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KRASNOSEL'SKIĬ–SCHAEFER TYPE METHOD IN THE EXISTENCE PROBLEMS

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Abstract. We consider a general integral equation satisfying algebraic conditions in a Banach space. Using Krasnosel'skiĭ–Schaefer type method and technical assumptions, we prove an existence theorem producing a periodic solution of some nonlinear integral equation.

1. Introduction and preliminaries

Let $(\mathcal{B}, \|\cdot\|)$ be the Banach space of continuous Γ-periodic functions $\varphi\colon \mathbb{R}\to\mathbb{R}$ with $\Gamma > 0$ and the supremum norm. In this paper, we study the following integral equation

$$\varphi(t) = f(t, \varphi(t)) - \int_{t-\alpha}^{t} D(t,s) g(s, \varphi(s))\, ds, \tag{1.1}$$

where $\alpha > 0$, $f, g, D\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are continuous functions satisfying the following assumptions:

- (A1) $f(t+\Gamma, x) = f(t,x)$, $D(t+\Gamma, s+\Gamma) = D(t,s)$, $g(t+\Gamma, x) = g(t,x)$ for all $s,t,x \in \mathbb{R}$,
- (A2) $D(t, t-\alpha) = 0$, $D_{st}(t,s) \leq 0$, $D_s(t,s) \geq 0$ for all $t \in \mathbb{R}$ and $s \in (t-\alpha, t)$,
- (A3) the function D_{st} is continuous,

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(A4) there exists $\gamma > 0$ such that

$$|f(t,x) - f(t,y)| \leq \frac{|x-y|}{1+\gamma|x-y|} \quad \text{for all } t,x,y \in \mathbb{R},$$

(A5) $xg(t,x) \geq 0$ for all $t,x \in \mathbb{R}$,

(A6) for each positive K such that $K > |f(t,0)|$ for all $t \in \mathbb{R}$ there exist $P > 0$ and $\beta > 0$ such that

$$\frac{-2\gamma|x|}{1+\gamma|x|}\, xg(t,x) \leq (-\beta - 2K)|g(t,x)| + P \quad \text{for all } t,x \in \mathbb{R}.$$

Using the Krasnosel'skiĭ–Schaefer type method (see Schaefer [4]), we prove an existence theorem producing a periodic solution of problem (1.1). There is a wide literature on the application of this methodology to the integral equations theory. We mention here the important papers of Burton [1], Burton and Kirk [2], Liu and Li [3], Wardowski [7], which are strongly related to our findings. In our approach we derive new fixed point tools useful to deal with abstract existence problems.

Recently Wardowski [6] initiated a concept of a contraction-type mapping, characterized by the possibility of overcoming various situations (by hybridization with well-known contractive conditions in the literature, see Vetro and Vetro [5] and the references therein). Here, we recall the following version of the notion of a Wardowski's contraction mapping and related fixed-point result for further use.

Let (X, d) be a metric space. A mapping $T\colon X \to X$ is called a (τ, F)-*contraction* if there exist the functions $F\colon (0,\infty) \to \mathbb{R}$ and $\tau\colon (0,\infty) \to (0,\infty)$ satisfying:

(i) F is strictly increasing,

(ii) $\lim\limits_{t\to 0^+} F(t) = -\infty$,

(iii) $\liminf\limits_{s\to t^+} \tau(s) > 0$ for all $t \geq 0$,

(iv) $\tau(\mathrm{d}(x,y)) + F(\mathrm{d}(Tx,Ty)) \leq F(\mathrm{d}(x,y))$ for all $x,y \in X$ such that $Tx \neq Ty$.

Remark 1.1. Observe that (i), (iv) and the fact that $\tau > 0$ immediately imply that for all $x,y \in X$ $\mathrm{d}(Tx,Ty) \leq \mathrm{d}(x,y)$, and this gives the continuity of T.

Theorem 1.2 ([7, Theorem 2.1]). *Let (X, d) be a complete metric space and let $T\colon X \to X$ be a (τ, F)-contraction. Then T has a unique fixed point.*

Finally, we recall the following Schaefer's result in [4].

Theorem 1.3. *Let X be a normed space, $H\colon X \to X$ a continuous mapping, compact on each bounded subset of X. Then either the equation $x = Hx$ has a solution or the set of all solutions of the equation $x = \lambda Hx$, for $0 < \lambda < 1$, is unbounded.*

2. Auxiliary results

We start from our fixed point tool, that is a local version of Theorem 1.2.

Theorem 2.1. *Let (X, d) be a complete metric space and let $T\colon B(x_0, r) \to X$ be a (τ, F)-contraction defined on the open ball centered at $x_0 \in X$ with radius $r > 0$. If F is left-continuous, τ nonincreasing and the following condition holds*

$$F(r) - F(r - \mathrm{d}(x_0, Tx_0)) < \tau(r),$$

then T has a fixed point.

Proof. Since F is left-continuous, we can find $0 < \varepsilon < r$ such that

$$F(\varepsilon) - F\big(\varepsilon - \mathrm{d}(x_0, Tx_0)\big) < \tau(r).$$

Consider the closed ball $C = \overline{B}(x_0, \varepsilon)$. We show that $T(C) \subset C$. Take $x \in C$ and observe that if $\mathrm{d}(Tx, x_0) - \mathrm{d}(Tx_0, x_0) \le 0$ then we immediately get

$$\mathrm{d}(Tx, x_0) \le \mathrm{d}(Tx_0, x_0) \le \varepsilon.$$

In other case we have the inequalities

$$\begin{aligned} F(\mathrm{d}(Tx, x_0) - \mathrm{d}(Tx_0, x_0)) \le F(\mathrm{d}(Tx, Tx_0)) &\le F(\mathrm{d}(x, x_0)) - \tau(\mathrm{d}(x, x_0)) \\ &\le F(\varepsilon) - \tau(r) \le F(\varepsilon - \mathrm{d}(x_0, Tx_0)). \end{aligned}$$

Hence, we get $\mathrm{d}(Tx, x_0) \le \varepsilon$, which means that $Tx \in C$. Completeness of C and Theorem 1.2 end the proof. □

Theorem 2.2. *Let X be a Banach space, U an open subset of X and let $T\colon U \to X$ be a (τ, F)-contraction. If F is left-continuous and τ nonincreasing then the mapping $h(x) = x - T(x)$, $x \in U$ maps homeomorphically U onto $h(U)$.*

Proof. The continuity of h is easily visible due to Remark 1.1. So, consider $u, v \in X$, $u \neq v$ and suppose that $h(u) = h(v)$. Consequently we get $T(u) \neq T(v)$. Moreover, we have

$$\|h(u) - h(v)\| = \|u - T(u) - v + T(v)\| \ge \|u - v\| - \|T(u) - T(v)\|.$$

So, we obtain

$$F(\|u - v\|) \le F(\|T(u) - T(v)\|) \le F(\|u - v\|) - \tau(\|u - v\|),$$

which is impossible since τ is positive. Therefore h is a one-to-one mapping.

In order to show the continuity of h^{-1}, we verify that h is open. Let Q be an open subset of U and let $w \in h(Q)$. Take $v \in Q$ such that $w = h(v)$. There exists $r > 0$ satisfying $B(v, r) \subset Q$. Since F is left-continuous, there exists $0 < \varepsilon < r$ for which we get

$$F(r) - F(r - \eta) < \tau(r) \quad \text{for every } \eta \in [0, \varepsilon]. \tag{2.1}$$

Now, take $y \in B(w,\varepsilon)$ and define the mapping $S\colon B(v,r) \to X$ as follows

$$S(x) := y + T(x).$$

It is obvious that S is a (τ,F)-contraction. Moreover, observe that, by (2.1), we have

$$\begin{aligned} F(r) - F(r - \|v - S(v)\|) &= F(r) - F(r - \|y + T(v) - v\|) \\ &= F(r) - F(r - \|y - h(v)\|) < \tau(r). \end{aligned}$$

Hence, applying Theorem 2.1 to the mapping S, there exists $x \in B(v,r)$ such that $x = S(x) = y + T(x)$. In consequence $y = h(x) \in h(B(v,r))$, which implies $B(w,\varepsilon) \subset h(B(v,r)) \subset h(Q)$. Thus $h(Q)$ is open. $\square$

Now, we are interested in a special case of (τ,F)-contractive mapping, where $F(t) = -1/t$ for all $t \in (0,\infty)$. We will show that the contractive condition for such F, in the setting of Banach space X, gives the possibility to obtain the extensions of some known results and new applications in the theory of integro-differential equations. So, putting $F(t) = -1/t$ in (iv), we get

$$\|T(x) - T(y)\| \leq \frac{\|x-y\|}{1 + \tau(\|x-y\|)\|x-y\|}, \quad x,y \in X,\ x \neq y. \tag{2.2}$$

Now, we prove the following propositions.

Proposition 2.3. *For every $0 < \lambda < 1$, if T satisfies inequality* (2.2)*, then $\lambda T(\,\cdot\,/\lambda)$ is (η,F)-contraction, with*

$$\eta(t) = \frac{1}{\lambda}\,\tau\bigg(\frac{t}{\lambda}\bigg) \quad \textit{and} \quad F(t) = -\frac{1}{t} \quad \textit{for all } t \in (0,\infty).$$

Moreover, for every $x \in X \setminus \{0\}$, we have

$$\bigg\|\lambda T\bigg(\frac{x}{\lambda}\bigg)\bigg\| \leq \frac{\|x\|}{1 + \eta(\|x\|)\|x\|} + \|T(0)\|\,.$$

Proof. For $x,y \in X$, $x \neq y$ we have

$$\begin{aligned} \bigg\|\lambda T\bigg(\frac{x}{\lambda}\bigg) - \lambda T\bigg(\frac{y}{\lambda}\bigg)\bigg\| &\leq \lambda \frac{\Big\|\dfrac{x}{\lambda} - \dfrac{y}{\lambda}\Big\|}{1 + \tau\bigg(\Big\|\dfrac{x}{\lambda} - \dfrac{y}{\lambda}\Big\|\bigg)\Big\|\dfrac{x}{\lambda} - \dfrac{y}{\lambda}\Big\|} \\ &= \frac{\|x-y\|}{1 + \dfrac{1}{\lambda}\,\tau\bigg(\dfrac{\|x-y\|}{\lambda}\bigg)\|x-y\|} = \frac{\|x-y\|}{1 + \eta(\|x-y\|)\|x-y\|}. \end{aligned}$$

Next, for every $x \in X \setminus \{0\}$, using the above inequality, we obtain

$$\bigg\|\lambda T\bigg(\frac{x}{\lambda}\bigg)\bigg\| \leq \bigg\|\lambda T\bigg(\frac{x}{\lambda}\bigg) - \lambda T(0)\bigg\| + \lambda\|T(0)\| \leq \frac{\|x\|}{1 + \eta(\|x\|)\|x\|} + \|T(0)\|. \quad \square$$

Following the Burton's ideas, we state and prove next result.

Theorem 2.4. *Let X be a Banach space, $T_1\colon X \to X$ continuous and mapping bounded sets into compact sets, and let $T_2\colon X \to X$ satisfy* (2.2) *with constant $\tau > 0$. Then one of the following holds*:

(a) *the equation $x = \lambda T_1(x) + \lambda T_2(x/\lambda)$ has a solution for $\lambda = 1$;*
(b) *the set $\{x \in X : x = \lambda T_1(x) + \lambda T_2(x/\lambda)$ for some $0 < \lambda < 1\}$ is unbounded.*

Proof. We observe that, by Proposition 2.3 and the fact that $\tau\lambda^{-1} > \tau$, the mapping $X \ni x \mapsto \lambda T_2(x/\lambda)$ satisfies (2.2). Moreover, considering any $y \in X$ the mapping $X \ni x \mapsto \lambda T_2(x/\lambda) + \lambda T_1(y)$ also satisfies the condition (2.2). Therefore, by Theorem 1.2, there exists exactly one $x \in X$ satisfying

$$x = \lambda T_1(y) + \lambda T_2\left(\frac{x}{\lambda}\right).$$

Putting $h(x) = x - T_2(x)$ and applying Theorem 2.2, the above equation can be rewritten in the form

$$x = \lambda(h^{-1} \circ T_1)(y).$$

By Theorem 1.3 (the Schaefer's result) the above equation with $y = x$ has a solution for $\lambda = 1$ or the set $\{x \in X : x = \lambda(h^{-1} \circ T_1)(x)$ for some $0 < \lambda < 1\}$ is unbounded, which ends the proof. □

3. Periodic solution of integral equations

On account of Theorem 2.4, we prove the existence of a periodic solution for the integral equation (1.1). Precisely, we build our existence theorem on two key-lemmas.

Lemma 3.1. *The mapping $T\colon \mathcal{B} \to \mathcal{B}$ given by the formula*

$$(T\varphi)(t) = f(t, \varphi(t)),$$

with f fulfilling (A4)*, satisfies* (2.2) *with constant $\tau = \gamma$.*

Proof. We observe that for all $\zeta, \eta \in \mathcal{B}$ and any $t \in \mathbb{R}$ we have

$$|(T\zeta)(t) - (T\eta)(t)| = |f(t, \zeta(t)) - f(t, \eta(t))| \le \frac{|\zeta(t) - \eta(t)|}{1 + \gamma|\zeta(t) - \eta(t)|}.$$

The function $[0, \infty) \ni s \mapsto s/(1 + \gamma s)$ is increasing, therefore we obtain

$$\begin{aligned} \|T\zeta - T\eta\| &= \sup_{t\in[0,\Gamma]} |(T\zeta)(t) - (T\eta)(t)| \\ &= \sup_{t\in[0,\Gamma]} |f(t, \zeta(t)) - f(t, \eta(t))| \le \sup_{t\in[0,\Gamma]} \frac{|\zeta(t) - \eta(t)|}{1 + \gamma|\zeta(t) - \eta(t)|} \\ &= \frac{\sup_{t\in[0,\Gamma]} |\zeta(t) - \eta(t)|}{1 + \gamma \sup_{t\in[0,\Gamma]} |\zeta(t) - \eta(t)|} = \frac{\|\zeta - \eta\|}{1 + \gamma\|\zeta - \eta\|}. \end{aligned}$$

□

Lemma 3.2. *There exists $C > 0$ such that if $\varphi \in \mathcal{B}$ is the solution of the equation*

$$\varphi(t) = \lambda f\bigg(t, \frac{\varphi(t)}{\lambda}\bigg) - \lambda \int_{t-\alpha}^{t} D(t,s) g(s, \varphi(s))\, ds, \tag{3.1}$$

for some $0 < \lambda < 1$, then $\|\varphi\| \leq C$.

Proof. Let $\varphi \in \mathcal{B}$ denote a solution of (3.1) for some $0 < \lambda < 1$. Using the Burton's method consider the following function of Liapunov type

$$V(t) = \lambda^2 \int_{t-\alpha}^{t} D_s(t,s) \bigg(\int_s^t g(v, \varphi(v)) dv \bigg)^2 ds.$$

We have

$$\begin{aligned} V'(t) = \lambda^2 \int_{t-\alpha}^{t} \bigg[& D_{st}(t,s) \bigg(\int_s^t g(v, \varphi(v))\, dv \bigg)^2 \\ & + 2 D_s(t,s) g(t, \varphi(t)) \int_s^t g(v, \varphi(v))\, dv \bigg] ds \\ & + \lambda^2 D_s(t,t) \bigg(\int_t^t g(v, \varphi(v))\, dv \bigg)^2 - \lambda^2 D_s(t, t-\alpha) \bigg(\int_{t-\alpha}^t g(v, \varphi(v))\, dv \bigg)^2. \end{aligned}$$

We get $\int_t^t g(v, \varphi(v))\, dv = 0$, moreover using (A2) we deduce that

$$\begin{aligned} V'(t) \leq 2\lambda^2 g(t, \varphi(t)) \int_{t-\alpha}^{t} D_s(t,s) \int_s^t g(v, \varphi(v))\, dv\, ds & \\ - \lambda^2 D_s(t, t-\alpha) \bigg(\int_{t-\alpha}^t g(v, \varphi(v))\, dv \bigg)^2 &. \end{aligned}$$

Due to $D_s(t, t-\alpha) \geq 0$ (see again (A2)) we get the following inequality

$$V'(t) \leq 2\lambda^2 g(t, \varphi(t)) \int_{t-\alpha}^{t} D_s(t,s) \int_s^t g(v, \varphi(v))\, dv\, ds.$$

Integration by part and $D(t, t-\alpha) = 0$ (see (A2)) yield

$$\begin{aligned} V'(t) &\leq 2\lambda^2 g(t, \varphi(t)) \bigg[\int_{t-\alpha}^{t} D(t,s) g(s, \varphi(s))\, ds - D(t, t-\alpha) \int_{t-\alpha}^{t} g(v, \varphi(v))\, dv \bigg] \\ &= 2\lambda^2 g(t, \varphi(t)) \int_{t-\alpha}^{t} D(t,s) g(s, \varphi(s))\, ds = 2\lambda g(t, \varphi(t)) \bigg[\lambda f\bigg(t, \frac{\varphi(t)}{\lambda}\bigg) - \varphi(t) \bigg]. \end{aligned}$$

Next, using (A4), we have

$$\begin{aligned}\left|\lambda f\left(t, \frac{\varphi(t)}{\lambda}\right)\right| &= \lambda\left|f\left(t, \frac{\varphi(t)}{\lambda}\right) - f(t,0) + f(t,0)\right| \\ &\le \lambda\left|f\left(t, \frac{\varphi(t)}{\lambda}\right) - f(t,0)\right| + |f(t,0)| \\ &\le \lambda\frac{\left|\dfrac{\varphi(t)}{\lambda}\right|}{1+\gamma\left|\dfrac{\varphi(t)}{\lambda}\right|} + K \le \frac{|\varphi(t)|}{1+\gamma|\varphi(t)|} + K\end{aligned}$$

for some $K > |f(t,0)|$. Observe that the choice of K does not depend on t since $f(\,\cdot\,, x)$ is periodic. By using (A5), it follows that

$$\begin{aligned}V'(t) &\le 2\lambda\left[\frac{\varphi(t)g(t,\varphi(t))}{1+\gamma|\varphi(t)|} + K|g(t,\varphi(t))| - \varphi(t)g(t,\varphi(t))\right] \\ &= 2\lambda\left[\frac{-\gamma|\varphi(t)|}{1+\gamma|\varphi(t)|}\varphi(t)g(t,\varphi(t)) + K|g(t,\varphi(t))|\right].\end{aligned}$$

From (A6) we obtain

$$V'(t) \le \lambda(P - \beta|g(t,\varphi(t))|).$$

From the above facts and since V is a periodic function, we have

$$\begin{aligned}0 = V(\Gamma) - V(0) &= \int_0^\Gamma V'(t)\,dt \\ &\le \lambda\int_0^\Gamma [P - \beta|g(t,\varphi(t))|]\,dt = \lambda\left[P\Gamma - \beta\int_0^\Gamma |g(t,\varphi(t))|\,dt\right].\end{aligned}$$

In consequence, we get

$$\int_0^\Gamma |g(t,\varphi(t))|\,dt \le \frac{P\Gamma}{\beta}.$$

Next, by (A1) and the continuity of g we have $g(t,\varphi(t)) \in \mathcal{B}$. Therefore, there exists $N > 0$ chosen independently from φ such that

$$\int_{t-\alpha}^t |g(s,\varphi(s))|\,ds \le N.$$

Denote $M := \max\limits_{-\alpha\le s\le t\le\Gamma} |D(t,s)|$. Summarizing we have

$$\begin{aligned}|\varphi(t)| &\le \left|\lambda f\left(t, \frac{\varphi(t)}{\lambda}\right)\right| + \lambda\int_{t-\alpha}^t |D(t,s)g(s,\varphi(s))|\,ds \\ &\le \frac{|\varphi(t)|}{1+\gamma|\varphi(t)|} + K + MN \le \frac{1}{\gamma} + K + MN.\end{aligned}$$

So, it is easy to deduce that

$$\|\varphi\| \le C := \frac{1}{\gamma} + K + MN,$$

which ends the proof. □

Before we announce the main result of this section, for the convenience of the reader, we recall the following result due to Burton and Kirk.

Lemma 3.3 ([2, Lemma 3.3]). *Let T be defined as follows*:

$$(T\varphi)(t) = -\int_{t-\alpha}^{t} D(t,s)\, g(s,\varphi(s))\, ds, \quad \varphi \in \mathcal{B}.$$

Then $T(\mathcal{B}) \subset \mathcal{B}$, T is continuous and T maps bounded sets into compact sets.

Theorem 3.4. *If* (A1)–(A6) *hold for some $\Gamma > 0$, then the equation* (1.1) *has a Γ-periodic solution.*

Proof. In the light of Theorem 2.4 we put

$$X = \mathcal{B}, \qquad (T_1\varphi)(t) = -\int_{t-\alpha}^{t} D(t,s)\, g(s,\varphi(s))\, ds, \qquad (T_2\varphi)(t) = f(t,\varphi(t)).$$

By Lemma 3.3, the mapping T_1 is continuous and maps bounded sets into compact subsets of $\mathcal{B}$. Using Lemma 3.2, we get that condition (b) of Theorem 2.4 does not hold and hence (a) is satisfied. □

Example 3.5. Consider the following equation:

$$\varphi(t) = \frac{|\varphi(t)|}{1+|\varphi(t)|} - \int_{t-\alpha}^{t} (s-t+\alpha)\, \varphi(s)\, ds, \tag{3.2}$$

where $\alpha > 0$ is taken arbitrarily and fixed. The existence of a Γ-periodic solution of the equation (3.2) for any $\Gamma > 0$ is guaranteed by Theorem 3.4. Indeed, it is enough to consider $f(t,x) = |x|/(1+|x|)$, $g(t,x) = x$, $D(t,s) = s-t+\alpha$ and $\gamma = 1$. Then, the conditions (A1)–(A5) are easy to be observed. In order to get (A6) it is enough for any $K > 0$ to take $\beta = K$ and $P = (9K^2 + 24K)/8$.

On the other hand, observe that the function f in the equation (3.2) does not allow to reduce the problem to the most known case of F-contraction, i.e. Banach contraction. If it was possible then we would have the existence of $k \in (0,1)$ such that

$$|f(t,x) - f(t,y)| \leq k|x-y| \quad \text{for all } x,y,t \in \mathbb{R}.$$

In consequence, taking $y = 0$ and any $x \neq 0$ we would have:

$$k \geq \frac{|f(t,x) - f(t,0)|}{|x|} = \frac{1}{1+|x|},$$

which is impossible.

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