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Reverse Mathematics and Completeness Theorems for Intuitionistic Logic

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Abstract In this paper, we investigate the logical strength of completeness theorems for intuitionistic logic along the program of reverse mathematics. Among others we show that ACA_0 is equivalent over RCA_0 to the strong completeness theorem for intuitionistic logic: any countable theory of intuitionistic predicate logic can be characterized by a single Kripke model.

1 Introduction

In this paper, we investigate the logical strength of completeness theorems for intuitionistic logic along the program of reverse mathematics. Several kinds of models have been invented for intuitionistic logic, for example, lambda calculus models, Kripke-Beth models, topological models, and so on. We here treat only Kripke models of intuitionistic logic. The completeness theorem of classical logic asserts that if Γ is consistent then Γ has a model. On the other hand, the (strong) completeness theorem for intuitionistic logic asserts that any countable theory in intuitionistic predicate logic can be characterized by a single Kripke model. The standard proof can be regarded as a generalization of Henkin construction for classical logic where the maximal filters of classical Lindenbaum Boolean algebras are replaced by presheaves of prime filters of intuitionistic Lindenbaum distributive lattices (Troelstra and van Dalen [4]). We show that ACA_0 is equivalent over RCA_0 to the strong completeness theorem for intuitionistic logic. The proof of the strong completeness theorem for intuitionistic logic in ACA_0 is essentially due to Ishihara et al. [2] and Gabbay [1] which are pioneer works on recursive model theory for intuitionistic logic.

The following definitions are made in RCA_0. A language $\mathcal{L}$ consists of countably many relation symbols and constant symbols but no function symbols. Logical symbols are given as usual. We use $\perp$ (the falsity) to define the negation $\neg\varphi$ as

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$\varphi \to \bot$ [4]. We identify terms and formulas with their Gödel numbers under a fixed primitive recursive coding. Then let $\mathrm{Form}_{\mathcal{L}}$ and $\mathrm{Snt}_{\mathcal{L}}$ be the sets of $\mathcal{L}$-formulas and $\mathcal{L}$-sentences, respectively. A set of sentences is often called a *theory*. By $\Gamma \vdash_i \varphi$, we mean that φ is intuitionistically deducible from Γ. Let Γ and Δ be two theories. The pair (Γ, Δ) is *consistent* if there are no finite sets $\Gamma_0 \subseteq \Gamma$ and $\Delta_0 \subseteq \Delta$ such that $\vdash_i \wedge\Gamma_0 \to \vee\Delta_0$. Here, we set $\wedge\varnothing \equiv \top$, $\vee\varnothing \equiv \bot$. Γ is *consistent* if $(\Gamma, \varnothing)$ is consistent.

2 *C*-saturated Theory

In this section, we show that WKL_0 is equivalent over RCA_0 to a version of the saturation lemma for intuitionistic logic.

Definition 2.1 The following definition is made in RCA_0. Let C be a set of constants. A theory Γ is *C-saturated* if it satisfies the following conditions:

1. $\Gamma \nvdash_i \bot$,
2. if $\Gamma \vdash_i \sigma$ then $\sigma \in \Gamma$,
3. if $\sigma \vee \tau \in \Gamma$ then $\sigma \in \Gamma$ or $\tau \in \Gamma$,
4. if $\exists x\varphi(x) \in \Gamma$ then $\exists c \in C(\varphi(c) \in \Gamma)$,

where the formula $\varphi(x)$ has no other free variable than x.

Lemma 2.2 (Saturation Lemma) *The following is provable in* WKL_0. *Suppose that a sentence σ_0 is not intuitionistically deducible from a theory Γ. Let C be an infinite set of constants not in the language $\mathcal{L}$ of Γ. Then there is a C-saturated set Γ' of sentences in the language $\mathcal{L}(C)$ such that $\Gamma \subseteq \Gamma'$ and $\Gamma' \nvdash_i \sigma_0$.*

Proof Let Γ be a theory such that $\Gamma \nvdash_i \sigma_0$. Form a tree $T_\Gamma \subseteq 2^{<\mathbb{N}}$ by putting $t \in T_\Gamma$ if and only if $\forall\sigma < \mathrm{lh}(t)(t(\sigma) = 1 \to \sigma \in \mathrm{Snt}_{\mathcal{L}(C)})$ and $\forall\sigma < \mathrm{lh}(t)(\sigma \in \Gamma \to t(\sigma) = 1)$ and $\forall p < \mathrm{lh}(t)\big((p$ is an intuitionistic proof $\wedge \forall i < \mathrm{lh}(p)(p(i)$ is a nonlogical axiom of $p \to t(p(i)) = 1)) \to \forall i < \mathrm{lh}(p)(p(i) \in \mathrm{Snt}_{\mathcal{L}(C)} \to t(p(i)) = 1)\big)$ and $\forall\sigma < \mathrm{lh}(t)\forall\tau < \mathrm{lh}(t)(t(\sigma \vee \tau) = 1 \to t(\sigma) = 1 \vee t(\tau) = 1)$ and $\forall n < \mathrm{lh}(t)\forall m < \mathrm{lh}(t)((n = \ulcorner\exists x\varphi(x)\urcorner, m = \ulcorner\varphi(c_\varphi)\urcorner$ and $t(n) = 1) \to t(m) = 1)$ and $\sigma_0 < \mathrm{lh}(t) \to t(\sigma_0) = 0$, where $c_\varphi \in C$ is the Henkin-constant of an $\mathcal{L}(C)$-formula $\varphi(x)$.

T_Γ exists by Δ^0_1 comprehension. Clearly T_Γ is an infinite 0-1 tree since $T_\Gamma \nvdash_i \sigma_0$. By weak König's lemma, T_Γ has a path P. Let $\Gamma' = \{x \in \mathrm{Snt}_{\mathcal{L}(C)} : P(x) = 1\}$. By the construction of T_Γ, Γ' is a C-saturation of Γ. ☐

Theorem 2.3 *The following assertions are pairwise equivalent over* RCA_0*:*

1. WKL_0,
2. *the saturation lemma for intuitionistic predicate logic,*
3. *the saturation lemma for intuitionistic propositional logic with countably many atoms.*

Proof Lemma 2.2 gives the implication (1) → (2). The implication (2) → (3) is straightforward. It remains to prove (3) → (1).

Now consider intuitionistic propositional logic with countably many atomic formulas $\langle a_n : n \in \mathbb{N}\rangle$. A set Δ of formulas is *saturated* if and only if Δ satisfies the conditions of C-saturation except (4).

The saturation lemma for intuitionistic propositional logic asserts that if $\Gamma \nvdash_i \sigma$ then there exists a saturated set Γ' such that $\Gamma \subseteq \Gamma'$ and $\Gamma' \nvdash_i \sigma$. We want to prove weak König's lemma from the saturation lemma. Let $T \subseteq 2^{<\mathbb{N}}$ be an infinite tree. For each $n \in \mathbb{N}$, form a propositional formula

$$\sigma_n \equiv \bigvee\{\bigwedge\{a_i^{s(i)} : i < n\} : s \in T, \mathrm{lh}(s) = n\}$$

where $a_i^1 = a_i, a_i^0 = \neg a_i$. Let $\Gamma = \{\sigma_n : n \in \mathbb{N}\}$. Since T contains a sequence of length n for each n, Γ is classically consistent, especially, intuitionistically consistent. From the saturation lemma, it follows that Γ has a saturation Γ'. Let $P(n) = 1$ if $a_n \in \Gamma'$; $P(n) = 0$ if $\neg a_n \in \Gamma'$. Then P is a path through T. This completes the proof of (3) $\rightarrow$ (1). □

Note that the proofs of Lemma 2.2 and Theorem 2.3 are almost identical to the proofs of the results in Section 4.3 in Simpson [3] where the completeness theorem of classical logic is discussed.

From the saturation lemma, we can show within WKL_0 the following version of the strong completeness theorem for intuitionistic logic in the usual way which is called the canonical model construction ([4], Definition 6.4 and Theorem 6.6). Let Γ be a theory such that $\Gamma \nvdash_i \bot$. Let $\widehat{K}$ be the class of theories Δ such that $\Gamma \subseteq \Delta$ and Δ is C_Δ-saturated where C_Δ is the set of all constant symbols of Δ. By $\Delta \Vdash \sigma$, we mean $\sigma \in \Delta$. Let D be a functional from $\widehat{K}$ such that $D(\Delta) = C_\Delta$. Then $(\widehat{K}, \subseteq, D, \Vdash)$ satisfies the conditions of a Kripke model of Γ and $\forall\sigma\big(\Gamma \nvdash_i \sigma \rightarrow \exists\Delta(\Delta \in \widehat{K} \wedge \Delta \nVdash \sigma)\big)$. This statement is indeed equivalent over RCA_0 to WKL_0 since it obviously implies the saturation lemma. Note that the symbols $\widehat{K}$ and D are informally used here. In the next section, we consider another version of the completeness theorem in which Kripke models are defined as sets in the language of second-order arithmetic.

3 The Strong Completeness Theorem

In this section, we first define Kripke models and show that ACA_0 is equivalent over RCA_0 to the strong completeness theorem for Intuitionistic Logic.

Definition 3.1 The following definition is made in RCA_0. Let $K(\subseteq \mathbb{N})$ be a nonempty set of possible worlds and $\leq_K$ a partial order on K. Let D be a function assigning a domain to each world of K. Let $\Vdash$ be a binary relation on $K \times \mathrm{Snt}$ where Snt is the set of sentences in the language extended with constant symbols for each element of $\bigcup_{k\in K} D(k)$. Then $\mathcal{K} = (K, \leq_K, D, \Vdash)$ is a (code for a) *Kripke model* if $\mathcal{K}$ obeys the familiar conditions: for any $k, k' \in K$,

1. if $k \leq_K k'$ then $D(k) \subseteq D(k')$;
2. if $k \Vdash \sigma$ then σ is a sentence in $\mathcal{L} \cup D(k)$;
3. if $k \leq_K k'$ and $k \Vdash \sigma$ then $k' \Vdash \sigma$;
4. $k \nVdash \bot$;
5. $k \Vdash \sigma \wedge \tau$ iff $k \Vdash \sigma \wedge k \Vdash \tau$;
6. $k \Vdash \sigma \vee \tau$ iff $k \Vdash \sigma \vee k \Vdash \tau$;
7. $k \Vdash \sigma \rightarrow \tau$ iff $\forall k'' \in K(k \leq_K k'' \rightarrow (k'' \Vdash \sigma \rightarrow k'' \Vdash \tau))$;
8. $k \Vdash \exists x\varphi(x)$ iff $k \Vdash \varphi(c)$ for some $c \in D(k)$;
9. $k \Vdash \forall x\varphi(x)$ iff $\forall k'' \geq_K k \forall c \in D(k'')(k'' \Vdash \varphi(c))$.

Definition 3.2 The following definition is made in RCA_0. Let σ_0 be an $\mathcal{L}$-sentence. A theory Γ of $\mathcal{L}$ is *$\mathcal{L}$-maximal with respect to σ_0* if it satisfies the following conditions:

1. $\Gamma \nvdash_i \sigma_0$;
2. if $\Gamma \vdash_i \sigma$ then $\sigma \in \Gamma$;
3. if $\sigma \vee \tau \in \Gamma$ then $\sigma \in \Gamma \vee \tau \in \Gamma$;
4. if $\exists x \varphi(x) \in \Gamma$ then $\varphi(c) \in \Gamma$ for some constant c in $\mathcal{L}$;
5. if $\Gamma \cup \{\sigma\} \nvdash_i \sigma_0$ then $\sigma \in \Gamma$.

Lemma 3.3 *The following is provable in* ACA_0. *Suppose that a sentence σ_0 is not intuitionistically deducible from a theory Γ. Let C be an infinite set of constants not in L. Then there is an $\mathcal{L}(C)$-maximal Γ' with respect to σ_0 such that $\Gamma \subseteq \Gamma'$.*

Proof Let $\langle \tau_n : n \in \mathbb{N} \rangle$ be a one-to-one enumeration of all the sentences in $L \cup C$. We may assume that $\sigma \vee \tau$, $\sigma \to \tau$, $\varphi(c_\varphi)$, and so on appear after σ, τ and $\exists x \varphi(x)$ in the enumeration, where c_φ is the Henkin constant of $\varphi(x)$.

By arithmetical comprehension, there exists a set $\Gamma^* = \{\sigma : \Gamma \vdash_i \sigma\}$. Define a function $f : \mathbb{N} \to \{0, 1\}$ by primitive recursion as follows:

$$f(n) = \begin{cases} 1 & \text{if } \Gamma \cup \{\tau_l : f(l) = 1 \wedge l < n\} \cup \{\tau_n\} \nvdash_i \sigma_0 \\ 0 & \text{otherwise.} \end{cases}$$

Let $\Gamma' = \{\tau_n : f(n) = 1\}$. We prove that Γ' is an $L \cup C$-maximal theory with respect to σ_0. We need to prove all the conditions of Definition 3.2. It is obvious that Γ' satisfies the conditions (1), (2), and (5) of Definition 3.2 and $\Gamma \subseteq \Gamma'$.

Suppose that $\sigma \notin \Gamma'$ and $\tau \notin \Gamma'$. Since $\sigma \vee \tau$ is enumerated after σ and τ, $\Gamma \cup \{\tau_l : f(l) = 1 \wedge l < n\} \cup \{\sigma\} \vdash_i \sigma_0$ and $\Gamma \cup \{\tau_l : f(l) = 1 \wedge l < n\} \cup \{\tau\} \vdash_i \sigma_0$. Then $\Gamma \cup \{\tau_l : f(l) = 1 \wedge l < n\} \cup \{\sigma \vee \tau\} \vdash_i \sigma_0$, that is, $\sigma \vee \tau \notin \Gamma'$. This implies that Γ' satisfies condition (3) of Definition 3.2.

Finally, we show that Γ' satisfies condition (4) of Definition 3.2. Suppose that $\exists x \varphi(x) \in \Gamma'$ and $\varphi(c_\varphi) = \tau_n$. Since $\exists x \varphi(x)$ is enumerated before $\varphi(c_\varphi)$, we have $\exists x \varphi(x) \in \{\tau_l : f(l) = 1 \wedge l < n\}$. Then $\Gamma \cup \{\tau_l : f(l) = 1 \wedge l < n\} \cup \{\varphi(c_\varphi)\} \nvdash_i \sigma_0$, that is, $\varphi(c_\varphi) \in \Gamma'$. $\square$

Remark 3.4 In the proof of Lemma 3.3, arithmetical comprehension axioms are sufficient to show the existence of $\Gamma^* = \{\sigma : \Gamma \vdash_i \sigma\}$. Fix any $\mathcal{L}$-theory Γ and any set C of new constants. Then, in ACA_0, we can prove that there exists an $\mathcal{L}(C)$-maximal extension Γ' of Γ with respect to σ_0 which is $\Gamma^* \oplus C$-recursive. We identify a $\Gamma^* \oplus C$-recursive set Δ with its $\Gamma^* \oplus C$-recursive index i_Δ. Then we can show the following generalization of Lemma 3.3.

Lemma 3.5 *The following is provable in* ACA_0. *Suppose that an $\mathcal{L}$-sentence σ_0 is not intuitionistically deducible from an $\mathcal{L}$-theory Γ. Let $\langle C_n : n \in \mathbb{N} \rangle$ be an infinite sequence of pairwise disjoint infinite sets of constants not in $\mathcal{L}$. We put $\mathcal{L}_0 = \mathcal{L}$, $\mathcal{L}_{n+1} = \mathcal{L}_n \cup C_n$, and $C = \bigcup_{n \in N} C_n$. Then there is a partial function Φ from the set of $\Gamma^* \oplus C$-recursive theories to itself which satisfies the following: if a $\Gamma^* \oplus C$-recursive theory Δ in $\mathcal{L}_n$ is closed under intuitionistic deduction and an $\mathcal{L}_n$-sentence ψ is not intuitionistically deducible from Δ, then $\Phi(n, \psi, \Delta)$ is an $\mathcal{L}_{n+1}$-maximal extension of Δ with respect to ψ.*

Lemma 3.6 (Strong Completeness Theorem) *The following is provable in* ACA_0. *Let Γ be an intuitionistically consistent theory. Then there exists a Kripke model $(K, \leq_K, \Vdash)$ such that $\forall \sigma (\Gamma \vdash_i \sigma \leftrightarrow \forall k \in K\ k \Vdash \sigma)$.*

Proof Fix any intuitionistically consistent theory Γ. By Lemma 3.5, we have $\langle F_n : n \in \mathbb{N}\rangle$ such that

$$F_0 = \{\Phi(0, \psi, \Delta) : \psi \text{ is an } \mathcal{L}_0\text{-sentence not deducible from } \Delta \text{ and } \Delta \text{ is a finite extension of } \Gamma\}$$

and

$$F_n = \{\Phi(n, \psi, \Delta) : \psi \text{ is an } \mathcal{L}_n\text{-sentence not deducible from } \Delta \text{ and } \Delta \text{ is a finite extension of some theory in } F_n\}.$$

Let K be the set of (indices of) theories in $\bigcup_{n\in\mathbb{N}} F_n$. For each $\Delta, \Delta' \in K$, we have $\Delta \leq_K \Delta'$ if $\Delta \subseteq \Delta'$. If $\Delta \in F_n$, then $D(\Delta) =$ the set of all constants of $\mathcal{L}_{n+1}$. $\Delta \Vdash \sigma$ if $\sigma \in \Delta$. Then it is easy to check that $(K, \leq_K, D, \Vdash)$ is a Kripke model by the usual way. By the construction of K, we see that

$$\forall\sigma(\Gamma \vdash_i \sigma \leftrightarrow \forall k \in K\ k \Vdash \sigma).$$

This completes the proof of Lemma 3.6. □

Theorem 3.7 *The following assertions are pairwise equivalent over* RCA_0*:*

1. ACA_0*,*
2. *the strong completeness theorem for intuitionistic predicate logic,*
3. *the strong completeness theorem for intuitionistic propositional logic with countably many atoms.*

Proof Lemma 3.6 gives the implication (1) → (2). The implication (2) → (3) is straightforward. It remains to prove (3) → (1).

Now consider intuitionistic propositional logic with countably many atomic formulas $\langle a_n : n \in \mathbb{N}\rangle$. A triple $(K, \leq_K, \Vdash)$ is *Kripke model* if and only if it satisfies the conditions of Definition 3.1 except (1), (2), (8), and (9).

The strong completeness theorem for intuitionistic propositional logic asserts that if a set Γ of formulas is intuitionistically consistent then there exists a Kripke model $(K, \leq_K, \Vdash)$ such that $\forall\sigma(\Gamma \vdash_i \sigma \leftrightarrow \forall k \in K\ k \Vdash \sigma)$. It is enough to show Σ^0_1-comprehension from the strong completeness theorem by Lemma 3.1.3 in [3].

Let $\varphi(n)$ be a Σ^0_1 formula. Write $\varphi(n)$ as $\exists x\theta(x, n)$ where $\theta(x, n)$ is Σ^0_0. Let

$$\Gamma = \{a_n \vee a_n \vee \cdots \vee a_n : \exists m \leq \ulcorner a_n \vee a_n \vee \cdots \vee a_n \urcorner \theta(m, n)\}.$$

Γ exists by Δ^0_1-comprehension. Clearly, Γ is intuitionistically consistent. By the strong completeness theorem, there exists a Kripke model $(K, \leq_K, \Vdash)$ such that $\forall\sigma(\Gamma \vdash_i \sigma \leftrightarrow \forall k \in K\ k \Vdash \sigma)$. Then

$$\forall k \in K(k \Vdash a_n) \leftrightarrow \Gamma \vdash_i a_n \leftrightarrow a_n \vee a_n \vee \cdots \vee a_n \in \Gamma \leftrightarrow \varphi(n).$$

The left-hand side of this equivalence is Π^0_1. Hence by Δ^0_1-comprehension, we obtain $\exists X\forall n(n \in X \leftrightarrow \varphi(n))$. This completes the proof of (3) → (1). □

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