathfrak{t}_1''$, with initial items A and B, such that $\mathfrak{t}RE\mathfrak{t}_1''$; and a new tableau $\mathfrak{t}_2''$, with initial items A and C, such that $\mathfrak{t}_1''RE\mathfrak{t}_2''$. In T'' do the same as in T'', but with B and C interchanged.
22. IS3. If $\mathsf{I}A$, $\mathsf{S}B$, $\mathsf{S}C$, and $\mathsf{S}D$ appear in $\mathfrak{t} \in T$, proceed as in rule IS2, replacing T by the thirteen alternative sets corresponding to the thirteen disjuncts in the axiom TIS3.
23. ISn. $n > 3$ Again analogous to IS2, the number of alternative sets required being equal to the number of disjuncts in axiom TISn.

This completes the list of R-rules. We now stipulate that a tableau *closes* iff it contains a pair of mutually contradictory items A and $\sim A$, that a set of tableaux closes iff one of its tableaux closes, and that an R-construction closes iff all its alternative sets close.

Theorem 2 *If A is valid, then the R-construction for $\sim A$ closes.*

Proof: Assume that the R-construction for $\sim A$ does not close. As is pointed out by Kripke ([2], p. 77) this assumption assures us only of an open alternative set at each stage in the (possibly infinite) construction—not that there is a set open for the whole construction. Note, however, that an R-construction may be infinite either through possessing an open infinite set, in which case we have what we require, or through possessing an infinite number of alternative sets.[8] In the latter case, we can construct what amounts to an open infinite set by diagramming in tree form the splits in alternative sets as follows:



etc.

Each set "contains" each set above it on a branch of the tree in the sense that it was formed from it by first copying it out and then adding something to it (as is specified by the rules $\sim\&$, S1, IS2, etc.) If the R-construction comprises an infinite number of sets the tree will be infinite, and by Koenig's lemma, since it forks finitely, it will contain an infinite branch. This branch defines what we may call an open infinite quasi-set: by following down the branch we may specify the quasi-set to any length desired.

The R-construction for $\sim A$ being completed and open, we now define a countermodel $\mathfrak{M}$ for A as follows. Select any open alternative set or quasi-set T_i, and define $\mathfrak{M}$ as $\langle W, L, v\rangle$, where W is the set of tableaux $\mathfrak{t}_j'$ comprising T_i; L is defined as the union of RL with the converse of RE (i.e., $L\mathfrak{t}_j\mathfrak{t}_k$ iff $\mathfrak{t}_jRL\mathfrak{t}_k$ or $\mathfrak{t}_kRE\mathfrak{t}_j$); branches are maximal L-chains in T_i; and v is defined for propositional variables p_k as follows:

$$v(p_k, \mathsf{t}_j) = \mathbf{T} \text{ iff } p_k \in \mathsf{t}_j.$$

We then show, for any formula A and any tableau $\mathsf{t} \in T_i$:

Lemma 1 *If* $A \in \mathsf{t}$, *then* $v_{\mathfrak{M}}(A, \mathsf{t}) = \mathbf{T}$.

Proof: By induction on the length of A. The basis of the induction follows from the definition of v. The induction step breaks down into cases as follows:

Case 1. A is $\sim B$, where B is atomic. Since T_i is open, and since $\sim B \in \mathsf{t}$, $B \notin \mathsf{t}$. Hence $v_{\mathfrak{M}}(B, \mathsf{t}) = \mathbf{F}$, hence $v_{\mathfrak{M}}(A, \mathsf{t}) = \mathbf{T}$.
Case 2. A is of the form $\sim\sim B$. Since the R-construction is completed, $B \in \mathsf{t}$ by an application of the rule $\sim\sim$. Hence $v_{\mathfrak{M}}(B, \mathsf{t}) = \mathbf{T}$ by the inductive hypothesis, hence $v_{\mathfrak{M}}(A, \mathsf{t}) = \mathbf{T}$.
Case 3. A is B & C. By rule &, $B \in \mathsf{t}$ and $C \in \mathsf{t}$, hence $v_{\mathfrak{M}}(B, \mathsf{t}) = v_{\mathfrak{M}}(C, \mathsf{t}) = \mathbf{T}$. Hence $v_{\mathfrak{M}}(A, \mathsf{t}) = \mathbf{T}$.
Case 4. A is $\sim(B$ & $C)$. By rule $\sim$&, $\sim B \in \mathsf{t}$ or $\sim C \in \mathsf{t}$, hence $v_{\mathfrak{M}}(\sim B, \mathsf{t}) = \mathbf{T}$ or $v_{\mathfrak{M}}(\sim C, \mathsf{t}) = \mathbf{T}$. Hence $v_{\mathfrak{M}}(B, \mathsf{t}) = \mathbf{F}$ or $v_{\mathfrak{M}}(C, \mathsf{t}) = \mathbf{F}$, hence $v_{\mathfrak{M}}(B \,\&\, C, \mathsf{t}) = \mathbf{F}$, hence $v_{\mathfrak{M}}(A, \mathsf{t}) = \mathbf{T}$.
Case 5. A is $\sim \mathsf{F}B$. By the rule $\sim\mathsf{F}$. $\mathsf{G}\sim B \in \mathsf{t}$, and $v_{\mathfrak{M}}(A, \mathsf{t}) = v_{\mathfrak{M}}(\mathsf{G}\sim B, \mathsf{t})$. This case reduces to case 12 below.
Case 6. A is $\sim\mathsf{G}B$. Reduced to case 11 below.
Case 7. A is $\sim\mathsf{P}B$. Reduced to case 14 below.
Case 8. A is $\sim\mathsf{H}B$. Reduced to case 13 below.
Case 9. A is $\sim\mathsf{S}B$. Reduced to case 16 below.
Case 10. A is $\sim\mathsf{I}B$. Reduced to case 15 below.
Case 11. A is $\mathsf{F}B$. The rule F guarantees that there exists a tableau t' such that $\mathsf{t}RE\mathsf{t}'$, i.e., $L\mathsf{t}'\mathsf{t}$, and that $B \in \mathsf{t}'$. By the inductive hypothesis $v_{\mathfrak{M}}(B, \mathsf{t}') = \mathbf{T}$, hence $v_{\mathfrak{M}}(A, \mathsf{t}) = \mathbf{T}$.
Case 12. A is $\mathsf{G}B$. The rule G guarantees that for every tableau t' such that $\mathsf{t}RE\mathsf{t}'$ or $\mathsf{t}'RL\mathsf{t}$, i.e., $L\mathsf{t}'\mathsf{t}$, $B \in \mathsf{t}'$. By the inductive hypothesis $v_{\mathfrak{M}}(B, \mathsf{t}') = \mathbf{T}$, hence $v_{\mathfrak{M}}(A, \mathsf{t}) = \mathbf{T}$.
Case 13. A is $\mathsf{P}B$. Similar to case 11.
Case 14. A is $\mathsf{H}B$. Similar to case 12.
Case 15. A is $\mathsf{S}B$. The rules IS, $\mathsf{IS2}$, . . . guarantee that, if t contains any I-items, the item B will occur somewhere on every branch of the R-construction which is "future" relative to t. Furthermore, the rule S1 guarantees that if, at any given stage in the R-construction, there exists a future branch relative to t on which B does not occur, B will eventually be added to that branch. For suppose there is a future branch k nodes long on which B does not occur. Let t' be the first node of this branch. An application of S1 will result in three new alternative sets:

(i) In the first set, we have $B \in \mathsf{t}'$.
(ii) In the second set, we have $\mathsf{S}B \in \mathsf{t}'$. Relative to t', there is now a future branch $k - 1$ nodes long on which B does not occur. If $k = 1$ (i.e., if t' is the last node of the branch) an application of rule S provides a "tip" to the branch on which B occurs. If $k > 1$, we repeat the procedure until eventually the branch is furnished with a B-bearing tip.

(iii) In the third set, a new tableau t'' is constructed which constitutes the new first node of the branch and which contains B. Hence in every case, where $SB \in t$, application of the R-rules ensures that B holds somewhere on every future branch of t. Hence $v_{\mathfrak{M}}(SB, t) = \mathbf{T}$.

Case 16. A is IB. If t contains no S-items, application of the rule I ensures that B occurs everywhere on at least one future branch of t. If t contains k S-items, the rule ISk will be used once for each distinct I-item of t, and will ensure the desired result in each alternative set. If, at a later stage in the R-construction, a new $(k + 1)$th S-item is added to t, application of the rule IS$k + 1$ produces the same result. Hence in every case, $v_{\mathfrak{M}}(IB, t) = \mathbf{T}$.

This completes the proof of the lemma. We note now that $\sim A$, the initial item of the R-construction, is the initial item of the main tableau t_1 of T_i. Hence $v_{\mathfrak{M}}(\sim A, t_1) = \mathbf{T}$ by the Lemma 1. It follows that A is not valid, which completes the proof of Theorem 2.

Theorem 3 *If the R-construction for $\sim A$ closes, then the* **R***-construction for* $\sim A$ closes.

Proof: We must first define the difference between an R-construction and an **R**-construction. Following Kripke [2], the relations **RE** and **RL** of **R**-constructions will not reflect any special properties of the relation L: in particular, unlike RE and RL, they will not be transitive. The rules for **R**-constructions will differ slightly from those of R-constructions: we substitute **RE** and **RL** throughout for RE and RL, and re-write rules G, H, and ISn as follows:

G. If GA appears in t, put GA and A in any tableau t' such that $t\,\mathbf{RE}\,t'$ or $t'\,\mathbf{RL}\,t$.
H. Add HA as additional transferred item.
IS. Add IA as additional initial item.
IS*n*, $n \geqslant 2$. Similar to the re-writing of IS.

The re-written rules plainly give the *effect* of transitivity (*cf.* [2], p. 81). Their advantage over R-rules is that, in the proof of Theorem 4 below, we need only consider, for any given application of the rules, the changes wrought on a tableau by its immediate neighbors. (By constrast, R-rules may affect far-away tableaux.) For the proof of Theorem 4, it suffices to note that for any closed R-construction there will be a closed **R**-construction with precisely corresponding tableaux and sets of tableaux.

Theorem 4 *If the* **R***-construction for $\sim A$ closes, then $\vdash A$.*

Proof: We begin with some definitions.

Definition 1 Rank of a tableau t in a set T.

(i) t has rank 0 if there is no t' such that $t\,\mathbf{RE}\,t'$ or $t\,\mathbf{RL}\,t'$.
(ii) If $t_1, \ldots t_n$ are all the immediate **RE**- or **RL**-descendants of t, then $\mathrm{Rank}(t) = \mathrm{Max}\{\mathrm{Rank}(t_i)\} + 1$.

Definition 2 The *associated formula* (a.f.) of a tableau t at a stage in an $\mathbf{R}$-construction is the conjunction of all the items of t.

Definition 3 The *characteristic formula* (c.f.) of a tableau t at a stage is defined inductively as follows:

(i) If $\mathsf{Rank}(\mathsf{t}) = 0$, then the c.f. is the a.f.
(ii) $\mathsf{Rank}(\mathsf{t}) > 0$. Suppose t bears $\mathbf{RE}$ to $\mathsf{t}_1, \ldots \mathsf{t}_m$, and $\mathbf{RL}$ to $\mathsf{t}_{m+1}, \ldots, \mathsf{t}_n$. Let B_i be the c.f. of t_i, and let A be the a.f. of t. Then the c.f. of t is:

$$A \mathbin{\&} \mathsf{F}B_1 \mathbin{\&} \ldots \mathbin{\&} \mathsf{F}B_m \mathbin{\&} \mathsf{P}B_{m+1} \mathbin{\&} \ldots \mathbin{\&} \mathsf{P}B_n.$$

Definition 4 The c.f. of a set T at a stage is the c.f. of the main tableau of T. The c.f. of an $\mathbf{R}$-construction comprising sets $T_1, \ldots T_r$ at a stage is $D_1 \vee D_2 \vee \ldots \vee D_r$, where D_i is the c.f. of T_i.

Lemma 2 *If A_0 is the* c.f. *of the initial stage of an* R-*construction, and B_0 the* c.f. *of any stage, then* $\vdash A_0 \supset B_0$.

Proof: The proof, which is by induction on stages, proceeds exactly as in [2], pp. 83-85. The proof is by cases, according to which $\mathbf{R}$-rule produces the $(m + 1)$th stage from the mth stage, and the tense-logical theses (corresponding to Kripke's modal theses) needed to justify the cases generated by the rules $\sim\sim$, **&**, and $\sim$**&** are the following:

1. $\mathsf{G}(p \supset q) \supset (\mathsf{F}p \supset \mathsf{F}q)$
2. $\mathsf{H}(p \supset q) \supset (\mathsf{P}p \supset \mathsf{P}q)$
3. $\mathsf{F}(p \vee q) \supset (\mathsf{F}p \vee \mathsf{F}q)$
4. $\mathsf{P}(p \vee q) \supset (\mathsf{P}p \vee \mathsf{P}q)$.

In addition, the rules RG and RH are needed. The following cases are new:

Case 4. The $(m + 1)$th stage comes by the rule $\sim$**F**. Justified by

5. $\sim\mathsf{F}p \supset \mathsf{G}\sim p$.

Cases 5-9. $\sim$**G**, $\sim$**P**, $\sim$**H**, $\sim$**S**, $\sim$**I**. Similar to case 4.
Case 10. **F**. The c.f. of tableau t at stage m is $X \mathbin{\&} \mathsf{F}A$, and at stage $m + 1$ it is $X \mathbin{\&} \mathsf{F}A \mathbin{\&} \mathsf{F}A$. Justified by $\vdash p \supset (p \mathbin{\&} p)$.
Case 11. **G**. The c.f. of t at stage m is $\mathsf{G}A \mathbin{\&} X \mathbin{\&} \mathsf{F}B_1 \mathbin{\&} \ldots \mathbin{\&} \mathsf{F}B_n$

(i) If there is no t' such that $\mathsf{t}'\mathbf{RL}\mathsf{t}$, the c.f. of t at stage $m + 1$ is

$$\mathsf{G}A \mathbin{\&} X \mathbin{\&} \mathsf{F}(B_1 \mathbin{\&} \mathsf{G}A \mathbin{\&} A) \mathbin{\&} \ldots \mathbin{\&} \mathsf{F}(B_n \mathbin{\&} \mathsf{G}A \mathbin{\&} A),$$

and the case is justified by repeated use of

6. $(\mathsf{G}p \mathbin{\&} \mathsf{F}q) \supset \mathsf{F}(q \mathbin{\&} \mathsf{G}p \mathbin{\&} p)$.

(ii) If there is a t' such that $\mathsf{t}'\mathbf{RL}\mathsf{t}$, the c.f. of t' at stage m is

$$Y \mathbin{\&} \mathsf{P}(\mathsf{G}A \mathbin{\&} X \mathbin{\&} \mathsf{F}B_1 \mathbin{\&} \ldots \mathbin{\&} \mathsf{F}B_n).$$

The c.f. of t' at stage $m + 1$ is

$$\mathsf{G}A \mathbin{\&} A \mathbin{\&} Y \mathbin{\&} \mathsf{P}[\mathsf{G}A \mathbin{\&} X \mathbin{\&} \mathsf{F}(B_1 \mathbin{\&} \mathsf{G}A \mathbin{\&} A) \mathbin{\&} \ldots \mathbin{\&} \mathsf{F}(B_n \mathbin{\&} \mathsf{G}A \mathbin{\&} A)]$$

and the case is justified by 6,

7. $\mathsf{P}(p \,\&\, q) \supset \mathsf{P}p$
8. $\mathsf{PG}p \supset \mathsf{G}p$

and

9. $\mathsf{PG}p \supset p$.

Case 12. **GN**. The c.f. of $\mathfrak{t}$ at stage m is $X \,\&\, \mathsf{G}A$. At stage $m+1$ it is $X \,\&\, \mathsf{G}A \,\&\, \mathsf{F}A$. Justified by

10. $\mathsf{G}p \supset \mathsf{F}p$

Case 13. **P**. Similar to case 10.
Case 14. **H**. Similar to case 11, using

11. $(\mathsf{H}p \,\&\, \mathsf{P}q) \supset \mathsf{P}(q \,\&\, \mathsf{H}p \,\&\, p)$
12. $\mathsf{F}(p \,\&\, q) \supset \mathsf{F}p$
13. $\mathsf{FH}p \supset \mathsf{H}p$
14. $\mathsf{FH}p \supset p$.

Case 15. **HN**. Similar to case 12, using

15. $\mathsf{H}p \supset \mathsf{P}p$

Case 16. **S**. The c.f. of $\mathfrak{t}$ at stage m is $X \,\&\, \mathsf{S}A$. At stage $m+1$ it is $X \,\&\, \mathsf{S}A \,\&\, \mathsf{F}A$. Justfied by

16. $\mathsf{S}p \supset \mathsf{F}p$

Case 17. **S1**. (i) Suppose $\mathfrak{t} \in T$ and there is a tableau $\mathfrak{t}'$ "next" to $\mathfrak{t}$ such that $\mathfrak{t}\mathbf{RE}\mathfrak{t}'$. The c.f. of $\mathfrak{t}$ at stage m is

$$E\colon\ X \,\&\, \mathsf{S}A \,\&\, \mathsf{F}B$$

and at stage $m+1$ T, with c.f. D_j, is replaced by the three alternative sets T', T'', and T''', with c.f.'s D_{j1}, D_{j2}, and D_{j3}. The c.f. of $\mathfrak{t}$

$$\begin{aligned} &\text{in } T' \text{ is } E'\colon && X \,\&\, \mathsf{S}A \,\&\, \mathsf{F}(B \,\&\, A)\\ &\text{in } T'' \text{ is } E''\colon && X \,\&\, \mathsf{S}A \,\&\, \mathsf{F}(B \,\&\, \mathsf{S}A)\\ &\text{in } T''' \text{ is } E'''\colon && X \,\&\, \mathsf{S}A \,\&\, \mathsf{F}(A \,\&\, \mathsf{F}B)\end{aligned}$$

(Note that in T''' we have inserted a new tableau $\mathfrak{t}''$ between $\mathfrak{t}$ and $\mathfrak{t}'$ and thereby increased the rank of $\mathfrak{t}$ by one.) Using

17. $(\mathsf{S}p \,\&\, \mathsf{F}q) \supset [\mathsf{F}(q \,\&\, p) \vee \mathsf{F}(q \,\&\, \mathsf{S}p) \vee \mathsf{F}(p \,\&\, \mathsf{F}q)]$

we can show that $\vdash E \supset (E' \vee E'' \vee E''')$, and by an argument similar to Kripke's for the case of the rule $\sim\&$ we derive eventually

$$\vdash D_j \supset (D_{j1} \vee D_{j2} \vee D_{j3}).$$

(ii) Suppose there is a tableau $\mathfrak{t}'$ next to $\mathfrak{t}$ such that $\mathfrak{t}'\mathbf{RL}\mathfrak{t}$. The c.f. of $\mathfrak{t}'$ at stage m is

$$E\colon\ Y \,\&\, \mathsf{P}(X \,\&\, \mathsf{S}A)$$

and at stage $m + 1$ the c.f. of t'

in T' is E': $A \;\&\; Y \;\&\; P(X \;\&\; SA)$
in T'' is E'': $SA \;\&\; Y \;\&\; P(X \;\&\; SA)$
in T''' is E''': $Y \;\&\; P(A \;\&\; P(X \;\&\; SA))$.

(Note that in T''' we have increased the rank of t' by one.) To prove $\vdash E \supset (E' \vee E'' \vee E''')$ we need

18. $P(p \;\&\; Sq) \supset [(q \;\&\; P(p \;\&\; Sq)) \vee (Sq \;\&\; P(p \;\&\; Sq)) \vee P(q \;\&\; P(p \;\&\; Sq))]$.

Case 18. **I**. The c.f. of t at stage m is $X \;\&\; IA$, and at stage $m + 1$ it is $X \;\&\; IA \;\&\; FA$. Justified by

19. $Ip \supset Fp$.

Case 19. **IS**. The c.f. of t at stage m is $X \;\&\; IA \;\&\; SB$, and at stage $m + 1$ it is $X \;\&\; IA \;\&\; SB \;\&\; F(IA \;\&\; A \;\&\; B)$. Justified by

20. $(Ip \;\&\; Sq) \supset F(Ip \;\&\; p \;\&\; q)$.

Case 20. **IS2**. The c.f. of t at stage m is

$$E\colon\ X \;\&\; IA \;\&\; SB \;\&\; SC,$$

and at stage $m + 1$ the c.f. of t

in T' is E': $X \;\&\; IA \;\&\; SB \;\&\; SC \;\&\; F(IA \;\&\; A \;\&\; B \;\&\; C)$
in T'' is E'': $X \;\&\; IA \;\&\; SB \;\&\; SC \;\&\; F(IA \;\&\; A \;\&\; B \;\&\; F(IA \;\&\; A \;\&\; C))$
in T''' is E''': $X \;\&\; IA \;\&\; SB \;\&\; SC \;\&\; F(IA \;\&\; A \;\&\; C \;\&\; F(IA \;\&\; A \;\&\; B))$.

We derive $\vdash E \supset (E' \vee E'' \vee E''')$ by means of the thesis

21. $(Ip \;\&\; Sq \;\&\; Sr) \supset [F(Ip \;\&\; p \;\&\; q \;\&\; r) \vee F(Ip \;\&\; p \;\&\; q \;\&\; F(Ip \;\&\; p \;\&\; r))$
$\vee F(Ip \;\&\; p \;\&\; r \;\&\; F(Ip \;\&\; p \;\&\; q))]$.

Case 21. **ISn**, **n** > 2. Similar to case 20, using a derivative of axiom TISn in place of 21.

The theses 1-21 are all derivable in TNK_{ts}.

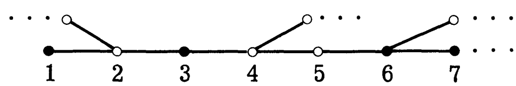
This completes the proof of Lemma 2. The proof of Theorem 4 follows as in [2], p. 86, and our completeness proof for TNK_{ts} is ended.

5 *Decision procedure for* $\mathbf{TNK_{ts}}$ The use made of semantic tableaux in the completeness proof allows a decision procedure to be devised for $\mathbf{TNK_{ts}}$. Since all items occurring in all tableaux in **R**-constructions are sub-formulae of the initial item of the main tableau, they are finite in number. Hence the number of distinct tableaux (disregarding redundant items) is also finite, and any sufficiently large **R**-construction will contain "repetitive" tableaux. The existence of such tableaux may be used to prevent the growth of infinite **R**-constructions, so that all non-theorems of $\mathbf{TNK_{ts}}$ will have finite countermodels.

Every alternative set of every **R**-construction has the form of a tree, each tableau at every node of the tree being related to its successor-tableaux by one of the relations **RE** or **RL**. Branches of this tree have one of the following three forms:

(i) An **RE**-chain of tableaux: i.e., a sequence each member of which is related to its successor by the relation **RE**.
(ii) An **RL**-chain.
(iii) An **R**-chain which is neither an **RE**- nor an **RL**-chain: i.e., a sequence some members of which bear **RE** to their immediate successors and some **RL**. We call such a chain an **R***-chain.[9]

The tableaux which make up a chain of any one of the above types may be divided into "parent" and "offspring" tableaux, as the following picture of an **RE**-chain indicates:



Tableaux 1, 3, 6, and 7 are "parent" tableaux, which is to say that at one stage in the construction they formed the end of the chain and other members of the chain were subsequently started from them. For example, 2 and 3 might have been started from 1 by **S1**; 4, 5, and 6 from 3 by **IS3**, etc. No tableaux on the chain are started from the "offspring" tableaux 2, 4, and 5, although tableaux on other chains may be (in which case, relative to these chains, the tableaux in question become "parent" tableaux).[10]

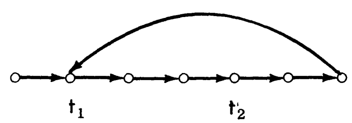
It will be evident that all items in all tableaux on a chain are subformulae of items in any "parent" tableau above them. (This is not true of "offspring" tableaux: tableau 3 for example in the diagram above may contain items which are not subformulae of tableau 2.) This fact leads to the following

Theorem *No* **R***-*chain in any* **R**-*construction can be infinite, unless it culminates in an infinite* **RE**- *or* **RL**-*chain.*

Proof: **R***-chains consist of alternating sections of **RE**-chains and **RL**-chains. Let t and t' be two members of an **R***-chain such that each tableau is both the last tableau of an **RE**-section and the first tableau of an **RL**-section. Furthermore let t' be below t. Both t and t' are "parent" tableaux. Defining the *power* **P** of a tableau as the number of symbols in the longest item it contains, we see that $\mathbf{P}(t) > \mathbf{P}(t')$. (If the longest item of t (which need not be unique) is either a truth-function or begins with F, G, P, S, or I, it will not appear in any tableau below t. If it begins with H, it will appear in all tableau belonging to the **RL**-section below t, but it will not appear in t'.) Since the power of the first tableau of any **R***-chain is finite, no such chain can have more than a finite number of **RE**- and **RL**-sections. Hence any infinite **R***-chain must terminate in an infinite **RE**- or **RL**-chain.

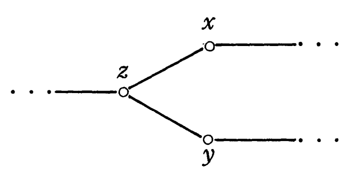
We are now in a position to see how to prevent the growth of infinite **R**-constructions. Let us call two tableaux *equivalent* if (ignoring redundancies) they contain the same items. We can limit the growth of

RE- and **RL**-chains by running together equivalent tabelaux as follows. Suppose t_1 and t_2 are equivalent tableaux on the same **RE**- or **RL**-chain, formed from the same initial formula by the application of the same **R**-rules at each stage in their construction. Suppose at some later stage a new tableau t_3 is about to be started, with the same initial formula, the same R-rules again being applicable at each stage. Plainly all and only those items which went into t_1 and t_2 will go into t_3, so instead of starting t_3 as a new tableau, we relate its "parent" tableau back to either t_1 or t_2, using whichever of the relations **RE** or **RL** is appropriate.[11]



What has been done is to provide the chain with a terminal loop, and when this has been accomplished for each **RE**- and **RL**-chain the resulting **R**-construction will be finite. Note that model structures with cyclical branches are quite compatiable with the transitivity and non-endingness of the relation L of our semantics, so that the models resulting from reducing infinite to finite **R**-constructions are still $\mathbf{TNK_{ts}}$-models.

6 *Non-finite axiomatizability of* $\mathbf{TNK_{ts}}$ Consider the set of axioms 1-15 of section **3**. Let the sequence of systems $S_1, S_2, \ldots, S_n, \ldots$ be defined as follows. Each system S_i is closed under the rules of $\mathbf{TNK_{ts}}$, S_1 is axiomatized by the set of axioms 1-14, and $S_n = S_{n-1} \cup \{TISn\}$. We shall show that the sequence $\{S_n\}$ constitutes a chain of systems of increasing strength. To show this, take any axiom A of $\mathbf{TNK_{ts}}$ and replace S by F and I by G throughout to obtain A^*. Let $\{S_n^*\}$ be the corresponding sequence of systems obtained by so doing. To show that S_n^* is a proper supersystem of S_{n-1}^*, we see that the following model falsifies TIS_n^*, while satisfying all the axioms of S_{n-1}^*:



(Let $p_1, p_2, \ldots, p_{n+1}$ be all the variables of $TISn^*$, and p_1, p_2, p_3 those of axioms 1-14. Let $Gp_1, Fp_2, \ldots, Fp_{n+1}$ be true at z; $p_1, p_2, \ldots, p_n$ true at x; and p_1 and p_{n+1} true at y. Then the antecedent of $TISn^*$ is true but the consequent false. Since p_{n+1} occurs in $TISn^*$ but not in $TISn - 1^*$, and since the other axioms of S_n^* hold in transitive non-beginning, non-ending model structures, the linear model $\{\ldots, z, x, \ldots\}$ satisfies all the axioms of S_{n-1}^*.)

It can be shown by induction on the length of proof that if any formula B is provable in S_n, then B^* is provable in S_n^*. Since $TISn^*$ is not provable in S_{n-1}^*, $TISn$ is not provable in S_{n-1}. Hence $S_1 \subset S_2 \subset \ldots \subset S_n \subset \ldots$, and we have that, for all n, $S_n \neq \mathbf{TNK_{ts}}$.

Lemmon has shown in [3] that a necessary and sufficient condition for any system S not to be finitely axiomatizable is that there be an infinity of systems $S_1, S_2, \ldots, S_n, \ldots$ such that (i) $S_n \subseteq S_{n+1}$, (ii) $S_n \neq S$ for all n, and (iii) $S = \bigcup S_n$. These conditions are satisfied by the sequence $\{S_n\}$ defined above, hence we conclude that $\mathbf{TNK_{ts}}$ is not finitely axiomatizable.

NOTES

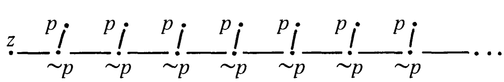
1. In [8], p. 270, Thomason conceives the point at issue to be whether it is *meaningful* to assert the existence of an ontologically distinguished future.

2. It does not matter that there is no unique way of defining a world-state of mutually simultaneous events; *any* 3-dimensional cross-section comprised of points separated by space-like intervals will do.

3. The truth-conditions given here for S differ in one important respect from those found in [4]. In [4] there is no quantification over branches, and the truth-conditions for S are:

$$v_{\mathfrak{M}}(SA,z) = \mathsf{T} \text{ iff } (x)[Lxz \supset (\exists y)(Bxy \;\&\; Lyz \;\&\; v_{\mathfrak{M}}(A,y) = \mathsf{T})]$$

where "Bxy" is defined as "$Lxy \vee Lyx \vee x = y$". It was pointed out to the author, however, by Mr. Alasdair Urquhart, that these truth-conditions allow for the following:

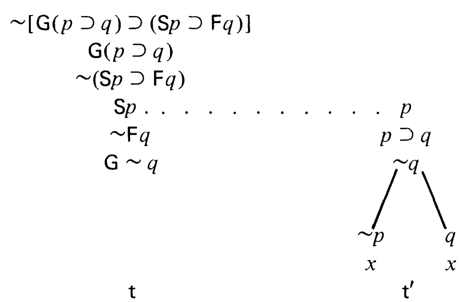


Sp is true at z, but p is never true on the main branch. Hence the future tense operator of [4] might be better described as representing the idea of a "hopeful" future, where p is always "just around the corner," rather than of a future in which p is definitely going to be true.

4. Prior would say, complete with respect to transitive, non-beginning and non-ending *time*. But it is intuitively more plausible to regard time as having the structure of a one-dimensional continuum, and to consider the four-dimensional universe of events as being non-ending, branched, etc.

5. For $n = 4$, this number is 75; for $n = 5$, 541; for $n = 6$, 4683.

6. Jeffrey [1] credits Smullyan (see, e.g., [7]) with the invention of one-sided tableaux. For convenience, instead of writing different alternative sets on different pieces of paper, one can divide tableaux into branches as Jeffrey does. For example:



7. Since IA means that A is true everywhere on some branch, and SB that B is true somewhere on every branch, the rule IS expresses what may be called the principle of the arrow and the net: the net always catches the arrow, and the arrow always pierces the net.

8. R-constructions for G Fp and G(p v Fp) exemplify these two possibilities.

9. In term of model structures, **RE**- and **RL**-chains correspond to branches, and **R***-chains to zig-zag paths directed alternately past- and future-wards.

10. Although further tableaux belonging to a chain may be started by an application of **S1** (alternative set no. 3) from a tableau already introduced by **S1**, the latter tableau does not qualify as a "parent" tableau because at no time did it form the end of the chain.

11. The reason why we waited until t_1 began to repeat itself for the *second* time, rather than the first, will become apparent upon examining the R-construction for $\sim$(G$Hp \supset$ FGp). Again, it should not be thought that if t_1 and t_2 are equivalent, their "parent" tableaux must also be equivalent, as is shown by the R-construction for G(Fp & Fq).

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