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Broadening the Iterative Conception of Set

Mark F. Sharlow

Abstract The iterative conception of set commonly is regarded as supporting the axioms of Zermelo-Fraenkel set theory (ZF). This paper presents a modified version of the iterative conception of set and explores the consequences of that modified version for set theory. The modified conception maintains most of the features of the iterative conception of set, but allows for some non-wellfounded sets. It is suggested that this modified iterative conception of set supports the axioms of Quine's set theory NF.

1 Introduction

The iterative conception of set (see Boolos [2] and [3]) is an intuitive view of sets that provides a convenient picture of the structure of the set theoretic universe. The central idea of this conception is that the domain of all sets can be partitioned into levels in a specific manner. The bottom level of this partition contains any objects that are not sets. Above the bottom level, the sets in each successive level are thought of as being "built up" by the gathering together of objects from lower levels. When an infinite series of levels have been built up in this way, the union of these levels is taken to be the next level so that the "process" of adding new levels can continue into the transfinite.

This intuitive picture of the set theoretic universe as a hierarchy of levels, with sets built up via metaphorical acts of gathering or collecting, has found widespread use in set theory. The iterative conception of set commonly is regarded as supporting the axioms of Zermelo-Fraenkel set theory (ZF).[1] Indeed, the familiar concept of the set theoretic hierarchy corresponds closely to the system of levels portrayed by the iterative conception. As an informal picture of the universe of set theory, the iterative conception of set is both useful and perspicuous. One of its nicer characteristics is that it rules out the existence of the sets which give rise to the paradoxes of set theory ([2], p. 218).

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2 Preliminaries

In this paper I will study what happens when one makes a slight change in the iterative conception of set. This change involves the weakening of one assumption implicit in the iterative conception; namely, that the gathering together of objects to form a set always forms a set of higher level than the objects one began with. In the modified conception which I will describe, sets are built up by the gathering together of other sets, just as in the iterative conception. However, the set which one gets from such an act of gathering need not always be a "new" set; under some circumstances, this set can turn out to be identical to one of the sets with which one began. This modification may seem strange to those of us reared on the iterative conception of set; hence my first task will be to argue that the modified view of sets is coherent. This modified picture of the creation of sets does have a certain conceptual advantage: it allows for the existence of non-wellfounded sets. Such sets have been the objects of a great deal of research (see, for example, Aczel [1]) and are interesting in many ways, but the iterative conception of set rules them out across the board (see [1], pp. xvii–xviii; [2], pp. 219–20).

An obvious objection to this modification of the iterative conception is the possibility that it will reintroduce the paradoxes of set theory which the standard iterative conception of set neatly eliminates. I will argue that the modification, if carried out carefully, does not restore any of the known paradoxes. Indeed, it is only through consideration of the paradoxes that we can see how the modification should be carried out. I will argue that the modified iterative conception of set implements Russell's vicious circle principle (discussed in Gödel [7], pp. 453–59) in a way slightly different from the way in which the iterative conception implements that principle. Although the modified iterative conception allows some non-wellfounded sets to exist, it does not allow sets that have a clear potential for generating paradoxes.

After exploring this modified iterative conception of set, I will ask what sort of set theoretic axioms it might motivate. No complete answer will be forthcoming. However, I will present two arguments suggesting that the conception lends plausibility to Quine's set theory NF.

3 Iteration and the Paradoxes

As background for the main argument of this paper, I will spend the next several paragraphs discussing the paradoxes of set theory. This digression into an all-too-familiar topic may seem to be a waste of time, but actually it will provide a useful starting point for the rest of the paper. (For general information on the paradoxes, see Fraenkel et al. [6], Chapter 1.)

The iterative conception of set is known to forestall the paradoxes of set theory. The sets central to the paradoxes, such as the set of all non-self-membered sets in Russell's paradox, cannot be created via the set-building procedure used in the iterative conception of set. Hence a set theory motivated solely by the iterative conception of set should be free of the customary paradoxes. This much is well known (see [2], p. 218).

How does the iterative conception of set forestall the paradoxes? One well-known answer to this question is that the sets allowed by the iterative conception obey a principle of limitation of size (see Hallett [9], Chapters 4, 5, and 6). However, there is another well-known possible answer: that the paradoxes involve self-reference of a certain sort ([6], p. 11). If one thinks of the sets that give rise to the paradoxes as being built up through the gathering of objects, then those sets must be built up, directly or indirectly, from themselves (see [6], p. 11). Russell proposed that comprehension principles for sets must obey a *vicious-circle principle*—a restriction which says, more or less, that a collection cannot have an element which is built up from that same collection (see [7], pp. 453–59). Regardless of what one thinks of Russell's formulation of this principle (or of my paraphrase of his formulation), it is clear that the paradoxes of set theory all involve self-reference in essential ways.

Let us look, in a little more detail, at the way in which self-reference enters into the paradoxes. When one examines the paradoxes closely, one can discern in them a certain general logical structure. Each of the standard paradoxes begins with the assumption that a particular set exists—a set S of all objects with P, for some specific property P which is different for each paradox. Next, the existence of this set S is shown to imply that either (1) there is a member of S that *does not* satisfy P, or (2) there is an object that is not a member of S but that *does* satisfy P. In either case a contradiction ensues, because S was defined as the set of all Ps.

When a property P fits the description given in the preceding paragraph, we will say that P is *self-defeating*. That is, a property P is self-defeating if and only if the existence of a set S of all Ps entails that either some member of S does not have P, or that some nonmember of S has P.

The following restatements of two of the paradoxes illustrate how some of the better known paradoxes fit into the logical structure just described and how they involve the use of self-defeating properties as comprehension conditions.

Russell's paradox:

1. Let P be the property of not being a member of oneself.
2. Suppose that there is a set S of all Ps.
3. Then there is an object x such that either (1) x is a member of S, and x is not a P or (2) x is not a member of S, and x is a P. (This object x is S itself. To restate (3) in more familiar terms: if $x \in S$, then $x \notin x$; but if $x \notin S$, then $x \in x$.)

Burali-Forti paradox:

1. Let P be the property of being an ordinal.
2. Suppose that there is a set S of all Ps.
3. Then there is an object x such that either (1) x is a member of S, and x is not a P or (2) x is not a member of S, and x is a P. (In this case, the object x is the order type of S. To restate (3) in more familiar and detailed terms: if the order type x of the set S of ordinals is a member of S, then $x < x$ in the usual ordering of the ordinals, so x is not an ordinal; but if x is not a member of S, then S is well-ordered, and hence x is an ordinal.)

These two examples, though stated briefly, point up some details of the common logical structure to which I have referred. Each paradox involves a property P, a set S of all objects having P, and an object x (which may or may not be identical to S, depending on which paradox one is considering). If one assumes that x is a member of S, then one finds that x does not have P—but if one assumes that x is not a member of S, then one finds that x does have P. Either of these conclusions is false, since S is just the set of all objects having P.

Once we notice that the paradoxes have this logical structure, we can offer a more specific answer to the question, How does the iterative conception of set forestall the paradoxes? According to the iterative conception, the creation of a set S cannot change the domain of objects from which the members of S are gathered. Thus, if S is any set defined by means of a comprehension property P, then the existence of S cannot imply the existence of a member x of S which does not have P, or of an object x which is not a member of S but which has P.

These remarks, which actually contain little that is new, provide one way (not the only or the "best" way) of looking at the fact that the iterative conception of set forestalls the paradoxes. Viewed in this way, all the paradoxes have a common logical structure. Each of them involves a comprehension property having a certain logical feature. The iterative conception of set does not allow properties with this feature to determine sets.

4 Loosening Up the Hierarchy

According to the iterative conception of set, the creation of a set does not alter the levels of the universe lower than the level of that set. We have seen how this restriction prevents the paradoxes: it blocks the creation of sets whose existence would imply that there are lower-level objects of certain problematical sorts.

Now I am going to ask the reader to consider a novel possibility—one which the iterative conception of set does not allow. This is the possibility that the gathering of some objects to form a set might yield a set which turns out to be *identical* to one of the objects being gathered together.

The iterative conception of set posits set-building "acts" by which new sets are built up from *preexisting* sets. Thus, if we accept the unmodified iterative picture of sets, we will find it impossible to build up a set S from objects if one of those objects is S itself.[2] However, if we weaken one assumption of the iterative conception just slightly, we can allow for a set-gathering act which gathers up the very set that it is creating. The assumption that must be weakened is that the act of set formation always forms a set which *was not created prior* to that act. To weaken this assumption, we need only allow the possibility of gathering together objects which already have been gathered into a set at some earlier stage. An act of set "creation" of this sort would not really add a new set, but would merely re-create (or rediscover) a set that already is there. Such an act of set formation would not truly create a new set, but would *point out* a set that already exists. One can think of a technological analogue of this kind of set-forming act: it is much like the creative act of an inventor who unwittingly reinvents a known machine, thereby merely "pointing to" an existing machine instead of introducing a new machine into the universe.

The iterative conception of set, as we know it, allows only for acts of creation that create *new* sets. In the familiar temporal metaphor of the iterative conception, the act of gathering objects always yields a set which did not exist "before" the objects with which the process started. But one also can envision set-forming acts of another kind—acts analogous to reinventions instead of inventions. In an act of this latter sort, available objects are gathered together into a set, but the set thus formed is a set which already existed beforehand. To envision acts of this second sort, one need not abandon the basic mental picture behind the iterative conception of set. One still can think of sets as objects formed by acts of gathering together of previously available objects. But one must allow the possibility that the set thus formed will be one which already exists, instead of inevitably being a new set.

This possibility lies outside the usual iterative conception of set, but it does not lie as far outside as one might think. In a sense, the iterative conception *already* allows the recreation of previously created sets. Once a set is created in the iterative hierarchy, it is created again and again at each successive level (see [2], pp. 220–22; [3], p. 6). For example, the set $\{\varnothing\}$ exists at level 2, but it is created again at levels 3, 4, ... Thus, the iterative conception of set already allows for the formation of sets which previously have been formed. It allows for set-forming acts which do not yield truly new sets, but simply pick out old ones. What is new in our modified iterative conception is not the re-creation of sets, but the possibility that the set being built may be found among the objects from which that set is built up.

Once we allow this possibility, we can begin to see how an act of set formation could yield a self-membered set. One might gather together some existing sets, form a set from them—and then find that this "new" set actually is one of the sets that one started with. Such a set will be a member of itself. Thus, our modification of the iterative conception of set allows (at least in principle) for the possibility of non-wellfounded sets.

So far, this proposed modification of the iterative conception remains rather murky. As yet, I have done very little to explain it. To make this idea clearer and to fill in the details, I will show how one can visualize the modified iterative conception, in a way closely analogous to the familiar way of visualizing the iterative hierarchy.

People often explain the iterative hierarchy of sets with the help of a familiar cone-shaped diagram. A schematic version of this diagram is shown in Figure 1.

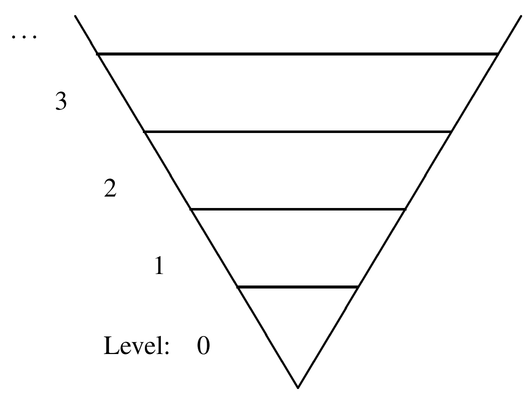

Figure 1 The set theoretic hierarchy.

In this diagram, the horizontal subdivisions represent levels in the universe. Of course, the iterative hierarchy does not consist of levels alone; it consists of sets, and the sets are arranged into levels. Thus, it would be more accurate to depict the iterative hierarchy as a lattice of individual sets as indicated in Figure 2. The thin solid lines in that figure join pairs of sets connected by the membership relation; if $a \in b$, then a and b are at the ends of a solid line segment, with b higher in the diagram than a. Even if we ignored the level structure of the universe, we still could classify sets as "earlier" and "later" by using the membership relation alone: if $a \in b$, we can say that a is earlier than b in the hierarchy, and that b is later than a. (The lattice in Figure 2, and parts of the later figures as well, resemble the graphs used in [1] to represent sets.)

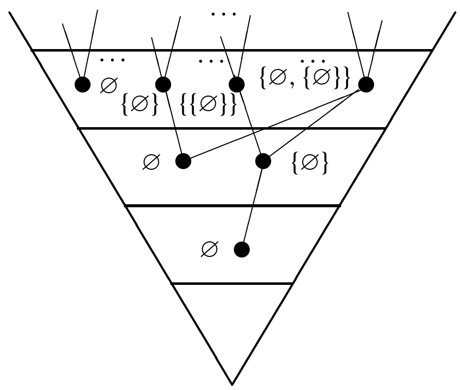

Figure 2 Set-building steps in the set theoretic hierarchy. The thin solid lines represent the membership relation with lower sets the members of higher ones in the diagram.

The iterative conception of set rests on the assumption (at least partially metaphorical) that sets are built up by means of set-building "acts" of some sort and that such an act involves the gathering together of sets earlier than the set being created. At bottom, this means that a set S cannot have as a member a set which is equal to S or later than S in the order induced by the $\in$-relation. The iterative conception allows us to build sets such as the set A in Figure 3(a). To perform the set-building act that creates A, we grab the sets $\varnothing$ and $\{\varnothing\}$, which are not later than or equal to A, and gather them together into a new set. (The new set A is just the von Neumann ordinal 2.)

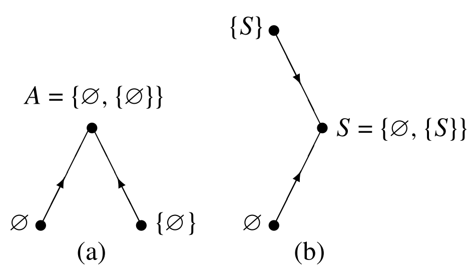

Figure 3 A set-building step: (a) creating a wellfounded set; (b) creating a non-wellfounded set.

In the modified iterative conception of set that I am suggesting, we allow all of the set-building acts which the iterative conception of set allows. In addition, we allow some set-building acts that build a set from *later* sets—or, as a special case, from the set being built by the act itself.

An example of a set-building act of this latter sort is shown in Figure 3(b). The set S is non-wellfounded; it is equal to $\{\varnothing, \{S\}\}$. It is created by means of the following set-building act: grab the empty set $\varnothing$, grab the singleton of S, and gather these two sets together into a new set. We are easily able to perform this act, because even though $\{S\}$ does not "preexist" S, we know that the set $\{S\}$ is going to exist at a later stage of set-building. While we are creating S, we already know that the set $\{S\}$ is going to come along at a later stage. We know this because we know that every set has a singleton—or, more precisely, because we know that any set can be gathered up and used to build its singleton in the usual iterative manner. Thus, we can include $\{S\}$—the singleton of the set we are about to create—among the sets we are gathering together. Instead of simply gathering earlier sets together, as we did in Figure 2, we also gather a *later* set and make it a member of the set we are building.

Note that this idea of including later sets actually is a variant of the idea of re-creating earlier sets that I mentioned earlier. For example, suppose that to form a set A, we must gather together objects $x_1, x_2, x_3, \ldots$ where one of the objects $x_1, x_2, x_3, \ldots$ is A. Then we have gathered together a later, yet-uncreated set, namely A, while we were forming A. However, we also have re-created a set A which was among the original objects $x_1, x_2, x_3, \ldots$. Thus, we can regard the creation of a set by use of a *later* set, as the re-creation of a set which became available *earlier* for use in set-building processes.

Let us look at the creation of our set S more closely. We can think of the gathering of $\{S\}$ into S in two ways, both seemingly coherent. If we think of sets as not existing until they are created, then we can think of the act of placing $\{S\}$ in S as an act of dropping a *placeholder* into the set S and making a mental note to replace the placeholder with $\{S\}$ after $\{S\}$ comes into existence. Alternatively, if we want to think of all sets as existing at once, then we can think of ourselves as adding to S an already existing element which we know must exist once we have created S. I will not try to rule on which of these mental pictures is the best; either one will do equally well for our purposes.

We can expand the diagram of Figure 2 to include set-building steps of this sort. The resulting diagram is shown schematically in Figure 4. All we have done is admit loops into the diagram. In Figure 4, the transitive closure of the membership relation no longer is a partial order. Instead, it is merely a transitive relation (not irreflexive or antisymmetric). Instead of resembling a tree or a lattice, the diagram now looks more like a thicket.

Figure 4 shows, in a schematic way, the proposed new picture of the set theoretic universe—the picture that we get when we relax the restriction against building sets from equal or later sets. In this new, modified version of the iterative picture, we still can assign sets to levels. The new picture does not allow for a neat level structure with linearly ordered levels (the same can be said for any picture of sets that allows for non-wellfounded sets). However, sets still have levels, as shown in Figure 4. Let level 0 be empty (or contain all nonsets, if we allow nonsets). Let level 1 contain the empty set, together with any sets that do not have the empty set anywhere in their $\in$-ancestry. Let level 2 contain all sets created by gathering together sets available at level 1. (Level 2 contains $\{\varnothing\}$ but now also contains a set S with $S = \{\varnothing, \{S\}\}$, and may possibly contain many other sets as well.) Let level n+1 contain all sets created by gathering together sets available at level n. This conception of levels gives us all of the finite-numbered levels. In this paper we will not assume that there are, or are not, transfinite levels as well.

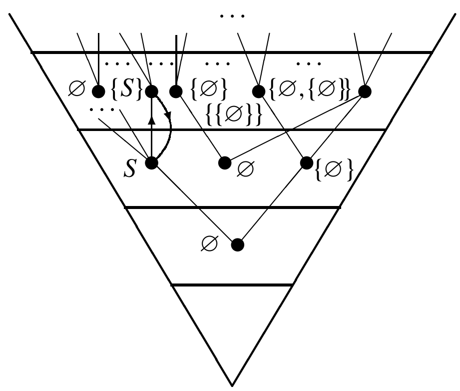

Figure 4 The universe with non-wellfounded sets added. (Thin solid lines represent the membership relation. Arrowheads indicate the direction of the membership relation; thin solid lines without arrowheads are assumed to point upward.)

The levels defined in this way resemble the levels in the standard iterative conception in some respects but not in others. The crucial difference is that, in the modified conception, there can be sets of higher level that are members of sets of lower level. (Example: the set $\{S\}$ just discussed.) A set need not be created at a particular level to be available for gathering at the next level above.

Earlier I hinted at two ways to think of the formation of non-wellfounded sets by set-gathering acts. We can think of a non-wellfounded set as being built by a set-building act that grabs later sets. On the other hand, we can think of a non-wellfounded set as not being fully formed by the act which creates it; on this view, we create a non-wellfounded set by leaving a place for each later set, and filling in these places later on. (Recall my earlier remark about the "placeholder.") The first of these two views seems more compatible with the view that all sets really exist at once. The second view seems more compatible with the iterative conception's suggestion that a set should not be thought of as existing until a set-building step creates it. On the second view, a non-wellfounded set is created by one set-building act and subsequently is completed by later acts. Collections of this "unfinished" sort are nothing new; they have long been known to set theory. Proper classes often are thought of as collections which cannot be created by any single set-building step. Indeed, there are versions of set theory that allow for the possibility of objects which are created at one step and completed at other steps (see Sharlow [14]). But I will not try to adjudicate between these two views in this paper, since doing so will not be necessary for my project.

Let me now summarize the proposed new variant of the iterative conception of set. In this variant, as in the standard iterative conception, sets are built up from sets or urelements by means of (metaphorical) acts of set formation. In each such act, some objects (zero or more objects) are gathered together into a single set. The objects which are gathered together are thought of as being available for use prior to the act of set formation. However, this new conception differs from the standard iterative conception in a crucial way: in the new conception, an act of set formation may produce a set which incorporates an element that is not yet created. Such a later element can, after all, be available for use by the set-gatherer even before the set-gatherer actually creates that object. All that is required is that the set-gatherer can anticipate with certainty the object's later existence. Thus, an act of set formation may make use of later sets as well as earlier ones. Such an act may either create a new set or pick out an old set. It need not always create a new set.

This variant of the iterative conception of set is less restrictive than the standard iterative conception. It can be thought of as the iterative conception of set minus one constraint: it lacks the prohibition against the use, in a set-building process, of a set which is created with the help of that process. This new conception of sets contains all the essential elements of the iterative conception—particularly the creation of sets from previously available sets—but without this strong (and unnecessary) restriction. Let us call the new conception the *minimally iterative conception of set*.[3]

5 Whither the Paradoxes?

The idea that acts of set creation may utilize later sets creates a prima facie threat of paradox. Acts of this sort have a certain self-referential character; in principle, such acts might give rise to paradoxical sets such as the Russell set. However, this threat is more apparent than real. Consider, for example, what one would have to do to create the Russell set in this way. One would have to gather together all and only those sets which are not members of themselves. It is easy to show that such a set-forming act *is impossible to carry out under any circumstances*. Suppose, by way of reductio, that it is possible to carry out such a set-forming act. Then the Russell set R is among the sets from which the members of R are selected. If one carries out the act that creates the set R, then either one gathers the available object R into that set, or one does not. If the object R is not self-membered, then one must include R in the set being formed. However, once one does this, R becomes a member of itself; hence it no longer belongs in the set. On the other hand, if R is a member of itself, then one should leave R out of the set being formed; but if one does this, then R is no longer a member of itself, and hence should be placed in the set. It is clear that one *cannot* gather together all and only the sets which are not members of themselves. If one tries to do so, one will, at best, get a set a little different from the set that one is trying to construct. Therefore, it is impossible to carry out any set-forming act that gives rise to the Russell set. This is impossible, not because the existence of the Russell set would logically imply a contradiction, but for a deeper reason: the set-forming act needed to create that set cannot even be completed in the first place.

It is important to note that the preceding argument for the nonexistence of the Russell set does not actually make use of the fact that the Russell set is paradoxical. That argument should not be confused with the following, somewhat similar-sounding argument: "Suppose that there were a Russell set. Then one could prove that $R \in R$ implies $R \notin R$, but one also could prove that $R \notin R$ implies $R \in R$. Therefore, there cannot be a Russell set." This is not what our argument says. The argument in the preceding paragraph is not about the logical consequences of the assumption "There exists a Russell set." Instead, it is about the impossibility of the *set-forming act* required to *create* the Russell set. The iterative conception of set, either in its standard form or in our modified form, portrays sets as products of set-creating acts or steps. The argument in the preceding paragraph tells us that no such act can create the Russell set. This is very different from telling us that the assumption of the existence of the Russell set implies a paradox. One might fairly say that the argument in the last paragraph is similar in structure to the derivation of Russell's paradox and that we would not have discovered this argument had we not already known of Russell's paradox. But these observations do not change the fact that the argument does not actually make use of Russell's paradox at all. The argument is not about the paradoxical consequences of certain set existence assumptions but about the impossibility of certain set-forming acts. The argument asks us to consider carefully what happens when set-forming acts are performed—not to explore the formal consequences of the assumption that there is a Russell set.

Note that our argument against the existence of the Russell set works just as well within the minimally iterative conception of set as it does within the standard iterative conception. That argument does not depend upon the standard iterative conception's restriction against using sets of higher level to build up new sets. According to either conception of set, the required set-forming act simply cannot be done.

Our earlier analysis of the paradoxes suggests that the preceding argument about Russell's paradox can be extended, mutatis mutandis, to all of the standard paradoxes. Each paradox involves a property P such that the existence of a set S of all Ps entails the existence of an object x which either is a member of S and lacks P or is a nonmember of S and has P. If we try to build such a set S by means of a set-forming act, then we find (either at that set-building step or at some later step) that we also build the object x. If we gather x into S, we end up with a non-P in S. If we leave x out of S, we end up with a P outside of S. In either case, the set that we end up forming cannot be the set we wanted—the set of all and only the Ps. Therefore, the set-building act cannot be carried out.

Set-building acts of the new sort discussed in the preceding section—acts which gather later as well as earlier sets—cannot build the sets which the paradoxes require. They cannot do this any more than can the conventional set-building acts used in the standard iterative conception. Hence if we extend the iterative conception of set by allowing acts of the new sort, we will not reintroduce the paradoxes. The minimally iterative conception of set should be free of the usual paradoxes of set theory.

When we form sets in the minimally iterative conception, we are not restricted to using sets that were constructed earlier; we may also utilize sets that will be constructed later. To illustrate how this conception works, I have given examples of sets built from earlier sets plus particular, specified later sets. For example, I described a set S equal to $\{\varnothing, \{S\}\}$, which one forms by gathering together the earlier set $\varnothing$ and the later set $\{S\}$. However, we need not restrict ourselves to gathering together particular, named later sets, such as $\{S\}$. We also might form sets by gathering together later or earlier sets of a particular *kind*. For example, instead of forming a set S by gathering together $\varnothing$ and $\{S\}$, we might form a set by gathering together all sets which are subsets of some particular set $\{A, B, C\}$ or which are elements of elements of a particular set D. Of course, we cannot gather together sets of just any old kind; our earlier discussion of impossible set-building acts showed that for some choices of P, we simply *cannot* gather together all sets having P. But we should be able to gather together all sets of a particular kind, provided that the kind is not the extension of a self-defeating property. That is, we should be able to gather together all sets with P, provided that P is not self-defeating. A set-gathering act of this sort amounts to an act of picking out a lot of specific objects which have a given property instead of just picking out specific named objects like $\varnothing$ and $\{S\}$. We can do this provided that we don't encounter objects for which we cannot decide whether the object should be gathered—that is, provided that no self-defeating property is involved in the set-building act.

6 Motivating a Set Theory: A First Argument

In this section I will ask and attempt to answer the question, What axioms of set theory does the minimally iterative conception of set support? I will argue that this conception can be used to justify some interesting known axioms of first-order set theory. As one might expect, the axioms in question will turn out to be somewhat different from those of ZFC.

Let us consider the iterative hierarchy constructed according to the minimally iterative conception of set. At level 1 in the hierarchy, we get the empty set just as we do within the standard iterative conception. According to our earlier definition of levels, we may also get some non-wellfounded sets at this level. At level 2 we get the set $\{\varnothing\}$ just as in the standard iterative conception, but we also get various other sets. One of these other sets is a set S such that $S = \{\varnothing, \{S\}\}$. Level 2 also may contain sets besides S and $\{\varnothing\}$.

The fact that we get non-wellfounded sets at level 2 is no surprise in view of what we have learned about the minimally iterative conception of set. However, one of the sets that we get at level 2 may come as a complete surprise. *At level 2, we get the universal set.*

The following argument shows how this happens. Consider a trivially true property P, say the property of self-identity (expressed by '$x = x$'). Suppose that we try to form a set S of all sets that have P. To form S, we must gather together all objects that have P—that is, all objects of level lower than that of S that have P, together with all objects of level equal to or higher than that of S that have P. But *all* objects have P. Therefore, S contains all sets.

Is the set-building act that creates S really possible? According to the minimally iterative conception of set, we can form the set of all objects of a given kind, provided that the act of gathering together all of those objects is an act that can be completed. (Recall what I said about self-defeating properties in the preceding section.) Thus, to find out whether the set S can be formed, we must ask whether the property of self-identity (P) is a self-defeating property. The readily available answer is that it is not. When we are gathering together sets, we can never be in any doubt about whether a particular set we are thinking of gathering has P. The property P is trivially true, so every set has P, and that's that. The set-gatherer's decision to put a particular set x into S or not to put that set x into S cannot change the fact that x has P. Hence we can never run into a situation like those encountered in our study of the paradoxes—a situation in which placing a set x in S makes x lose the property P or leaving a set x out of S makes x gain the property P which it would not otherwise have. To doubt that some particular set has P, or to worry that the formation of the set S might bear on whether some particular set has P, is to forget that P is a trivially true property.

Thus, the minimally iterative conception of set allows the universal set. If we can gather together earlier and later sets to form a set, then we can just as well gather together *all* earlier and later sets to form a set.

At what level of the hierarchy does the universal set first appear? We can form this set once we have the empty set, which is a member of the universal set. Thus, we can form it at level 2. The universal set is a set of level 2.

For those of us schooled in ZFC, the preceding claim may seem counterintuitive, shocking, and downright odd. It seems this way for two independent reasons. The first reason is that we have gotten the universal set from an intuitive picture that is a modified version of the iterative conception of set. The *un*modified iterative conception of set does not have a universal set; hence, the fact that we got such a set from a variant of the iterative conception may be counterintuitive. The second reason is that we got the universal set at a very low level in the iterative buildup. According to all of our training in ZFC, large sets should appear at high levels of the hierarchy—not right at the bottom at level 2.

Actually, both of these sources of surprise stem from intuitions which, though perfectly sound for the universe of the standard iterative conception, are inappropriate in the universe of the minimally iterative conception. Although these two conceptions have much in common, we cannot assume that they will handle very large sets in the same way. The fact that the minimally iterative conception gives a universal set—and not just at the top of the level structure but way down at level 2—is simply a reminder of the extent of these differences.

The discovery that the minimally iterative conception allows the universal set may lead us to wonder again whether that conception gives any paradoxical sets. We already have answered this question; the answer is "no"—or at least "we have no reason to think so." It is known that the existence of a universal set does not, in itself, give rise to any paradoxes. It does not even give rise to Cantor's paradox, which makes explicit use of the universal set (see, for example, Forster [5], p. 22 and also p. 1). The minimally iterative conception of set does not allow for sets determined by self-defeating comprehension properties. It disallows such sets, not because those sets have been found to give paradoxes, but for a more fundamental reason: because no set-building act can create those sets. All of the classical paradoxes appear to involve sets determined by self-defeating properties. Therefore, the minimally iterative conception is not expected to give rise to the classical paradoxes. The fact that the minimally iterative conception gives the universal set, which is more inclusive than any of the putative paradoxical sets, does not change this in the least.

Now I am ready to suggest a partial answer to the question What axioms of set theory does the minimally iterative conception of set support? The answer that I will suggest may be something of a surprise: *the minimally iterative conception of set supports all the axioms of Quine's set theory NF*. (For background on NF, see [5] and Quine [13].)

To back up this claim, I will use the minimally iterative conception of set to justify each of the set existence axioms in Hailperin's [8] finite axiomatization of NF. In these arguments, I will state each axiom verbally (although I am following roughly the formal version given by [5], p. 25) and will show how the minimally iterative conception of set gives the set required by the axiom. All ordered pairs used in these arguments are Wiener-Kuratowski.

Axiom 6.1 For any x and y, there is a set z of all objects which either are not in x or are not in y.

Argument Let x and y be two sets. (Note that these are not class abstracts but given, fixed sets. When we obtain them, we know that they exist, equipped with all their members of earlier or later level. Once we have picked out these two sets from the universe, there is no chance that some subsequent operation will show us that they cannot exist, as happens with the putative Russell class.) We can build either of the sets x and y by gathering together objects (these objects may be either earlier or later than x or y). Hence we also can gather together all the objects that were gathered together during the creation of x and also during the creation of y. In this way, we can build the binary intersection $w = x \cap y$. If we can build w, and we can build the universal set, then we also can build the absolute complement of w: just gather together all earlier and later sets as if forming the universal set, but leave out any set that is in w. This complement of w is the set z required by Axiom 6.1.

Note that this argument does not involve any self-defeating properties. Once a set-builder picks out two fixed sets x and y, he effectively has all of the members of x and y available for further gathering, regardless of whether those members are earlier or later than the set that he is building. In picking out x and y, he is not picking out class abstracts whose extensions may change when new sets are created. He is simply picking out fixed, given, known-to-exist sets. Since the members of these sets x and y already have been picked out (to form x and y), the set-builder can pick them out again to form the binary intersection w. Similarly, if he can pick out all objects to form the universal set, then he can pick out all objects besides the elements of w, to form z.

Another argument for Axiom 6.1 is as follows. In the standard iterative conception of set, any pair of sets has a binary intersection and a set difference. That is, if we can build up sets x and y by gathering objects together, then we can build up $x \cap y$ and $x - y$ too. This fact carries over to the minimally iterative conception; none of the new features of that conception could stop us from forming $x \cap y$ and $x - y$ in the familiar iterative way. But the minimally iterative conception, unlike the standard iterative conception, has a universal set V. Therefore, the minimally iterative conception also gives a set $V - (x \cap y)$, which is the set z required by Axiom 6.1.

Axiom 6.2 For any s, there is a set t such that for any x and y, t contains the ordered pair $\langle\{x\}, \{y\}\rangle$ if and only if s contains the ordered pair $\langle x, y\rangle$.

Argument Since the set s can be created by the iterative set-gathering procedure, it follows that all of the ordered pairs $\langle x, y\rangle$ in s can be created also. Hence so can all the sets x and y which are first and second members of those ordered pairs. But if we can build those sets x and y, then we also can build each of the singletons $\{x\}$ and $\{y\}$ of those sets, and hence each of the ordered pairs $\langle\{x\}, \{y\}\rangle$ made from those singletons. And just as we gathered all the ordered pairs $\langle x, y\rangle$ (perhaps along with other objects) to form s, so we also can gather all the ordered pairs $\langle\{x\}, \{y\}\rangle$ (perhaps along with other objects) to form a new set t. In this argument, we did not define any sets with the aid of self-defeating comprehension properties. We only built up new sets from elements that we already knew to exist. Those already known elements were the sets x and y whose ordered pairs are in the given fixed set s.

Axiom 6.3 For every s, there is a set t such that for any x, y, and z, t contains the ordered triple $\langle x, y, z\rangle$ if and only if s contains the ordered pair $\langle x, y\rangle$.

Argument Since the set s can be created by the iterative set-gathering procedure, it follows that all of the ordered pairs $\langle x, y\rangle$ in s can be created also. Hence so can all the sets x and y which are first and second members of those ordered pairs. But if we can create all such sets x and y, then we can go further and create from them all the ordered triples $\langle x, y, z\rangle$, where z is any set in the universal set. And just as we gathered all sets into a universal set, we can gather all these ordered triples $\langle x, y, z\rangle$ into a new set t. Again, no self-defeating properties have been deployed. We simply built up new sets from old sets (represented by x, y, z) that we already knew to exist.

Axiom 6.4 For every s, there is a set t such that for any x, y, and z, t contains the ordered triple $\langle x, z, y\rangle$ if and only if s contains the ordered pair $\langle x, y\rangle$.

Argument The argument is very similar to the argument for Axiom 6.3.

Axiom 6.5 For every s, there is a set t such that for all x and y, t contains the ordered pair $\langle x, y\rangle$ if and only if s contains x.

Argument Since the set s can be created by the iterative set-gathering procedure, it follows that all of the elements x of s can be created also. If we can create all such sets x, then we can go further and create from them all the ordered pairs $\langle x, y\rangle$ where y is any set in the universal set. And just as we gathered all sets into a universal set, we can gather all these ordered pairs $\langle x, y\rangle$ into a new set t. Again, no self-defeating properties were used.

Axiom 6.6 For every s, there is a set t of all sets x such that for all z, $\langle z, \{x\}\rangle$ is in s.

Argument Since the set s can be created by the iterative set-gathering procedure, it follows that all of the ordered pairs $\langle z, \{x\}\rangle$ in s can be created also. Hence so can all the sets x which are members of second members of those ordered pairs. But if we can build those sets x and gather them (after other steps like ordered pair formation) into the set s, then we also can gather all of those sets x together directly to form a new set t. Again, no self-defeating properties were used.

Axiom 6.7 For every s, there is a set t such that for all x and y, the ordered pair $\langle x, y\rangle$ is in t if and only if the ordered pair $\langle y, x\rangle$ is in s.

Argument This is much like the argument for Axiom 6.3.

Axiom 6.8 There is a set of all singletons.

Argument Suppose that we try to collect together all sets (earlier or later) that are singletons. Since there are at least two singletons (say $\{\varnothing\}$ and $\{\{\varnothing\}\}$), it follows that the set S we are building is not a singleton. S will not begin to be a singleton just because we have placed S, or some other set or sets, in S. There is no other available set which would cease (or begin) to be a singleton just because we placed it in S; the construction of S only changes the size of S. Thus, the creation of a set of all singletons is not a self-defeating process. Hence, by the minimally iterative conception of set, such a set exists.

Another intuitive argument for Axiom 6.8 is as follows. We can build up every object x and form a universal set from these objects. For every object x, we can build up the singleton $\{x\}$. Thus we can take the singletons $\{x\}$ and build up a set from them in exactly the same way as we built the universal set. This will give us the set of all singletons.

Axiom 6.9 There is a set t such that for all x and y, the ordered pair $\langle\{x\}, y\rangle$ is in t if and only if $x \in y$.

Argument For any set y, we can build up a pair $\langle y, y\rangle$. Given any $x \in y$, we can build a similar pair with all elements but x left out of the first member. Just as we built up the universal set from all sets y, we can take these pairs and build up a set of them.

It is important to note that informal arguments like those just given cannot be used to obtain the familiar paradoxical sets. We cannot, for example, motivate the Russell set by saying "Take all the sets x in the universal set; pick out only the ones that are not members of themselves, and form a set from them." The reason we cannot do this is that the property of non-self-membership is a self-defeating property. We know that we cannot use the minimally iterative conception to motivate a set defined by a self-defeating comprehension property. If we try to build up such a set, we will encounter objects that cannot be placed in the set but cannot be left outside the set—just as we did in our study of Russell's paradox. There simply is no set-building act that would create the Russell set.

Note also that our arguments for Axioms 6.1 – 6.9 do not depend logically upon the hindsight observation that the sets involved are nonparadoxical. Instead, these arguments show something different: that the required sets can be created by set-building acts. Consider the argument for Axiom 6.8 as an example. That argument contains the claim that the property of being a singleton is not a self-defeating property. That argument does *not* utilize the premise that the set of all singletons is nonparadoxical—although this last conclusion easily can be drawn from the finding that the property of being a singleton is not self-defeating. The distinction between these two conclusions—that the set is nonparadoxical and that the property is not self-defeating—may seem small, but it is an important distinction. Our argument for Axiom 6.8 does not merely say, "The set of all singletons is not paradoxical, therefore that set exists." Instead, the argument says that the set of all singletons can be created by a set-building act that is not self-defeating and therefore is possible to carry out. The arguments for Axioms 6.1 – 6.9 do not actually make use of the formal paradoxes of set theory—although, of course, our knowledge of the paradoxes helped us greatly in arriving at the concepts of self-defeating property and of self-defeating set formation.

The above arguments for Hailperin's axioms of NF make use of comprehension properties only in simple ways that can be argued not to be self-defeating. With the exception of Axioms 6.1 and 6.8, we did not just form a set of all objects having a given formal property. Instead, we formed a set of all objects that were explicitly *built up from preexisting objects* in certain ways. For Axioms 6.2 – 6.7, we built up sets from elements that were obtained by the disassembly (e.g., repeated unioning) of preexisting sets. Even when we used elements drawn from the universal set, we built things up from these elements indiscriminately without picking out the elements that have some specific property. (Recall the kind of language we used: "all the ordered triples $\langle x, y, z\rangle$, where z is *any* set in the universal set" [emphasis added]). For Axiom 6.9, we began with all sets rather than with a particular preexisting set, but we did not screen the sets we used to see whether they had some particular property. We just took arbitrary sets x, found their elements, and then built up other sets from the resulting grab bag of objects. The arguments for Axioms 6.1 and 6.8 were different from the rest; they involved the use of comprehension properties in a nontrivial fashion. For Axiom 6.1, we argued that we can build the set of all objects with the property of not being in $u \cap v$. For Axiom 6.8, we argued that there is a set of all x such that x is a singleton. However, in both cases, we provided subsidiary arguments for the claim that the properties used were not self-defeating. Both of the properties used were of a simple, non-self-referential sort that would not give rise to self-defeating acts of set formation.

7 Motivating a Set Theory: A Second Argument

In Section 6, I tried to answer the question, What sort of set theory can the minimally iterative conception of set motivate? In this section, I will try to answer this question again but by an entirely different route. Interestingly, this new method will give exactly the same result as the method presented in the last section.

The minimally iterative conception of set allows us to assume that there is a set of all objects of a given kind, provided that no self-defeating properties are used in the creation of that set. Therefore, we should be able to create a set theory compatible with the minimally iterative conception by adopting the following system of axioms:

1. the axiom of extensionality,
2. a restricted axiom schema of comprehension: For every property P, there is a set of all objects having P, provided that P is not a self-defeating property and the creation of the set of all objects having P can be carried out without the use of any self-defeating properties.

Unfortunately, it appears that we cannot translate the restriction on P in (2) into the language of first-order set theory. Thus, we cannot use axiom (2) directly as an axiom schema of a first-order set theory. Instead, we must try to devise a restriction on P which is formalizable within first-order set theory, and which rules out, not only sets determined by self-defeating properties P, but also sets whose construction involves, in any way, the use of self-defeating properties for the selection of sets to gather together.

In this section, we will attempt to find a restriction of this kind. Instead of seeking a direct restriction on comprehension properties P, we will search for a suitable syntactical restriction on the comprehension formulas which express the properties P. That is, we will try to find a restriction on the form of the well-formed formula $F(x)$ such that, if $F(x)$ obeys this restriction, then the creation of the set $\{x|F(x)\}$ does not involve self-defeating properties in any way. As we have seen, the minimally iterative conception of set allows a set of all objects having P provided that the creation of this set does not involve any use of self-defeating properties. Therefore, the minimally iterative conception will allow the existence of $S = \{x|F(x)\}$ if $F(x)$ obeys a syntactical restriction of the type just mentioned. If we can find a restriction of this sort (call it **R**), then the minimally iterative conception of set will support the following axioms of set theory:

1. the axiom of extensionality,
2. a restricted axiom schema of comprehension: If $F(x)$ obeys **R**, then there is a set of all x such that $F(x)$.

For the sake of definiteness, we adopt the following conventions about the language of first-order set theory. Throughout this section, the language of first-order set theory is taken to be $\{\in, =\}$. However, our conclusions easily can be adapted to first-order set theory in the language $\{\in\}$ with equality defined.[4] $F(x)$ is a schematic wff of first-order set theory in which the variable x occurs free. The letters 't' through 'z' are variables; 'q' through 's' are metavariables. In discussing wffs I will resort to use-mention confusion where this seems to do no harm. With these preliminaries out of the way, we can work on formulating the syntactical restriction that we need.

There are many wffs $F(x)$ which do not specify or even suggest self-defeating properties. Some examples of these have been given earlier in this paper. However, there is one family of wffs $F(x)$ which could well lead to self-defeating comprehension. These are the wffs $F(x)$ which state that there is a relationship of *identity* between the object x and some object which is stated to be a *nonmember* of x. Wffs of this sort create difficulties for the following reason. If we select an arbitrary wff $F(x)$ of this kind, then we cannot, in the general case, rule out the possibility that $F(S)$ is true. If $F(S)$ is true, then S is identical to a nonmember of S. Hence S is an object which satisfies $F(x)$ but is not a member of S. Then the act of forming the set S cannot be carried out; any attempt to form the set S will be self-defeating.

The clearest example of a wff of this sort is $(\exists y)[y = x \wedge y \notin x]$. This is the most direct formal translation of "x is identical to some nonmember of x." Of course, one can say the same thing more concisely with the Russell's paradox wff, $x \notin x$. Both of these wffs are ruled out as comprehension conditions by the minimally iterative conception of set—not because these wffs are known in hindsight to give rise to paradoxes, but because these wffs express self-defeating properties. However, there are many other similar wffs that present the same danger of self-defeating comprehension. To be sure that no hijinks of this sort occur, we should attempt to rule out all instances of $F(x)$ which are formal equivalents of the statement that x is identical to some nonmember of x. Hence, in addition to $x \notin x$ and $(\exists y)[y = x \wedge y \notin x]$, we must ban all possible formal paraphrases of these wffs. Such paraphrases can be complicated and numerous, for the following reason. In first-order language, there are many different equivalents of "x is identical to y."[5] Some of these equivalents, such as $(\forall z)[z \in x \leftrightarrow z \in y]$ and $(\forall z)[x \in z \leftrightarrow y \in z]$, can be shown to be equivalent to $x = y$ in the presence of some plausible set theoretic axioms (see [6], pp. 26–29). These paraphrases of equality can be manipulated logically into forms which do not look like equality; one example of such a form is $(\forall z)(\exists w)[[w = z \wedge w \in x \wedge z \in y] \vee [w = z \wedge \neg[z \in x \vee w \in y]]]$. If $G(x, y)$ and $H(x, y)$ are two such paraphrases of $x = y$, then a wff like $(\exists z)(\exists w)[G(x, z) \wedge H(x, w) \wedge z \notin w]$ can be used as a paraphrase of $x \notin x$. A wff like this can be scrambled further through the use of more logic steps and extra variables. For example, one can change the wff just mentioned into

$$\neg(\forall z)(\forall w)[\neg[\neg G(x, z) \vee \neg H(x, w)] \rightarrow (\forall t)[G(w, t) \rightarrow z \in t]].$$

In this way, one can create paraphrases of $x \notin x$ that do not look like paraphrases of $x \notin x$ at all. If we want to ban all instances of $F(x)$ that might lead to self-defeating comprehension, then we must ban all of these paraphrases of $(\exists y)[y = x \wedge y \notin x]$.

To make things worse, the ban on equivalents of $x \notin x$ is not sufficient. We also must ban any wff $F(x)$ such that logical manipulations of $F(x)$ can lead to a wff which is an equivalent of $x \notin x$. An example of such a wff is the formal version of "x has no member which is a member of a member of x." Although this wff is not an equivalent of $x \notin x$, it implies $x \notin x$. Wffs of this kind may get us into trouble when we begin to manipulate and combine sets; we can easily pass from sets defined by wffs of this kind to sets defined by self-defeating properties. To avoid wffs like this, we should avoid any wff $F(x)$ which states that there is a relationship of identity between x and any object from which x is built up (a member of x, a member of a member of x, etc.).

There is a straightforward way to avoid all of these hazardous wffs, along with any other wff that might reasonably be expected to imply them. I will outline this way in the next few paragraphs. The method of classifying sets that I am about to describe bears more than a passing resemblance to some ideas put forth by Holmes ([11]; the ideas I will discuss are on pp. 37–45). I will cite Holmes's ideas in more detail later.

First, we note that for each set x, the universe contains a number of different kinds or "levels" of objects *relative* to x. For example, there are members of x, members of members of x, members of members of members of x, and so forth. These can be regarded as levels of sets "below" x. There also are levels above x: the sets to which x belongs as a member, the sets which have as members the sets to which x belongs, and so forth. This classification of objects by *relative* level should not be confused in any way with the system of absolute levels used in the iterative conception of set or even in the minimally iterative conception of set. The levels relative to a given set x need not be either mutually disjoint or mutually exhaustive of the universe.

It is intuitively clear that wffs of set theory can be thought of as making statements about sets and their members. In particular, some wffs $F(x)$ can be thought of as making statements about objects of specific levels relative to x. For example, $(\exists y) y \in x$ says something about the members of the set x, whereas $(\exists y)(\exists z)[z \in y \wedge y \in x]$ says something about the members of the members of x. This notion of a wff "saying something" about objects of a particular relative level can be made more precise as follows.

First, note that a wff can be read as a statement about objects of a particular level only if that wff contains variables positioned in certain specific ways. For example, $F(x)$ may say something about the members of x only if $F(x)$ contains a subwff of the form $y \in x$. The presence of this subwff does not guarantee that $F(x)$ conveys any information about members of x; it could turn out that $F(x)$ says something trivial about the members of x, and therefore really says nothing about them. (For example, $F(x)$ could be $(\forall y)[y \in x \leftrightarrow y \in x]$. Also, it could turn out that the instance of x in the subwff is bound in $F(x)$.) But if $F(x)$ does contain the subwff $y \in x$, then $F(x)$ is at least a candidate for saying something about the members of x. If we know nothing else about $F(x)$ besides the fact that it contains a subwff $y \in x$, then for all we know, $F(x)$ *might* be the formalized equivalent of a natural-language sentence about the members of x.

Similarly, if $F(x)$ has a subwff $x \in y$, then $F(x)$ is a candidate for making a statement about the sets to which x belongs. If $F(x)$ has subwffs $z \in y$ and $y \in x$, then $F(x)$ is a candidate for making a statement about the members of members of x, and so forth.

The possibility of classifying variables in terms of which objects they are able to describe is very much like a distinction drawn by Holmes ([11], pp. 37–45). Holmes has distinguished different "roles" which an object may play in a set theoretic property; these roles include those of element, set, set of sets, and so forth. The argument I am about to give next is formally close to the informal part of Holmes's argument ([11], pp. 37–38); my main conclusion also will be the same as the main conclusion of Holmes's argument. The substantive difference between my argument and Holmes's lies in the philosophical motivation: my argument begins with a modified version of the iterative conception of set, while Holmes's argument begins from a very different conception of set, which I will mention briefly later.

Intuitively, we can think of each variable in $F(x)$ as having a certain expressive power—or, as we might call it, an *expression class*. For example, the occurrence of y in $(\forall y) y \in x$ makes that wff a candidate for saying something about the members of x. We can say that y has as its expression class the members of x. This is just another way of saying that, because of the presence of the variable y in the wff, the wff is a candidate for being a statement about the members of x. (It emphatically does *not* mean that y is restricted to members of x.) A variable in $F(x)$ also may have as an expression class the members of members of x, or the sets to which x belongs, or the like.

To avoid further long-windedness, we will index the expression class of a variable with a number. First, number the levels of objects relative to a given set a, so that the members of a are at level -1 relative to a; the members of members of a are at level -2 relative to a; and so forth. Similarly, stipulate that the sets to which a belongs are at level 1 relative to a; the sets to which these sets belong are at level 2 relative to a; and so forth. The set a itself is at level 0 relative to a. Using this numbering, we can say that the y in the wff $(\exists y)y \in x$ has expression class -1 relative to x; that the y in the wff $(\exists y)x \in y$ has expression class 1 relative to x; and so forth. Note that the expression class of one variable relative to another variable in a wff need not be unique. A variable can have more than one expression class relative to x (like the z in $(\exists y)(\exists z)[z \in y \wedge z \in x \wedge y \in x]$) or no expression class at all relative to x (like the z in $(\exists y)[y \in x \wedge (\exists z)z = z]$). It is important to note that the expression class of a variable does not determine the range of a variable; all variables still range over all sets. A variable's expression class only tells us what role that variable may play in the expression of propositions or facts about sets.

Using this numbering system, we can define rigorously the expression class of a variable in a wff relative to another variable in that wff.

Definition 7.1 Let r and s be variables that occur in a wff F. Let n be a natural number. Say that n is an expression number of r relative to s in F (notation: $nEn(r, s, F)$) if it can be shown to be so by application of the following rules:

1. If r is the same variable as s, then $0En(r, s, F)$.
2. If $nEn(r, s, F)$, and there is a variable q such that $q \in r$ is a subwff of F, then $(n-1)En(q, s, F)$.
3. If $nEn(r, s, F)$, and there is a variable q such that $r \in q$ is a subwff of F, then $(n+1)En(q, s, F)$.
4. If $nEn(r, s, F)$, and there is a variable q such that $s \in q$ is a subwff of F, then $(n-1)En(r, q, F)$.
5. If $nEn(r, s, F)$, and there is a variable q such that $q \in s$ is a subwff of F, then $(n+1)En(r, q, F)$.
6. If $nEn(r, s, F)$, and there is a variable q such that either $r = q$ or $q = r$ is a subwff of F, then $nEn(q, s, F)$.
7. If $nEn(r, s, F)$, and there is a variable q such that either $s = q$ or $q = s$ is a subwff of F, then $nEn(r, q, F)$.

Now let us return to the problem of deciding which wffs can safely be used as comprehension conditions for sets. From our earlier discussion, we conclude that we must exclude any wff $F(x)$ that might possibly express identity (or nonidentity) between sets of different levels relative to the set x. To do this, it suffices to exclude all wffs which make statements equating sets of different levels relative to x. And to do that, it is sufficient to exclude all wffs in which any variable has more than one expression number relative to any other.

Fortunately, there is a known family of wffs that satisfies this condition. This is the family of *stratified* wffs: that is, wffs for which a consistent assignment of simple type indices to variables can be made.[6] If $F(x)$ is stratified, then we can give the variables in $F(x)$ an assignment of type indices, with no variable receiving more than one index. In a stratified wff, each variable will have at most one expression number relative to any other variable. To prove this, pick an arbitrary variable s in a stratified wff $F(x)$ and assign expression numbers relative to s to as many variables as possible in $F(x)$. If any variable gets more than one expression number, then it also would get more than one type index if we assigned type indices starting with 0 at s.

It is important to remember that our restriction on comprehension conditions $F(x)$ arises entirely from the restriction against the use of self-defeating properties to build sets. To underscore this point, I will briefly retrace the route by which we got the restriction on $F(x)$. First, we saw that the minimally iterative conception of set bans sets defined by self-defeating properties. Second, we asked how we must restrict the comprehension schema of first-order set theory to rule out all sets defined by self-defeating properties. Third, we decided that to do this, we must ban all comprehension conditions $F(x)$ which state that x is identical to a nonmember of itself or which can be twisted around logically so as to imply the same. Fourth, we used a rather long-winded argument to show that we can ban all such wffs by banning all nonstratified wffs. Thus, if we restrict $F(x)$ to stratified wffs only, then we will not be allowing any comprehension conditions which foreseeably can give rise to self-defeating comprehension.

This conclusion lets us give a partial and tentative answer to the question, What sort of set theory does the minimally iterative conception of set motivate? The set theory which has as axioms all comprehension axioms with stratified comprehension conditions, plus the axiom of extensionality, is NF.

Earlier I mentioned a classification of variables, due to Holmes, which is essentially the same as my classification of variables according to expression number. The argument in which Holmes uses this classification is, in fact, an argument for the intuitive motivation of NF or NFU. Holmes proposes a rationale for restricting each variable in a comprehension condition to only one of the "roles" of element, set, set of sets, and so forth ([11], pp. 37–38). Holmes uses this rationale to argue for the plausibility of stratified comprehension. From a formal standpoint, my expression classes amount to a rigorized version of what Holmes calls "roles." However, the intuitive motivation which I have given is substantively different from the one which Holmes uses. In my argument, the assignment of variables to unique roles arises *from the iterative conception of set*—albeit from a modified version thereof. Holmes justifies this assignment by way of an argument likening sets to abstract data types. Thus, my intuitive motivation for the assignment is, in a sense, closer to commonly held ideas about set theory. However, Holmes's motivation for distinguishing "roles" is of independent interest and importance and does not conflict with the motivation which I have suggested for NF.

One even could argue that the minimally iterative conception of set motivates a proper extension of NF rather than NF itself. It appears that every set-building act allowed by the standard iterative conception of set also is allowed by the minimally iterative conception. Thus, one could argue that the minimally iterative conception motivates an extension of NF in which the wellfounded sets satisfy a theory motivated by the standard iterative conception—namely, ZF. The consistency of such an extension is unknown; at the last report of which I am aware, the consistency of "there is an infinite wellfounded set" with NF remained an open question ([5], p. 42).[7] But in any case, this liberal extension of my argument for the motivation of NF is much more speculative than the argument itself.

8 Concluding Remarks

In this paper, we have examined a way of modifying the iterative conception of set to accommodate non-wellfounded sets. The outcome of this study was twofold. First, we found that the iterative conception of set is more flexible than we might have thought. By altering that conception slightly, we can arrive at a picture of the set theoretic universe which accommodates certain non-wellfounded sets without apparent paradoxes. Second, we found that this new picture of the universe provides some support for the axioms of NF.

It often is said that NF lacks a clear intuitive motivation—that there is no intuitive concept of the set theoretic universe which makes the axioms of NF seem plausible, in the same way that the concept of the iterative hierarchy makes the axioms of ZF seem plausible (see, for example, [6], p. 164). If the arguments presented in this paper are correct, then such an interpretation may not be as far away as it seems. Our results, preliminary as they are, may or may not lead to a definitive solution to the problem of the intuitive motivation of NF. However, they do show that the minimally iterative conception of set provides at least some intuitive support for that intriguing theory.

Notes

1. For various views on the adequacy of this support, see [2], 228–30; [3]; and [9], p. xiii and Chapter 6.
2. See [2], pp. 219–20, where it is argued that set-building of essentially this sort is impossible.
3. This name is not meant to suggest a contrast with what Wang has called "[t]he (maximum) iterative concept" (Wang [15], p. 183; see also p. 181).
4. See [6], pp. 25–30, for background on the treatment of equality in set theory.
5. See [6], pp. 25–30, for mention or discussion of some of these.
6. Stratification is defined on p. 78 of the paper in which Quine introduced NF: [13]. A more up-to-date reference is [5], p. 7.
7. See [5], pp. 39–43, for a discussion of work in this general area. For other relevant or related work see Hinnion [10] and Prati [12].

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Department of Chemistry and Biochemistry (for correspondence)
and Institute of Geophysics and Planetary Physics
University of California, Los Angeles
405 Hilgard Avenue
Los Angeles CA 90095
msharlow@usermail.com