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C_1 Is Not Algebraizable

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Abstract The purpose of this brief note is to give a very short proof of a fact, first proved by Mortensen, that illustrates the strength of the theory of algebraizability of deductive systems developed by Blok and Pigozzi. We build a small matrix model of C_1 for which the Leibniz operator is not 1-1.

In [3], da Costa introduces a family of paraconsistent deductive systems C_n, $n = 1,2,3,\ldots$; among other things he raises the question of algebraizability of these systems.

In [4] Mortensen proves that the paraconsistent deductive systems C_n are not algebraizable by showing that in the absolutely free formula algebra for C_1 there is no nontrivial congruence compatible with the set of theorems of the system. Since Mortensen gives no formal definition of algebraizability and a related theory of algebraizability, his proof is long and somewhat complicated.

Blok and Pigozzi develop such a theory in [1] using a generalization of the usual Lindenbaum-Tarski algebraization process. The reader is referred to that paper, especially Chapter 5, for the justification of our proof. The idea is very simple, it is proven (see Theorem 5.1) that if a deductive system is algebraizable and $\mathfrak{A} = \langle \boldsymbol{A},\mathfrak{D}\rangle$ is a matrix model, then there is an isomorphism between the lattice of filters of $\mathfrak{A}$ and the lattice of congruences of $\mathfrak{A}$. Moreover, this isomorphism is given by the function which assigns to each filter F the largest compatible congruence, that is, the largest congruence θ such that if $a\theta b$ and $a \in F$, then $b \in F$. Thus, in order to prove nonalgebraizability of a system, it is enough to find a matrix model in which this function, called the Leibniz operator, is not an isomorphism.

In Blok and Pigozzi [2], the question of the existence of a small matrix model for the system C_1, in which the Leibniz operator is not an isomorphism, is posed. We answer this question affirmatively, thus proving the nonalgebraizability of C_1. Obviously, since the system C_1 is an extension of the systems C_n, $n = 1,2,3,\ldots$, none of these are algebraizable.

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The system The system C_1 has the following axioms and rules, where x^0 stands for $\neg(x \wedge \neg x)$.

A1 $x \to (y \to x)$

A2 $(x \to y) \to ((x \to (y \to z)) \to (x \to z))$

A3 $\dfrac{x \quad x \to y}{y}$

A4 $(x \wedge y) \to x$

A5 $(x \wedge y) \to y$

A6 $x \to (y \to (x \wedge y))$

A7 $x \to (x \vee y)$

A8 $y \to (x \vee y)$

A9 $(x \to z) \to ((y \to z) \to ((x \vee y) \to z))$

A10 $x \vee \neg x$

A11 $\neg\neg x \to x$

A12 $y^0 \to ((x \to y) \to ((x \to \neg y) \to \neg x))$

A13 $(x^0 \wedge y^0) \to (x \wedge y)^0$

A14 $(x^0 \wedge y^0) \to (x \vee y)^0$

A15 $(x^0 \wedge y^0) \to (x \to y)^0$.

The example Consider the matrix $\langle \boldsymbol{A}, \mathfrak{D} \rangle$ where $\boldsymbol{A}$ is the algebra $\langle A; \vee, \wedge, \to, \neg, 0, 1 \rangle$ with domain $A = \{0, a, b, 1, u\}$, underlying lattice $\langle A; \vee, \wedge \rangle$ as depicted in Figure 1, $\mathfrak{D} = \{u, 1\}$ are the designated elements and the operations $\to$ and $\neg$ are defined in the tables below.

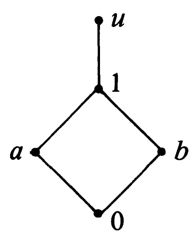

Figure 1.

$\to$	u	1	a	b	0	$\neg x$	$\neg\neg x$	$x^0 = \neg(x \wedge \neg x)$
u	u	u	a	b	0	1	0	0
1	u	1	a	b	0	0	1	1
a	u	1	1	b	b	b	a	1
b	u	1	a	1	a	a	b	1
0	u	1	1	1	1	1	0	1

Remark Notice that the sublattice $B = \{0, a, b, 1\}$ with $\neg$ as a complement is a Boolean algebra and for these elements, $x \to y = \neg x \vee y$.

We first have to check that this is a matrix model of C_1. This is not difficult. Observe that the partition $\{0\}, \{a\}, \{b\}, \{1, u\}$ of A is a $\langle \vee, \wedge, \rightarrow, 0, 1 \rangle$-congruence and that the corresponding quotient is a Heyting algebra. This takes care of Axioms 1 to 9. Axioms 10, 11, 13, 14, and 15 are immediate. The reader can easily check that Axiom 12 is also valid.

Next observe that there are no nontrivial congruences. Let θ be a congruence.

If $u\theta x$, $x \neq u$, then $0 = \neg\neg u \theta \neg\neg x = x$, so $u\theta 0$ and $\theta = \nabla$.
If $1\theta x$, $x \neq u$, 1, then $u = u \rightarrow 1\theta u \rightarrow x = x$, so $u\theta x$ and $\theta = \nabla$.
If

$$a\theta b, \text{ then } 1\theta b, \text{ so } \theta = \nabla$$

$$a\theta 0, \text{ then } b\theta 1, \text{ so } \theta = \nabla$$

$$b\theta 0, \text{ then } a\theta 1, \text{ so } \theta = \nabla.$$

So either $\theta = \nabla$ or $\theta = \Delta$.

Finally, just check that $F_1 = \{a, 1, u\}$ and $F_2 = \{b, 1, u\}$ are filters, and the largest compatible congruence with both of them is Δ.

REFERENCES

[1] Blok, W. J. and D. Pigozzi, "Algebraizable Logics," *Memoirs of the American Mathematical Society*, vol. 77, no. 396, 1989.

[2] Blok, W. J. and D. Pigozzi, *Lectures in Algebraic Logic*, unpublished manuscript.

[3] da Costa, N., "On the theory of inconsistent formal systems," *Notre Dame Journal of Formal Logic*, vol. 15, no. 4 (1976), pp. 497–510.

[4] Mortensen, C., "Every quotient algebra for C_1 is trivial," *Notre Dame Journal of Formal Logic*, vol. 21, no. 4 (1980), pp. 694–700.

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