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MULTIPLICITY AND CONCENTRATION FOR KIRCHHOFF TYPE EQUATIONS AROUND TOPOLOGICALLY CRITICAL POINTS IN POTENTIAL

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Abstract. We consider the multiplicity and concentration of solutions for the Kirchhoff Type Equation

$$-\varepsilon^2 M\bigg(\varepsilon^{2-N}\int_{\mathbb{R}^N}|\nabla v|^2\,dx\bigg)\Delta v+V(x)v=f(v)\quad \text{in } \mathbb{R}^N.$$

Under suitable conditions on functions M, V and f, we obtain the existence of positive solutions concentrating around the local maximum points of V, which gives an affirmative answer to the problem raised in [21]. Moreover, we also obtain multiplicity of solutions which are affected by the topology of critical points set of potential V.

1. Introduction

In this paper, we focus on the following Kirchhoff type equations:

$$\begin{cases} -\varepsilon^2 M\bigg(\varepsilon^{2-N}\displaystyle\int_{\mathbb{R}^N}|\nabla v|^2\,dx\bigg)\Delta v+V(x)v=f(v) & \text{in } \mathbb{R}^N,\\ v\in H^1(\mathbb{R}^N),\quad v>0, \end{cases}\tag{1.1}$$

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where $N \geq 3$, $\varepsilon > 0$ is a small parameter and M is a positive continuous function. Due to the presence of $M\big(\varepsilon^{2-N}\int_{\mathbb{R}^N}|\nabla v|^2\,dx\big)$, equation (1.1) is a nonlocal problem. Our main purpose is to consider the existence and asymptotic behavior of positive solutions for (1.1) when the potential $V(x)$ possesses local maximal points. This problem was raised by Figueiredo et al. in [21]. Moreover, without the oddness on nonlinearity f, we are also interested in considering the effect of the set's topology of critical points in potential V on multiplicity of solutions.

When $\varepsilon = 1$, $V(x) = 0$, $M(t) = a + bt$ and $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain in (1.1), it becomes

$$-\bigg(a + b\int_\Omega |\nabla v|^2 dx\bigg)\Delta v = f(x,v) \quad \text{in } \Omega. \tag{1.2}$$

This type of equation is related to the stationary analogue of the equations

$$v_{tt} - \bigg(a + b\int_\Omega |\nabla v|^2\,dx\bigg)\Delta v = f(x,v) \quad \text{in } \Omega, \tag{1.3}$$

which refers to free vibrations of elastic strings. For more details, see [3], [30], [37], [45]. Lions, in the pioneering work [33], proposed functional analysis approach to consider equation (1.3), then various researchers began to focus on this problem. During the past years, there have been many methods to investigate the existence of solutions for (1.2), see [13], [31], [32], [38], [41], [51].

When $M(t) \equiv 1$ in (1.1), it becomes the well-known nonlinear Schrödinger equation

$$-\varepsilon^2\Delta v + V(x)v = f(v) \quad \text{in } \mathbb{R}^N. \tag{1.4}$$

In the past two decades, a great deal of work has been devoted to the study of semiclassical standing waves for (1.4). It is worth mentioning that from [22], [39], [40], [48], authors observed that concentration of any family of solutions with uniformly bounded energy may occur only at critical points of $V(x)$. So we focus on types of potential $V(x)$ which are closely relating to this paper. [18], [19] devised a penalization approach to obtain the existence of single or several spikes solution located around the prescribed single or finite sets of local minimum in potential $V(x)$. Further, through similar approach, [20] handled potential $V(x)$ containing topologically nontrivial critical points, which can be captured by local minimax argument. Ruiz et al. [2] took different minimax argument on topological cone to cope with the potential possessing isolated local maximum or saddle point.

However, Byeon and Jeanjean [9] constructed a localized deformation argument obtaining positive solutions for (1.4), which concentrate to local minimal of $V(x)$ under the almost optimal conditions on $f \in C(\mathbb{R})$:

(f1) $\lim\limits_{s\to 0^+} f(s)/s = 0$.

(f2) There exist $C > 0$ and $p \in (1, (N+2)/(N-2))$, such that

$$|f(s)| \le C(1 + |s|^p).$$

(f3) There exists a constant $s_0 > 0$ such that $m_0 s_0^2/2 < F(s_0)$, where

$$F(s) = \int_0^s f(t)\, dt.$$

These conditions were first introduced by Berestycki and Lions in [7]. A few years later, Byeon and Tanaka [10] developed a more complicated approach to handle the case that $V(x)$ possessing local maximum points or saddle points.

When $M(t) \not\equiv$ constant, take $N = 3$ and $M(t) = a + bt$ in (1.1) as the following form:

$$-\bigg(\varepsilon^2 a + \varepsilon b \int_{\mathbb{R}^3} |\nabla v|^2\, dx\bigg)\Delta v + V(x)v = f(v) \quad \text{in } \mathbb{R}^3. \tag{1.5}$$

Under nonlinearity $f(v)$ in sub-critical or critical growth, [26], [46] used Nehari manifold and minimax methods to prove least energy solutions concentrating global minimum of $V(x)$ and multiplicity results through Lusternik–Schnirelmann theory. In [47] they consider this problem with competitive potentials. For local minimum in potential $V(X)$, He and Li [27] considered (1.5) with $f(v) = \lambda|u|^{p-2}u + |u|^4 u$, using proper changes of variables and quantitative deformation approach developed in [9], [21], they obtain the corresponding results with $2 < p \le 4$ in larger range.

It is generalization of (1.5) in some sense, when $\varepsilon \equiv 1$, $V(x) \equiv 0$ and M is a positive function in (1.1),

$$-M\bigg(\int_{\mathbb{R}^N} |\nabla v|^2\, dx\bigg)\Delta v = f(x, v) \quad \text{in } \mathbb{R}^N. \tag{1.6}$$

One of the first works involving these equations was considered in functional analysis setting by Vasconcellos [45]. Afterwards there are some results of existence referring to [3, 37] et al., where [3] gives sufficient conditions on M for existence of (1.6). And we make the following assumptions:

(M1) For any $t \ge 0$, $M(t) \ge \underline{M_0} > 0$.
(M2) $\liminf\limits_{t\to\infty} \big\{\widehat{M}(t) - (1 - 2/N)M(t)t\big\} = \infty$, where $\widehat{M}(t) = \int_0^t M(t)\, dt$.
(M3) $M(t)/t^{2/(N-2)} \to 0$ as $t \to \infty$.
(M4) The function $M(t)$ is increasing in $[0, \infty)$.
(M5) The function $t \mapsto M(t)/t^{2/(N-2)}$ is strictly decreasing in $(0, \infty)$.
(M6) $M(t)$ is differentiable and $M(t) + (1 - N/2)M'(t)t \ne 0$.

Fortunately, it is easy to find an example of $M(t)$ satisfying all conditions, i.e. $M(t) = a_0 + \sum\limits_{i=1}^{k} a_i t^{s_i}$, where $k \in \mathbb{N}$, $a_i > 0$ for $0 \le i \le k$ and $0 < s_i < 2/(N-2)$. When taking $M(t) = a + bt$, (1.1) turns to standard Kirchhoff equation, (M1)–(M6) still hold for $N = 3$.

Figueiredo et al. [21] investigate (1.1) under the case of local minimum of $V(x)$. Precisely, they assume:

(V1) $V(x) \in C(\mathbb{R}^N, \mathbb{R})$ and $\underline{V} := \inf_{x \in \mathbb{R}^N} V(x) > 0$.

(V2′) there exists a bounded open set $\Omega \subset \mathbb{R}^N$ such that

$$V_0 := \inf_{\Omega} V(x) < \inf_{\partial\Omega} V(x),$$

and set $\mathcal{M}' := \{x \in \Omega : V(x) = V_0\}$. Under (M1)–(M5) and (f1)–(f3), they obtain positive solutions concentrating around the set $\mathcal{M}'$. In addition, they ask that if the potential contains local maximum or saddle points, whether (1.1) have the corresponding results or not.

In [14] the authors have proved some results about existence and concentration. Furthermore, it is natural and interesting to ask that how to relate multiplicity of solutions to the topology of the set with local maximum or saddle points of $V(x)$. To the best of our knowledge, there is little result about the multiple solutions for Kirchhoff type equation which is correlative with the potential possessing local maximum points. The aim of present paper is to give an affirmative answer to such a question. Here, we present our main results:

Theorem 1.1. *Assume that* (M1)–(M6) *and* $f \in C^1(\mathbb{R})$ *hold with* (f1)–(f3). *And* $V(x) \in C^1$ *holds for* $N \geq 3$ *with* (V1) *and* $\sup_{x \in \mathbb{R}^N} V(x) < +\infty$. *Moreover, there exist local maximum points in* $V(x)$, *i.e. there exists bounded set* $O \subset \mathbb{R}^N$ *such that*

$$\sup_{x \in \partial O} V(x) < \sup_{x \in O} V(x) = m_0,$$

and denote $\mathcal{M} = \{x \in O : V(x) = m_0\}$. *Then, for* $\varepsilon > 0$ *small, there exist at least* $\operatorname{cupl}(\mathcal{M}) + 1$ *positive solutions* $v_\varepsilon^{(i)}$ *for* (1.1), *where* $i = 1, \ldots, \operatorname{cupl}(\mathcal{M}) + 1$, *satisfying the following properties*: *let* $x_\varepsilon^{(i)}$ *be a maximum point of* $v_\varepsilon^{(i)}$,

(a) $\operatorname{dist}\big(x_\varepsilon^{(i)}, \mathcal{M}\big) \to 0$ *as* $\varepsilon \to 0$.

(b) *Up to subsequence,* $v_\varepsilon^{(i)}(\varepsilon x + x_\varepsilon^{(i)}) \to U$ *in* $H^1(\mathbb{R}^N)$, *where* U *is the positive least energy solution of*

$$-M\bigg(\int_{\mathbb{R}^N} |\nabla v|^2\,dx\bigg)\Delta v + V(x_0)v = f(v) \quad \text{in } \mathbb{R}^N,$$

where $x_\varepsilon^{(i)} \to x_0 \in \mathcal{M}$.

(c) *There exist* $C, c > 0$ *such that*

$$v_\varepsilon^{(i)}(x) \leq C \exp\bigg(-\frac{c}{\varepsilon}\,\big|x - x_\varepsilon^{(i)}\big|\bigg).$$

Remark 1.2. First, the assumption of upper bound on $V(x)$ makes many computations have a simpler form. Whereas this condition can be removed, then it suffices to choose the appropriate cut-off function to define the neighbourhood of approximate solutions.

Second, the notation $\mathrm{cupl}(\mathcal{M})$ is the cup-length which is defined using Alexander–Spanier cohomology with coefficients in the field $\mathbb{F}$. In particular, if $\mathcal{M} = \mathbb{S}^{N-1}$ the $N-1$ dimensional sphere in $\mathbb{R}^N$, $\mathrm{cupl}(\mathcal{M}) + 1 = 2$. If $\mathcal{M} = \mathbb{T}^N$, the N dimensional torus, $\mathrm{cupl}(\mathcal{M}) + 1 = N + 1$.

In Theorem 1.1, we observe that the topology of $\mathcal{M}$ plays an important role in multiplicity of solutions, which is motivated by these papers [10], [11], [15], [21], especially [15] in which they consider multiplicity of schrödinger equation (1.4) with local minimum in potential. By the way, it is cannot be ignored that the domain's topology is related to multiple solutions, see [5], [6], [16], [12], [50]. We also have an interesting consequence:

COROLLARY 1.3. *Assume that* (M1)–(M6) *and* $V(x) \in C^1(\mathbb{R}^N)$ *hold with* (V1) *and* $\sup_{x\in\mathbb{R}^N} V(x) < +\infty$. *If* $f \in C^1$ *satisfy with* (f1) *and* (f2) *for* $N \geq 3$. *Moreover, there exist mutually disjoint bounded domains* O_i $(i = 1, \dots, k)$ *and constants* $0 < c_1 < \dots < c_k$ *such that*

$$\sup_{\partial O_i} V(x) < \sup_{O_i} V(x) = c_i.$$

Denote the sets of critical values by $\mathcal{M}_i = \{x \in O_i : V(x) = c_i\}$. *Finally, there exist constants* $s_i > 0$ *such that* $c_i s_i^2/2 < F(s_i)$ *for each* i. *Then, for* $\varepsilon > 0$ *small,*

(a) *there exist at least* $\sum_{i=1}^{k} \mathrm{cupl}(\mathcal{M}_i) + k$ *families of positive solutions* $v_\varepsilon^{(j_i)}$ *for* (1.1), *where* $j_i = 1, \dots, \mathrm{cupl}(\mathcal{M}_i) + 1$;

(b) *let* $x_\varepsilon^{(j_i)}$ *be a maximum point of* $v_\varepsilon^{(j_i)}$,

$$\lim_{\varepsilon\to 0} V\big(x_\varepsilon^{(j_i)}\big) = c_i;$$

(c) *up to subsequence,* $v_\varepsilon^{(j_i)} \to U^{(i)}$ *in* $H^1(\mathbb{R}^N)$ *as* $\varepsilon \to 0$, *where* $U^{(i)}$ *is the least energy solution of*

$$-M\bigg(\int_{\mathbb{R}^N} |\nabla v|^2 dx\bigg)\Delta v + c_i v = f(v) \quad \text{in } \mathbb{R}^N;$$

(d) *there exist* $C, c > 0$ *such that*

$$v_\varepsilon^{(j_i)}(x) \leq C \exp\bigg(-\frac{c}{\varepsilon}\big|x - x_\varepsilon^{(j_i)}\big|\bigg).$$

We remark that solutions obtained in this corollary can be separated by ε small enough, since O_i are mutually disjoint. And this corollary describes a kind of multiple concentrating phenomena.

The main difficulties in proof lie in the presence of nonlocal term $M(\|\nabla u\|_2^2)$. Concretely, first, it is more delicate than (1.4) when constructing the invariant neighbourhood of approximate solutions and calculating some estimates. Secondly, unlike (1.4), the weak limit of Palais Smale (P.S. for short) sequences is not solution for the corresponding Kirchhoff type equation. Whereas, we are inspired by [10], [21] and devise a new invariant neighbourhood and refine the argument of P.S. sequences.

This thesis relies on variational arguments and is divided into four sections. In Section 2, we first briefly refine more properties of potential that will be used in the back. Afterwards we introduce some results about the limit equation and define the center of mass which is used to estimate multiplicity and concentration property. Subsection 2.3 is aimed to construct the invariant neighbourhood. Then we estimate the energy functional and its norm of gradient in Subsection 2.4. In the following, three maps on the localized neighbourhood are defined and some related properties are proved. Section 3 is devoted to iterating three maps to obtain the estimates of energy functional and the results of existence. Finally, we focus on the multiplicity concerning with relative category and cup-length in algebraic topology in Section 4.

2. Preliminaries

We observe that defining $u(x) = v(\varepsilon x)$, the equation (1.1) is equivalent to

$$-M\bigg(\int_{\mathbb{R}^N} |\nabla u|^2\,dx\bigg)\Delta u + V(\varepsilon x)u = f(u). \tag{2.1}$$

In what follows we will focus on this equivalent problem. And we use $\|\cdot\|$ and $\|\cdot\|_r$ to denote the norm of $H^1(\mathbb{R}^N)$ and $L^r(\mathbb{R}^N)$ by

$$\|u\| = \bigg(\int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2\,dx\bigg)^{1/2} \quad \text{and} \quad \|u\|_r := \bigg(\int_{\mathbb{R}^N} |u|^r\,dx\bigg)^{1/r}$$

for $r \in [1,\infty)$.

We look for critical points of the functional $\Gamma_\varepsilon(u) \in C^1\big(H^1\big(\mathbb{R}^N\big),\mathbb{R}\big)$ defined by

$$\Gamma_\varepsilon(u) = \frac{1}{2}\,\widehat{M}\big(\|\nabla u\|_2^2\big) + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x)u^2\,dx - \int_{\mathbb{R}^N} F(u)\,dx. \tag{2.2}$$

The critical point of Γ_ε is clearly the weak solution of (2.1). We assume that $f(t) = 0$ for $t \leq 0$, then, by Bony maximum principle [34], any nontrivial solution of (2.1) is positive.

Now we further refine the conditions on potential V. In [21], under the condition (V2′), we observe that the lower bound of energy functional for limit equation can be obtained by monotonicity of potential V around local minimum. Yet, this property does not hold for the case of local maximum. Thus it suffices to explore more subtle properties from our assumptions in Theorem 1.1.

The local maximum in $V(x)$ can be further captured by a local minimax characterization. Moreover, there exists a quantitative deformation flow near the local maximum. More precisely, from conditions of Theorem 1.1, we have the following two conditions:

(V2) There exist a connected bounded open set $O \subset \mathbb{R}^N$ with smooth boundary and a $N-1$ dimensional compact manifold $L_0 \subset O$ without boundary such that

$$\max_{x \in L_0} V(x) < m_0 = \inf_{\substack{\mathcal{B} \in \mathcal{L}(L_0) \\ \varphi \in \Lambda_{\mathcal{B}}}} \max_{x \in \mathcal{B}} V(\varphi(x)), \tag{2.3}$$

where $\mathcal{L}(L_0)$ is the set of connected (orientable) N dimensional compact manifolds $\mathcal{B}$ and their boundary $\partial\mathcal{B}$ are homeomorphic to L_0, and $\Lambda_{\mathcal{B}}$ is a set

$$\Lambda_{\mathcal{B}} := \{\varphi \in C(\mathcal{B}, O) : \varphi|\partial\mathcal{B} \to L_0 \text{ is homeomorphic}\}.$$

(V3) Taking a compact set of critical points $\mathcal{M} = \{x \in O : \ V(x) = m_0\}$ such that for small $d > 0$, the neighbourhood $\mathcal{M}^d \subset O$. There exist positive constants $a, \mu, \nu > 0$ and a map $\zeta \in C([0,1] \times O, O)$ satisfying:
(a) $V(\zeta(t,x))$ is non-increasing with respect to $t \in [0,1]$, for any $x \in O$.
(b) $\zeta(t,x) = x$ for $t = 0$ or $x \notin V_{m_0-\nu}^{m_0}$.
(c) $|\zeta(t_1,x) - \zeta(t_2,x)| \le \mu|t_1 - t_2|$ for $t_1, t_2 \in [0,1], \ x \in O$.
(d) $\limsup_{h \to 0^+}(V(\zeta(t+h,x)) - V(\zeta(t,x))/h \le -a$, uniformly for $t \in [0,1)$ and $x \in \big(\mathcal{M}^d \cap V_{m_0-\nu}^{m_0}\big) \setminus \mathcal{M}$, where

$$\mathcal{M}^d := \Big\{x \in \mathbb{R}^N : \inf_{y \in \mathcal{M}} |x-y| \le d\Big\},$$
$$V_{m_0-\nu}^{m_0} := \{x \in O : m_0 - \nu \le V(x) \le m_0\}.$$

For the proof of these two conditions, see [10] for more details.

2.1. Limit equations. For any $a > 0$, we define a functional

$$L_a(u) = \frac{1}{2}\widehat{M}\big(\|\nabla u\|_2^2\big) + \frac{a}{2}\int_{\mathbb{R}^N} u^2\,dx - \int_{\mathbb{R}^N} F(u)\,dx, \tag{2.4}$$

in $C^1\big(H^1\big(\mathbb{R}^N\big), \mathbb{R}\big)$, which is associated to the limit equation

$$-M\big(\|\nabla u\|_2^2\big)\Delta u + au = f(u). \tag{2.5}$$

In (f3), by continuity, we can have that there exist $m_1, m_2 > 0$ with $m_1 < m_0 < m_2$ such that any $m_0 \in [m_1, m_2]$, (f3) still holds. We define the least energy value for (2.5) and the set by:

$$E(a) = \inf\{L_a(u) : L_a'(u) = 0, \ u \ne 0\},$$

$$S_a = \Big\{U \in H^1(\mathbb{R}^N) \setminus \{0\} : L_a'(U) = 0, \ E(a) \le L_a(U) \le E(m_0), \ U(0) = \max_{x \in \mathbb{R}^N} U(x)\Big\}.$$

When (M1)–(M5) and (f1)–(f3) are satisfied, the existence of the least energy solution of (2.5) is proved in [21].

It is also showed that any solution of (2.5) satisfies the Pohozaev identity:

$$P(u) := \frac{N-2}{2} M\big(\|\nabla u\|_2^2\big)\|\nabla u\|_2^2 + N\int_{\mathbb{R}^N} \frac{a}{2}\,u^2 - F(u)\,dx = 0. \tag{2.6}$$

From this, we have the following lemma:

Lemma 2.1. *Assume* (f1)–(f3) *and* (M1)–(M6) *hold. Then there exists a minimizer U of $\inf\limits_{u\in\mathcal{P}} L_a(u)$ such that U is the least energy solution for* (2.5). *Hence*

$$E(a) = \inf_{u\in\mathcal{P}} L_a(u), \quad \textit{where } \mathcal{P} := \big\{u \in H^1\big(\mathbb{R}^N\big) \setminus \{0\} : P(u) = 0\big\}.$$

Here (M6) plays an important role in looking for least energy solutions on Pohozaev manifold. For the proof we refer to [14].

Denoting $\underline{m}(d) := \inf\{V(x) : x \in \mathcal{M}^d\}$ and $\overline{m}(d) := \sup\{V(x) : x \in \mathcal{M}^d\}$, we choose $d > 0$ small enough such that

$$m_1 \le \underline{m}(d) < m_0 \le \overline{m}(d) \le m_2.$$

Next we define the set

$$\mathcal{M}([-\nu_0, 0]) := \{x \in L : V(x) - m_0 \in [-\nu_0, 0]\},$$

here we choose small $\nu_0 \in (0, \nu]$ such that

$$\mathcal{M}([-\nu_0, 0]) \subset \mathcal{M}^d \quad \text{and} \quad \max_{x\in L_0} V(x) < m_0 - \nu_0.$$

Then we define

$$\widehat{S} = \bigcup_{a\in[m_0-\nu_0, m_0]} S_a.$$

From Proposition 2.19 in [21], we see that $\widehat{S}$ is compact in $H^1(\mathbb{R}^N)$ and there exist $C, c > 0$ such that, for any $U \in \widehat{S}$,

$$U(x) + |\nabla U(x)| \le C\exp(-c|x|), \quad x \in \mathbb{R}^N. \tag{2.7}$$

We define the neighbourhood of $\widehat{S}$: for $r > 0$

$$\mathcal{S}(2r) = \big\{u = U(x-y) + \varphi(x) \in H^1(\mathbb{R}^N) : U \in \widehat{S},\ y \in \mathbb{R}^N,\ \|\varphi\| \le 2r\big\}.$$

2.2. Center of mass in $\mathcal{S}(2r_0)$. In this subsection, we introduce a center of mass in $\mathcal{S}(2r_0)$ which will be frequently used in the following.

Lemma 2.2. *There exist positive constants $R_0, r_0 > 0$ and a map*

$$\Upsilon(u)\colon \mathcal{S}(2r_0) \to \mathbb{R}^N \quad \textit{such that} \quad |\Upsilon(u) - y| \le R_0,$$

for all $u = U(\,\cdot\, - y) + \varphi \in \mathcal{S}(2r_0)$ with $U \in \widehat{S}$, $y \in \mathbb{R}^N$ and $\|\varphi\| \le 2r_0$.

Proof. Set $r_* = \min_{u\in\widehat{S}} \|u\|$ and it follows from (2.7) that there exists $R_0 > 0$ such that, for any $u \in \widehat{S}$,

$$\|u\|_{H^1(|x|\le R_0/2)} > \frac{3}{4}\,r_* \quad \text{and} \quad \|u\|_{H^1(|x|\ge R_0/2)} < \frac{1}{16}\,r_*.$$

Define

$$d(z,u) = \inf_{\widetilde{U}\in\widehat{S}} \big\|u - \widetilde{U}(x-z)\big\|_{H^1(|x-z|\le R_0/2)}.$$

We choose cut-off function $\phi \in C_0^\infty(\mathbb{R},[0,1])$ such that $\phi(s)=0$ for $s\in[r_*/2,+\infty)$ and $\phi(s)=1$ for $s\in[0,r_*/4]$. Taking $r_0 = r_*/16$, we define the mass center map $\Upsilon(u)$ as following:

$$\Upsilon(u) = \frac{\displaystyle\int_{\mathbb{R}^N} \phi(d(z,u))z\,dz}{\displaystyle\int_{\mathbb{R}^N} \phi(d(z,u))\,dz}.$$

Then, for $u \in \mathcal{S}(2r_0)$, $u = U(\,\cdot\,-y)+\varphi$ with $U\in\widehat{S}$ and $\|\varphi\|\le 2r_0$. We observe that for any $|z-y|\ge R_0$ and $\widetilde{U}\in\widehat{S}$,

$$\begin{aligned}
&\big\|u - \widetilde{U}(\,\cdot\,-z)\big\|_{H^1(|x-z|\le R_0/2)}\\
&\qquad \ge \big\|\widetilde{U}\big\|_{H^1(|x|\le R_0/2)} - \|U(\,\cdot\,-y)\|_{H^1(|x-y|\ge R_0/2)} - \|\varphi\|\\
&\qquad \ge \frac{3}{4}\,r_* - \frac{1}{16}\,r_* - 2r_0 \ge \frac{r_*}{2}.
\end{aligned}$$

Then $\phi(d(z,u))=0$ and we find that $\operatorname{supp}\{\phi(d(z,u))\}\subset B(y,R_0)$, which implies that $\Upsilon(u)\in B(y,R_0)$. $\square$

2.3. New invariant neighbourhoods $\mathcal{R}(r)$. In this subsection, we introduce a new invariant neighbourhood $\mathcal{R}(2r_0)$, which can be seen as the refinement of $\mathcal{S}(2r_0)$ in some extend. Due to the presence of the nonlocal term M, our definition of $\mathcal{R}(2r_0)$ is different from [10]. This set is invariant under the maps which will be introduced in the following subsections. Denoting $V_\varepsilon := V(\varepsilon x)$, we first define, for $r\in(0,2r_0]$,

$$\begin{aligned}
\mathcal{R}_1(r) = \bigg\{ & u \in \mathcal{S}(2r_0) :\\
& \int_{|x-\Upsilon(u)|\le 1/\sqrt{\varepsilon}} |\nabla u - U(\,\cdot\,-y)|^2 + |u-U(\,\cdot\,-y)|^2\,dx \le \frac{r^2}{2}\\
& \text{for some } U\in\widehat{S},\ y\in\mathbb{R}^N \text{ and } \int_{D_\varepsilon} |\nabla u|^2 + u^2 - 2F(u)\,dx \le \frac{r^2}{2}\bigg\},
\end{aligned}$$

where $D_\varepsilon = \{x\in\mathbb{R}^N : |x-\Upsilon(u)|\ge 1/\sqrt{\varepsilon}\}$.

Next we define

$$
\begin{aligned}
\mathcal{R}_2(r) = \Big\{ & u \in \mathcal{S}(2r_0) : \\
& \int_{|x-\Upsilon(u)|\le 1/\sqrt{\varepsilon}} |\nabla u - U(\,\cdot\, - y)|^2 + |u - U(\,\cdot\, - y)|^2 \, dx \le \frac{r^2}{2} \\
& \text{for some } U \in \widehat{S},\ y \in \mathbb{R}^N \text{ and} \\
& \int_{\alpha_{\varepsilon,u}}^{\alpha_{\varepsilon,u} + \int_{D_\varepsilon} |\nabla u|^2 \, dx} M(t)\, dt + \int_{D_\varepsilon} V_\varepsilon u^2 \, dx - 2 \int_{D_\varepsilon} F(u)\, dx \\
& \qquad\qquad\qquad \le \frac{r^2}{2} \min \big\{ \underline{V}, \underline{M_0} \big\} \Big\},
\end{aligned}
$$

where $\alpha_{\varepsilon,u} = \int_{|x-\Upsilon(u)|\le 1/\sqrt{\varepsilon}} |\nabla u|^2\, dx$. Thus we set $\mathcal{R}(r) := \mathcal{R}_1(r) \cup \mathcal{R}_2(r)$. From the definition, there is some relationship between the neighbourhoods $\mathcal{S}(2r_0)$ and $\mathcal{R}(r)$. For simplicity, denote

$$
\|u\|^2_{M,D_\varepsilon} := \int_{\alpha_{\varepsilon,u}}^{\alpha_{\varepsilon,u} + \int_{D_\varepsilon} |\nabla u|^2 \, dx} M(t)\, dt + \int_{D_\varepsilon} V_\varepsilon u^2 \, dx.
$$

Lemma 2.3. *For $c, c' \in (0,1]$ and some $r_1 \in (0, r_0]$, there exist $q := q(r_1) > 0$ such that $q(r_1) \to 0$ as $r_1 \to 0$ and, for small $\varepsilon > 0$ independent of r_1, one has*

$$
\mathcal{R}((1-q)cr_1) \subset \mathcal{S}(cr_1) \quad \text{and} \quad \mathcal{S}(c'r_1) \subset \mathcal{R}\big((1+q)\sqrt{2}c'r_1\big), \tag{2.8}
$$

where $1 + q \le \sqrt{2}$.

Proof. First, we need to verify that there exist $q := q(r) > 0$ such that for small $\varepsilon > 0$ and $u \in \mathcal{S}(2r)$, one has

$$
\int_{D_\varepsilon} F(u)\, dx \le \frac{1}{2}\, q(r) \|u\|^2_{M,D_\varepsilon} \tag{2.9}
$$

where $r \in (0, r_0]$ and $q(r) \to 0$ as $r \to 0$. In fact, set $u = U(\,\cdot\, - y) + \varphi$ with $U \in \widehat{S}$, $y \in \mathbb{R}^N$ and $\|\varphi\| \le 2r$. By (2.7) and Lemma 2.2, we have

$$
\lim_{R\to\infty} \int_{|x-\Upsilon(u)|\ge R} |\nabla U(\,\cdot\, - y)|^2 + U^2(\,\cdot\, - y)\, dx = 0. \tag{2.10}
$$

Hence there exists $R' > R_0$ such that

$$
\int_{|x-\Upsilon(u)|\ge R'} |\nabla u|^2 + V_\varepsilon u^2 \, dx \le 5r^2.
$$

Moreover, by (f1) and (f2), one has for any $\eta > 0$, there exists $C_\eta > 0$ such that $F(u) \le \eta u^2 + C_\eta |u|^{p+1}$. Thus by Sobolev inequality, we have, for small $\varepsilon > 0$,

$$
\int_{D_\varepsilon} |u|^{p+1}\, dx \le C \bigg(\int_{D_\varepsilon} |\nabla u|^2 + u^2 \, dx \bigg)^{(p+1)/2} \le C(5r^2)^{(p-1)/2} \int_{D_\varepsilon} |\nabla u|^2 + u^2 \, dx.
$$

Then we have that

$$\int_{D_\varepsilon} F(u)\,dx \le \frac{1}{2}\,q^*(r)\int_{D_\varepsilon} |\nabla u|^2 + u^2\,dx \le \frac{q^*(r)}{2\min\{\underline{M_0},\underline{V}\}}\,\|u\|^2_{M,D_\varepsilon}.$$

Thus we choose $q(r)=\max\{1,\underline{M_0}^{-1},\underline{V}^{-1}\}q^*(r)$ to end the proof of (2.9) and also have that

$$\int_{D_\varepsilon} F(u)\,dx \le \frac{1}{2}\,q(r)\int_{D_\varepsilon} |\nabla u|^2 + u^2\,dx.$$

According to this and (2.9), for $u\in\mathcal{S}(2r_0)$, one has

$$(1-q)\int_{D_\varepsilon} |\nabla u|^2 + u^2\,dx \le \int_{D_\varepsilon} |\nabla u|^2 + u^2 - 2F(u)\,dx$$

and

$$\begin{aligned}(1-q)\min\{\underline{M_0},\underline{V}\}&\int_{D_\varepsilon} |\nabla u|^2 + u^2\,dx\\ &\le \int_{\alpha_{\varepsilon,u}}^{\alpha_{\varepsilon,u}+\int_{D_\varepsilon}|\nabla u|^2\,dx} M(t)\,dt + \int_{D_\varepsilon} V_\varepsilon u^2\,dx - 2\int_{D_\varepsilon} F(u)\,dx.\end{aligned}$$

Then, choosing $r_1\in(0,r_0]$ with $1+q\le\sqrt{2}$. For $u\in\mathcal{R}((1-q)cr_1)$, we have

$$\int_{D_\varepsilon} |\nabla u|^2 + u^2\,dx \le \frac{1}{2}\,(1-q)(cr_1)^2,$$

and note that, for some $U\in\widehat{S}$, $y\in\mathbb{R}^N$,

$$\int_{|x-\Upsilon(u)|\le 1/\sqrt{\varepsilon}} |\nabla(u-U(\,\cdot\,-y))|^2 + |u-U(\,\cdot\,-y)|^2\,dx \le \frac{((1-q)cr_1)^2}{2}.$$

Hence, combining with (2.10), for small $\varepsilon>0$, one has

$$\begin{aligned}\|u-U(\,\cdot\,-y)\|^2 &= \int_{|x-\Upsilon(u)|\le 1/\sqrt{\varepsilon}} |\nabla(u-U(\,\cdot\,-y))|^2 + |u-U(\,\cdot\,-y)|^2\,dx\\ &\quad + \int_{D_\varepsilon} |\nabla u|^2 + u^2\,dx + o(1)\\ &\le \frac{((1-q)cr_1)^2}{2} + \frac{1}{2}(1-q)(cr_1)^2 + o(1) \le (cr_1)^2.\end{aligned}$$

On the other hand for $u\in\mathcal{S}(c'r_1)$, then for some $U\in\widehat{S}$ and $y\in\mathbb{R}^N$, one has that

$$\begin{aligned}\int_{|x-\Upsilon(u)|\le 1/\sqrt{\varepsilon}} &|\nabla\big(u-U(\,\cdot\,-y)\big)|^2 + |u-U(\,\cdot\,-y)|^2\,dx\\ &\le (c'r_1)^2 \le \frac{1}{2}\big((1+q)\sqrt{2}c'r_1\big).\end{aligned}$$

And from (2.10), we have for small $\varepsilon>0$,

$$\int_{D_\varepsilon} |\nabla u|^2 + u^2 dx \le (1+q)\big(c'r_1\big)^2$$

Thus,

$$\int_{D_\varepsilon} |\nabla u|^2 + u^2 - 2F(u)\,dx \le \frac{1}{2}\big((1+q)\sqrt{2}c'r_1\big)^2.$$

Hence $u \in \mathcal{R}\big((1+q)\sqrt{2}c'r_1\big)$. □

2.4. Energy and gradient estimates. In this subsection, we shall show some properties of energy functional (2.2) and its gradient's estimates on invariant neighbourhood.

We first use the least energy solution of limit equation to estimate the energy functional (2.2). By (V2), we observe that for any $\varepsilon > 0$, there exist a manifold $\mathcal{L}_\varepsilon \in \mathcal{L}(L_0)$ and a map $\varphi_\varepsilon \in \Lambda_{\mathcal{L}_\varepsilon} \subset C(\mathcal{L}_\varepsilon, O)$ such that $V(\varphi_\varepsilon(z)) \le m_0$ for any $z \in \mathcal{L}_\varepsilon$. From the definition of $\Lambda_{\mathcal{B}}$ in (V2), we can assume $\partial\mathcal{L}_\varepsilon = L_0$ and $\varphi_\varepsilon(z) = z$, for all $z \in L_0$. From (V3), there is a continuous map $\zeta(s,z)\colon [0,1]\times O \mapsto O$ such that

$$\zeta(1,\varphi_\varepsilon(z)) = \zeta(1,z) = z \quad \text{for any } z \in L_0;$$

$$V(\zeta(s,\varphi_\varepsilon(z))) \le m_0 \quad \text{for any } z \in \mathcal{L}_\varepsilon \text{ and } s \in [0,1]; \tag{2.11}$$

$$V(\zeta(s,\varphi_\varepsilon(z))) \le m_0 - \nu_0 \quad \text{if } \varphi_\varepsilon(z) \notin \mathcal{M}([-\nu_0,0]). \tag{2.12}$$

Then, denoting $\gamma(z) := \zeta(1,\varphi_\varepsilon(z))$, we define the map $A(t,z)\colon (0,\infty)\times\mathcal{L}_\varepsilon \mapsto H^1(\mathbb{R}^N)$ by

$$A(t,z)(x) := U_0\bigg(\frac{x-\gamma(z)/\varepsilon}{t}\bigg),$$

where U_0 is the least energy solution of the equation

$$-M\big(\|\nabla u\|_2^2\big)\Delta u + m_0 u = f(u). \tag{2.13}$$

Next we have the following energy estimates for $\Gamma_\varepsilon(A(t,z))$:

Lemma 2.4. *Assume* (M1)–(M5) *hold.*

(a) *There exists* $T > 1$ *such that* $\Gamma_\varepsilon(A(T,z)) < 0$.

(b) $\lim\limits_{\varepsilon\to 0}\ \max\limits_{t\in[0,T],\, z\in\mathcal{L}_\varepsilon} \Gamma_\varepsilon(A(t,z)) \le E(m_0)$.

(c) *There exist* $t_0' > 0$ *and* $\delta_1' > 0$ *such that* $A(t,z) \in \mathcal{S}(2r_1)$ *for any* $t \in [1-t_0', 1+t_0']$ *and*

$$\limsup_{\varepsilon\to 0}\max_{z\in\mathcal{L}_\varepsilon}\big\{\Gamma_\varepsilon(A(t,z)) : t \in (0,T]\setminus(1-t_0',1+t_0')\big\} \le E(m_0) - \delta_1'/2.$$

(d) $\limsup\limits_{\varepsilon\to 0}\ \max\limits_{t\in[0,T]}\big\{\Gamma_\varepsilon(A(t,z)) : z \in \mathcal{L}_\varepsilon,\ \gamma(z) \notin \mathcal{M}([-\nu_0,0])\big\} < E(m_0)$.

Proof. Through changes of variable and exponential decay of U_0, we have that for small $\varepsilon > 0$,

$$\Gamma_\varepsilon(A(t,z)) = \frac{1}{2}\widehat{M}\big(t^{N-2}\|\nabla U_0\|_2^2\big) + \frac{t^N}{2}\int_{\mathbb{R}^N} V(\gamma(z))U_0^2\,dx - t^N\int_{\mathbb{R}^N} F(U_0)\,dx + o(1).$$

It follows from (2.11) and (2.6) that

$$\begin{aligned}\Gamma_\varepsilon(A(t,z)) &\le \frac{1}{2}\widehat{M}\big(t^{N-2}\|\nabla U_0\|_2^2\big) + t^N\int_{\mathbb{R}^N}\frac{m_0}{2}U_0^2 - F(U_0)\,dx + o(1)\\ &= \frac{t^N}{2}\bigg(\frac{\widehat{M}\big(t^{N-2}\|\nabla U_0\|_2^2\big)}{t^N} - \frac{N-2}{N}M\big(\|\nabla U_0\|_2^2\big)\|\nabla U_0\|_2^2\bigg) + o(1).\end{aligned}$$

By (M3) and changes of variable, we note that $\Gamma_\varepsilon(A(t,z)) \to -\infty$, as $t \to +\infty$. Then there exists $T > 1$ such that $\Gamma_\varepsilon(A(T,z)) < 0$ for any $z \in \mathcal{L}_\varepsilon$.

Moreover, we denote

$$L(t) := \frac{1}{2}\widehat{M}\big(t^{N-2}\|\nabla U_0\|_2^2\big) - \frac{N-2}{2N}t^N M\big(\|\nabla U_0\|_2^2\big)\|\nabla U_0\|_2^2.$$

Differentiating $L(t)$, one has

$$\frac{d}{dt}L(t) = \frac{N-2}{2}t^{N-1}\|\nabla U_0\|_2^2\bigg\{\frac{M\big(t^{N-2}\|\nabla U_0\|_2^2\big)}{t^2} + NM\big(\|\nabla U_0\|_2^2\big)\bigg\}.$$

By (M5) and changes of variable, we observe that

$$\frac{dL(1)}{dt} = 0, \quad \frac{dL(t)}{dt} > 0 \quad \text{for } t \in (0,1) \quad \text{and} \quad \frac{dL(t)}{dt} < 0 \quad \text{for } t \in (1,+\infty).$$

Thus, noting that the definition of U_0 and (2.4), we have

$$\max_{t\in[0,\infty)} L(t) = L(1) = L_{m_0}(U_0) = E(m_0).$$

Thus (b) holds.

For (c), we can choose small $t_0' > 0$ such that for any $t \in [1-t_0', 1+t_0']$, $\|U_0(x/t) - U_0(x)\| \le 2r_1$. Then, for any $t \in [1-t_0', 1+t_0']$,

$$A(t,z) = U_0(\,\cdot\, - \gamma(z)/\varepsilon) + \varphi_t \in \mathcal{S}(2r_1) \quad \text{with } \|\varphi_t\| \le 2r_1.$$

Moreover, for small $\varepsilon > 0$,

$$\Gamma_\varepsilon(A(t,z)) \le L_{m_0}(A(t,z)) + o(1).$$

From the proof of the part (b), taking $\delta_1' = \min\{E(m_0) - L_{m_0}(A(1\pm t_0', z))\}$, for small $\varepsilon > 0$, one has

$$\Gamma_\varepsilon(A(t,z)) \le E(m_0) - \frac{\delta_1'}{2} \quad \text{for any } t \in [0,T]\setminus(1-t_0', 1+t_0').$$

For (d), if $\gamma(z) \notin \mathcal{M}([-\nu_0, 0])$, it follows from (2.12) that for ε small,

$$\Gamma_\varepsilon(A(t,z)) \le L(t) - \frac{\nu_0}{2}t^N\int_{\mathbb{R}^N}U_0^2\,dx + o(1) < \max_{t\in[0,\infty)} L(t) = E(m_0).$$

Above all, we complete the proof of this lemma. $\square$

From the proof of this lemma, we define $A(0,z)=0$. Then $A(t,z)\colon[0,T]\times\mathcal{L}_\varepsilon\to H^1(\mathbb{R}^N)$ is continuous. Next we give the gradient's lower bound of (2.2) on the annular neighbourhood of $\mathcal{R}(r)$. We define

$$C_\varepsilon=\max_{t\in[0,T],z\in\mathcal{L}_\varepsilon}\Gamma_\varepsilon(A(t,z)). \tag{2.14}$$

From Lemma 2.4 (b) we observe that $\lim\limits_{\varepsilon\to0}C_\varepsilon\le E(m_0)$.

Lemma 2.5. *For some $r_2\in(0,r_1)$ and any $r'\in(0,r_2)$, there exists $\delta_1=\delta_1(r_2,r')>0$ such that for small $\varepsilon>0$ independent of δ_1,*

$$\|\Gamma'_\varepsilon(u)\|_{H^{-1}}\ge\delta_1,$$

for all $u\in\mathcal{R}(2r_2)\setminus\mathcal{R}(r')$ with $\Gamma_\varepsilon(u)\le C_\varepsilon$ and $\varepsilon\Upsilon(u)\in\mathcal{M}([-\nu_0,0])$.

Proof. Assume on the contrary that for some $r'\in(0,r_2)$, there exists $u_\varepsilon\in\mathcal{R}(2r_2)\setminus\mathcal{R}(r')$ with $\Gamma_\varepsilon(u_\varepsilon)\le E(m_0)$ and $\varepsilon\Upsilon(u_\varepsilon)\in\mathcal{M}([-\nu_0,0])$ such that $\Gamma'_\varepsilon(u_\varepsilon)\to0$ as $\varepsilon\to0$. Since $\mathcal{R}(2r_2)\subset\mathcal{S}(2r_2)$, from Lemma 2.2, $u_\varepsilon=U(\,\cdot\,-y_\varepsilon)+\varphi$ for some $y_\varepsilon\in\mathbb{R}^N$, $U\in\widehat{S}$ and $\|\varphi\|\le2r_2$. By (2.7), we have for small $\varepsilon>0$ that

$$\int_{\mathbb{R}^N\setminus B(y_\varepsilon,1/\sqrt{\varepsilon})}|\nabla u_\varepsilon|^2+u_\varepsilon^2\,dx\le5r_2^2. \tag{2.15}$$

Let $3k_\varepsilon^2\le\varepsilon^{-1/2}$ with $k_\varepsilon\to\infty$ as $\varepsilon\to0$.

$$A_{\varepsilon,j}=\{x\in\mathbb{R}^N:\varepsilon^{-1/2}+3jk_\varepsilon\le|x-y_\varepsilon|\le\varepsilon^{-1/2}+3(j+1)k_\varepsilon\}$$

for $j=0,\dots,k_\varepsilon-1$. Then one has

$$\sum_{j=0}^{k_\varepsilon-1}\int_{A_{\varepsilon,j}}|\nabla u_\varepsilon|^2+V_\varepsilon u_\varepsilon^2\,dx\le\int_{\mathbb{R}^N\setminus B(y_\varepsilon,\varepsilon^{-1/2})}|\nabla u_\varepsilon|^2+V_\varepsilon u_\varepsilon^2\,dx\le5r_2^2.$$

Hence there exist $j_\varepsilon\in\{0,\dots,k_\varepsilon-1\}$ such that

$$\int_{A_{\varepsilon,j_\varepsilon}}|\nabla u_\varepsilon|^2+V_\varepsilon u_\varepsilon^2\,dx\le\frac{5r_2^2}{k_\varepsilon}. \tag{2.16}$$

We choose $\chi_\varepsilon(x)\in C_0^\infty(\mathbb{R}^N,[0,1])$ such that $\chi_\varepsilon(x)=1$ for $|x|\le\varepsilon^{-1/2}+(3j_\varepsilon+1)k_\varepsilon$ and $\chi_\varepsilon(x)=0$ for $|x|\ge\varepsilon^{-1/2}+(3j_\varepsilon+2)k_\varepsilon$ and $|\nabla\chi_\varepsilon(x)|\le2\varepsilon^{1/2}$. Then we define $u_\varepsilon^{(1)}(x)=\chi_\varepsilon(x-y_\varepsilon)u_\varepsilon(x)$ and $u_\varepsilon^{(2)}(x)=u_\varepsilon(x)-u_\varepsilon^{(1)}(x)$.

Next we prove that $\|u_\varepsilon^{(2)}\|_\varepsilon\to0$, as $\varepsilon\to0$. By (f_1) and (f_2), for any $\eta>0$ there is $C_\eta>0$ such that

$$f(s)s\le\eta s^2+C_\eta|s|^{p+1}. \tag{2.17}$$

Take $\eta\in(0,\underline{V}/2)$ then

$$\begin{aligned}\Gamma'_\varepsilon(u)u&=M(\|\nabla u\|_2^2)\int_{\mathbb{R}^N}|\nabla u|^2\,dx+\int_{\mathbb{R}^N}V_\varepsilon u^2\,dx-\int_{\mathbb{R}^N}f(u)u\,dx\\&\ge\min\{\underline{M_0},\underline{V}/2\}\|u\|^2-C_{\underline{V}}\|u\|_{p+1}^{p+1}.\end{aligned}$$

Now, using sobolev inequality, we choose $r_2 \in (0, r_1)$ such that, for some $c > 0$,

$$\Gamma_\varepsilon'(u)u \geq \min\{\underline{M_0}, \underline{V}/2\}\|u\|^2 - C'C_{\underline{V}}\|u\|^{p+1} \geq c\|u\|^2 \quad \text{for } \|u\| \leq 4r_2. \tag{2.18}$$

By (2.16) and (2.18), one has

$$\begin{aligned}\Gamma_\varepsilon'(u_\varepsilon)u_\varepsilon^{(2)} = M\big(\|\nabla u_\varepsilon\|_2^2\big)\int_{\mathbb{R}^N}|\nabla u_\varepsilon^{(2)}|^2\,dx + \int_{\mathbb{R}^N} V_\varepsilon\big(u_\varepsilon^{(2)}\big)^2\,dx \\ - \int_{\mathbb{R}^N} f\big(u_\varepsilon^{(2)}\big)u_\varepsilon^{(2)}\,dx + o(1) \geq c\|u_\varepsilon^{(2)}\|^2 + o(1),\end{aligned}$$

and it follows from $\Gamma_\varepsilon'(u_\varepsilon) \to 0$ that $\|u_\varepsilon^{(2)}\| \to 0$. Thus, from $\Gamma_\varepsilon \in C^1$, we have $\Gamma_\varepsilon'(u_\varepsilon^{(1)}) \to 0$. Since u_ε is bounded in $H^1(\mathbb{R}^N)$, denoting $\widetilde{u}_\varepsilon^{(1)}(x) = u_\varepsilon^{(1)}(x+y_\varepsilon)$, we have $\widetilde{u}_\varepsilon^{(1)} \rightharpoonup W$ in $H^1(\mathbb{R}^N)$. By Lemma 2.2 and $\varepsilon\Upsilon(u_\varepsilon) \in \mathcal{M}\big([-\nu_0,0]\big)$, it follows that $\varepsilon y_\varepsilon \to y_0 \in \mathcal{M}([-\nu_0,0])$. Then W solves the equation

$$-\alpha_0\Delta W + V(y_0)W = f(W) \quad \text{in } \mathbb{R}^N, \tag{2.19}$$

where $\alpha_0 = \lim_{\varepsilon\to 0} M\big(\|\nabla u_\varepsilon\|_2^2\big) = \lim_{\varepsilon\to 0} M\big(\big\|\nabla u_\varepsilon^{(1)}\big\|_2^2\big)$.

Then, we claim that $\widetilde{u}_\varepsilon \to W$ in $H^1(\mathbb{R}^N)$. First, we shall show that

$$\lim_{\varepsilon\to 0}\sup_{z\in\mathbb{R}^N}\int_{B_1(z)}\big|\widetilde{u}_\varepsilon^{(1)} - W\big|^2\,dx = 0. \tag{2.20}$$

Supposing on the contrary, there exist $\{z_\varepsilon\} \subset \mathbb{R}^N$ such that

$$\lim_{\varepsilon\to 0}\int_{B_1(z_\varepsilon)}\big|\widetilde{u}_\varepsilon^{(1)} - big|^2\,dx > 0.$$

Since $\widetilde{u}_\varepsilon^{(1)} \to W$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$, $|z_\varepsilon| \to \infty$. Moreover, noting that $\operatorname{supp}\{\widetilde{u}_\varepsilon^{(1)}\} = B(0, 2\varepsilon^{-1/2})$, one has $\varepsilon(y_\varepsilon + z_\varepsilon) \to y_0$ in $\mathbb{R}^N$ and denotes $v_\varepsilon(x) := \widetilde{u}_\varepsilon^{(1)}(x+z_\varepsilon)$. Then $v_\varepsilon \rightharpoonup \widetilde{W} \neq 0$ in $H^1(\mathbb{R}^N)$ and $\widetilde{W}$ satisfies

$$-\alpha_0\Delta\widetilde{W} + V(y_0)\widetilde{W} = f(\widetilde{W}) \quad \text{in } \mathbb{R}^N. \tag{2.21}$$

Set

$$\begin{aligned}L_{\alpha_0,V(y_0)}(u) &= \frac{\alpha_0}{2}\int_{\mathbb{R}^N}|\nabla u|^2\,dx + \frac{V(y_0)}{2}\int_{\mathbb{R}^N}u^2\,dx - \int_{\mathbb{R}^N}F(u)\,dx, \\ E_{\alpha_0}(V(y_0)) &:= \inf\big\{L_{\alpha_0,V(y_0)}(u) : L'_{\alpha_0,V(y_0)}(u) = 0,\ u \in H^1(\mathbb{R}^N)\setminus\{0\}\big\}.\end{aligned}$$

By (f1)–(f3) and the result of [7] and [29], $E_{\alpha_0}(V(x_0))$ is well defined and is increasing with respect to α_0 and $V(y_0)$. From the corresponding Pohozaev identity to (2.21), we have

$$L_{\alpha_0,V(y_0)}(u) = \frac{\alpha_0}{N}\big\|\nabla\widetilde{W}\big\|_2^2 \geq E_{\alpha_0}(V(y_0)).$$

Then, observing that $\underline{M}_0 \leq \alpha_0 \leq \overline{M}_0$, we take large $R > 0$ such that

$$\int_{B_R(0)}\big|\nabla\widetilde{W}\big|^2\,dx \geq \frac{N}{2\alpha_0}E_{\alpha_0}(V(y_0)) \geq \frac{N}{2\alpha_0}E_{\underline{M}_0}(V(y_0)).$$

Moreover, since $|z_\varepsilon| \to \infty$, we can assume that $B_R(z_\varepsilon) \subset \mathbb{R}^N \setminus B\big(y_\varepsilon, 1/\sqrt{\varepsilon}\big)$ for small $\varepsilon > 0$. Then, by (2.15),

$$\begin{aligned}\int_{B_R(0)} |\nabla v_\varepsilon|^2 + V(\varepsilon x + \varepsilon y_\varepsilon + \varepsilon z_\varepsilon) v_\varepsilon^2\, dx \\ \leq \int_{B_R(z_\varepsilon)} |\nabla u_\varepsilon(x + y_\varepsilon)|^2 + V(\varepsilon x + \varepsilon y_\varepsilon) u_\varepsilon^2(x + y_\varepsilon)\, dx \leq 5\, r_2^2.\end{aligned}$$

It follows from the Fatou lemma that

$$\int_{B_R(0)} |\nabla \widetilde{W}|^2\, dx \leq 5\, r_2^2.$$

Then we can take $r_2 > 0$ small such that

$$r_2^2 < \frac{N}{15\alpha_0}\, E_{m_0}(V(y_0))$$

to get a contradiction. Hence by (2.20) and Lemma I.1 in [35], we have

$$\widetilde{u}_\varepsilon^{(1)} \to W \quad \text{in } L^s(\mathbb{R}^N) \text{ for } 2 < s < 2^*. \tag{2.22}$$

Next we prove that

$$\limsup_{\varepsilon \to 0} \int_{\mathbb{R}^N} f\big(\widetilde{u}_\varepsilon^{(1)}\big)\, \widetilde{u}_\varepsilon^{(1)}\, dx \leq \int_{\mathbb{R}^N} f(W) W\, dx. \tag{2.23}$$

For fixed $\eta > 0$ in (2.17), noting that $\widetilde{u}_\varepsilon^{(1)}$ are bounded in $L^2(\mathbb{R}^N)$, one has for any $\xi > 0$,

$$\limsup_{\varepsilon \to 0} \int_{|x| \geq \xi} f\big(\widetilde{u}_\varepsilon^{(1)}\big)\, \widetilde{u}_\varepsilon^{(1)}\, dx \leq \eta\, C + C_\eta \int_{|x| \geq \xi} |W|^{p+1}\, dx.$$

Then we take $\xi_\eta > 0$ large enough such that

$$\begin{gathered}\limsup_{\varepsilon \to 0} \int_{|x| \geq \xi_\eta} f\big(\widetilde{u}_\varepsilon^{(1)}\big)\, \widetilde{u}_\varepsilon^{(1)}\, dx \leq (C+1)\eta, \\ \bigg| \int_{|x| \geq \xi_\eta} f(W) W\, dx \bigg| \leq \eta.\end{gathered} \tag{2.24}$$

Moreover, by Strauss lemma in [44], we have

$$\lim_{\varepsilon \to 0} \int_{|x| \leq \xi_\eta} f\big(\widetilde{u}_\varepsilon^{(1)}\big)\, \widetilde{u}_\varepsilon^{(1)}\, dx = \int_{|x| \leq \xi_\eta} f(W) W\, dx. \tag{2.25}$$

It follows from (2.24) and (2.25) that

$$\begin{aligned}\limsup_{\varepsilon \to 0} \int_{\mathbb{R}^N} f\big(\widetilde{u}_\varepsilon^{(1)}\big)\, \widetilde{u}_\varepsilon^{(1)}\, dx \\ \leq (C+1)\eta + \int_{|x| \leq \xi_\eta} f(W) W\, dx \leq (C+2)\eta + \int_{\mathbb{R}^N} f(W) W\, dx.\end{aligned}$$

Letting $\eta \to 0$, we obtain (2.23). Observing the fact that $\Gamma'_\varepsilon(u_\varepsilon^{(1)}) \to 0$, we have

$$M\big(\|\nabla \widetilde{u}_\varepsilon^{(1)}\|_2^2\big) \int_{\mathbb{R}^N} \big|\nabla \widetilde{u}_\varepsilon^{(1)}\big|^2\,dx + \int_{\mathbb{R}^N} V(\varepsilon x + \varepsilon y_\varepsilon)\big(\widetilde{u}_\varepsilon^{(1)}\big)^2\,dx = \int_{\mathbb{R}^N} f\big(\widetilde{u}_\varepsilon^{(1)}\big)\,\widetilde{u}_\varepsilon^{(1)}\,dx + o(1). \tag{2.26}$$

Then, by (2.23) and Fatou lemma, one has

$$\begin{aligned}
&\alpha_0 \int_{\mathbb{R}^N} |\nabla W|^2\,dx + \int_{\mathbb{R}^N} V(y_0)W^2\,dx \\
&\quad\le \liminf_{\varepsilon\to 0}\bigg(M\big(\big\|\nabla \widetilde{u}_\varepsilon^{(1)}\big\|_2^2\big)\int_{\mathbb{R}^N}\big|\nabla \widetilde{u}_\varepsilon^{(1)}\big|^2\,dx + \int_{\mathbb{R}^N} V(\varepsilon x+\varepsilon y_\varepsilon)\big(\widetilde{u}_\varepsilon^{(1)}\big)^2\,dx\bigg) \\
&\quad\le \limsup_{\varepsilon\to 0}\int_{\mathbb{R}^N} f\big(\widetilde{u}_\varepsilon^{(1)}\big)\widetilde{u}_\varepsilon^{(1)}\,dx \le \int_{\mathbb{R}^N} f(W)W\,dx.
\end{aligned}$$

Thus, since W solves the equation (2.19), we have that, up to subsequence,

$$\begin{aligned}
\lim_{\varepsilon\to 0}&\bigg(M\big(\big\|\nabla \widetilde{u}_\varepsilon^{(1)}\big\|_2^2\big)\int_{\mathbb{R}^N}\big|\nabla \widetilde{u}_\varepsilon^{(1)}\big|^2\,dx + \int_{\mathbb{R}^N} V(\varepsilon x+\varepsilon y_\varepsilon)\big(\widetilde{u}_\varepsilon^{(1)}\big)^2\,dx\bigg) \\
&= \alpha_0\int_{\mathbb{R}^N}|\nabla W|^2\,dx + \int_{\mathbb{R}^N} V(y_0)W^2\,dx.
\end{aligned}$$

Now we claim that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}\big|\nabla \widetilde{u}_\varepsilon^{(1)}\big|^2\,dx = \int_{\mathbb{R}^N}|\nabla W|^2\,dx. \tag{2.27}$$

On the contrary, we assume that

$$h_0 := \lim_{\varepsilon\to 0}\bigg(\int_{\mathbb{R}^N}|\nabla \widetilde{u}_\varepsilon^{(1)}|^2\,dx - \int_{\mathbb{R}^N}|\nabla W|^2\,dx\bigg) > 0.$$

We choose large $R' > 0$ such that

$$\int_{|x|\ge R'} V(y_0)W^2\,dx < \frac{\alpha_0 h_0}{2}.$$

From (2.26), one has that

$$\begin{aligned}
\alpha_0 &\int_{\mathbb{R}^N}|\nabla W|^2\,dx + \int_{\mathbb{R}^N} V(y_0)W^2\,dx \\
&\le \alpha_0\int_{\mathbb{R}^N}|\nabla W|^2\,dx + \int_{|x|\le R'} V(y_0)\,W^2\,dx + \frac{\alpha_0 h_0}{2} \\
&\le \liminf_{\varepsilon\to 0}\bigg(M\big(\big\|\nabla \widetilde{u}_\varepsilon^{(1)}\big\|_2^2\big)\int_{\mathbb{R}^N}|\nabla \widetilde{u}_\varepsilon^{(1)}|^2\,dx \\
&\qquad + \int_{|x|\le R'} V(\varepsilon x + x_\varepsilon)\big(\widetilde{u}_\varepsilon^{(1)}\big)^2\,dx\bigg) - \frac{\alpha_0 h_0}{2} \\
&\le \int_{\mathbb{R}^N} f(W)W\,dx - \frac{\alpha_0 h_0}{2}.
\end{aligned}$$

Observing (2.19), the previous inequality leads to a contradiction. Hence we have (2.27), which also implies that $\alpha_0 = M(\|\nabla W\|_2^2)$ and, up to subsequence,

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N} \big|\nabla \widetilde{u}_\varepsilon^{(1)}\big|^2 + V(\varepsilon x+\varepsilon y_\varepsilon)\big(\widetilde{u}_\varepsilon^{(1)}\big)^2\,dx = \int_{\mathbb{R}^N} |\nabla W|^2 + V(y_0)W^2\,dx \tag{2.28}$$

and W solves the equation

$$-M\big(\big\|\nabla W\big\|_2^2\big)\Delta W + V(y_0)W = f(W) \quad \text{in } \mathbb{R}^N.$$

Above all, noting that $|\Upsilon(u)-y_\varepsilon| \le R_0$, $\varepsilon\Upsilon(u_\varepsilon)\in\mathcal{M}([-\nu_0,0])$, one has $m_0-\nu_0 \le V(y_0)\le m_0$. Since

$$E(m_0)\ge \Gamma_\varepsilon(u_\varepsilon) = \Gamma_\varepsilon\big(u_\varepsilon^{(1)}+u_\varepsilon^{(2)}\big) = \Gamma_\varepsilon\big(\widetilde{u}_\varepsilon^{(1)}\big)+o(1).$$

We have

$$E(m_0-\nu_0)\le E(V(y_0))\le L_{V(y_0)}(W)\le E(m_0).$$

Thus, setting $y_1\in\mathbb{R}^N$ such that $W(y_1) = \max_{x\in\mathbb{R}^N} W(x)$, $\widehat{W} := W(x+y_1)\in\widehat{S}$. Observing that

$$\big\|u_\varepsilon - \widehat{W}(\,\cdot\,-y_1-y_\varepsilon)\big\| \le \big\|\widetilde{u}_\varepsilon^{(1)} - W(\,\cdot\,-y_1-y_\varepsilon)\big\| + \big\|u_\varepsilon^{(2)}\big\| \to 0 \quad \text{as } \varepsilon\to 0,$$

Consequently, we can obtain that for small $\varepsilon>0$, $u_\varepsilon\in\mathcal{R}(r')$ which leads to a contradiction. □

2.5. A map on $\mathcal{R}(2r_2)$ from minimizing problems. In this part, we introduce a map on $\mathcal{R}(2r_2)$, which is obtained by solving minimizing problems on part of $\mathbb{R}^N$. Hence, this map has some good properties: it does not increase the energy of (2.2) and makes functions possess exponential decay away from the center of mass. This map is not special, which was initially proposed in [17].

For $u\in H^1(\mathbb{R}^N)$, some $y\in\mathbb{R}^N$ and $b\in(0,2]$, there exists $R>0$ such that

$$\int_{\mathbb{R}^N\setminus B(y,R)} |\nabla u|^2 + u^2\,dx \le \frac{b^2}{2}. \tag{2.29}$$

Define

$$H_{y,b}^R(u) := \bigg\{v\in H^1(\mathbb{R}^N) : v=u \text{ in } B(y,R) \text{ and } \int_{\mathbb{R}^N\setminus B(y,R)} |\nabla v|^2+v^2\,dx\le b^2\bigg\}.$$

Then we consider the minimization problem on $H_{y,b}^R(u)$:

$$\begin{aligned} I_{y,b}^R(u) = \inf\bigg\{\frac12\widehat{M}\bigg(\int_{B(y,R)}|\nabla u|^2\,dx + \int_D |\nabla v|^2\,dx\bigg) \\ + \frac12\int_D V_\varepsilon v^2\,dx - \int_D F(v)\,dx : v\in H_{y,b}^R(u)\bigg\}, \end{aligned}$$

where $D:=\mathbb{R}^N\setminus B(y,R)$. Arguing as in the proof in [17], we have the following lemma:

LEMMA 2.6. *For some $b \in (0,2]$, there exists a unique minimizer $v_\varepsilon = v_\varepsilon(u,y,R) \in H^R_{y,b}(u)$ of $I^R_{y,b}(u)$ and v_ε solves, for some $\alpha_0 > 0$,*

$$-\alpha_0 \Delta v + V_\varepsilon v = f(v) \quad \text{in } \mathbb{R}^N \setminus B(y,R) \quad \text{and} \quad v = u \quad \text{in } B(y,R). \tag{2.30}$$

Moreover, there exist $C, c > 0$ independent of $\varepsilon > 0$ such that

$$v_\varepsilon(x) \le C \exp(-c\,|x-y| - R - 1) \quad \text{for } |x-y| \ge R+1.$$

PROOF. For any $v \in H^R_{y,b}(u)$,

$$\begin{aligned} M\big(\|\nabla v\|_2^2\big) = M\bigg(\int_{B(y,R)} |\nabla u|^2\,dx + \int_D |\nabla v|^2\,dx\bigg) \\ \le M\bigg(\int_{B(y,R)} |\nabla u|^2\,dx + 4\bigg) := M_1. \end{aligned}$$

Then, combining with (M1) and denoting $\alpha := \int_{B(y,R)} |\nabla u|^2\,dx$ one has the following inequality:

$$\begin{aligned} \min\{\underline{M_0}, \underline{V}\} \int_D |\nabla u|^2 + u^2\,dx &\le \|u\|^2_{M,D} \\ &:= \int_\alpha^{\alpha + \int_D |\nabla u|^2\,dx} M(t)\,dt + \int_D V_\varepsilon u^2\,dx \\ &\le \max\big\{M_1, \overline{V}\big\} \int_D |\nabla u|^2 + u^2\,dx. \end{aligned} \tag{2.31}$$

For simplicity, we denote

$$I_{\varepsilon,D}(v) := \frac{1}{2}\,\widehat{M}\bigg(\alpha + \int_D |\nabla v|^2\,dx\bigg) + \frac{1}{2}\int_D V_\varepsilon v^2\,dx - \int_D F(v)\,dx.$$

By (f1) and (f2), there exist $C_{\underline{V}/16} > 0$ such that

$$\int_D F(v)\,dx \le \frac{\underline{V}}{16}\int_D v^2\,dx + C_{\underline{V}/16}\int_D |v|^{p+1}\,dx.$$

By Sobolev inequality,

$$\int_D |v|^{p+1}\,dx \le C\bigg(\int_D |\nabla v|^2 + v^2\,dx\bigg)^{(p+1)/2}.$$

Hence, for $v \in H^R_{y,b}(u)$ with $\int_D |\nabla v|^2 + v^2\,dx = b^2$, by (2.31), there exists $c > 0$ such that $\|v\|^2_{M,D} = cb^2$. Then, for $b > 0$ small, we have

$$I_{\varepsilon,D}(v) \ge \frac{1}{2}\,\widehat{M}(\alpha) + \bigg(\frac{7}{16} - C\big(cb^2\big)^{(p-1)/2}\bigg)\|v\|^2_{M,D} > \frac{1}{2}\,\widehat{M}(\alpha) + \frac{3}{8}\,cb^2,$$

here we use that

$$\widehat{M}(s+t) = \widehat{M}(s) + \int_s^{s+t} M(t)\,dt.$$

On the other hand, for $v \in H^R_{y,b}(u)$ with $\|v\|^2_{M,D} \le cb^2/2$, one has

$$I_{\varepsilon,D}(v) \le \frac{1}{2}\,\widehat{M}(\alpha) + \bigg(\frac{9}{16} + C\bigg(\frac{cb^2}{2}\bigg)^{(p-1)/2}\bigg)\|v\|^2_{M,D} < \frac{1}{2}\,\widehat{M}(\alpha) + \frac{3}{8}\,cb^2,$$

for $b > 0$ small. Consequently, we observe that the minimizer of $I_{\varepsilon,D}(v)$ is obtained in the interior of $H^R_{y,b}(u)$.

Let v_n be the minimizing sequence, then

$$\int_D |\nabla v_n|^2 + V_\varepsilon v_n^2\, dx \le b^2.$$

Henc,e up to subsequence, $v_n \rightharpoonup v_\varepsilon$ in $H^1(\mathbb{R}^N)$. Similar with the proof of (2.23), we can obtain

$$\limsup_{n\to\infty} \int_D F(v_n)\, dx \le \int_D F(v_\varepsilon)\, dx.$$

Combining with weakly lower semi-continuity, v_ε minimizes $I_{\varepsilon,D}$, i.e. solves (2.30), where $\alpha_0 := \lim\limits_{n\to\infty} M\big(\alpha + \int_D |\nabla v_n|^2\, dx\big)$.

Next we prove the minimizer is unique. We assume that v^*_ε is the other solution, then we denote $Z = v^*_\varepsilon - v_\varepsilon$. Through calculating, we have for some $c > 0$ and $\lambda \in [0,1]$,

$$c\int_D |\nabla Z|^2\, dx + \int_D Z^2\, dx \le \int_D f'(Z^\lambda) Z^2\, dx,$$

where $Z^\lambda = \lambda v^*_\varepsilon + (1-\lambda)v_\varepsilon$. Then by (f1), (f2) and Remark 2.6 in [14], for any $\eta > 0$, there exists $C_\eta > 0$ such that $|f'(t)t^2| \le \eta t^2 + C_\eta t^{2^*}$. Then we have, for some $C > 0$, such that

$$\int_D |\nabla Z|^2 + Z^2\, dx \le C\int_D \big(|v^*_\varepsilon|^{4/(N-2)} + |v_\varepsilon|^{4/(N-2)}\big) Z^2\, dx.$$

By Hölder inequality, one has

$$\int_D \big(|v^*_\varepsilon|^{4/(N-2)} + |v_\varepsilon|^{4/(N-2)}\big) Z^2\, dx \le \big(\|v^*_\varepsilon\|^{4/(N-2)}_{2^*} + \|v_\varepsilon\|^{4/(N-2)}_{2^*}\big)\|Z\|^2_{2^*}.$$

Hence, by Sobolev inequality, we have for some $C > 0$

$$\int_D |\nabla Z|^2 + Z^2\, dx \le C\big(b^2 \max\{1, m_0^{-1}\}\big)^{4/(N-2)} \int_D |\nabla Z|^2 + Z^2\, dx.$$

Consequently, for small $b > 0$, we have $Z \equiv 0$.

Finally, we prove that v_ε has the exponential decay. First we claim that for any fixed $\varepsilon > 0$, $\lim\limits_{|x|\to\infty} v_\varepsilon(x) = 0$. In fact, by absolute continuity of integral, it follows that

$$\lim_{R\to\infty} \int_{|x|\ge R} v_\varepsilon^2 + v_\varepsilon^{2^*}\, dx = 0. \tag{2.32}$$

Noting that $\underline{M_0} \le \alpha_0 \le M_1$ and

$$-\Delta v_\varepsilon + \frac{V_\varepsilon}{\alpha_0}\, v_\varepsilon = \frac{1}{\alpha_0}\, f(v_\varepsilon) \quad \text{in } \mathbb{R}^N \setminus B(y,R).$$

By argument of regularization in [8] (also see in [36]), we have $v_\varepsilon \in L^s(\mathbb{R}^N)$ and $\|v_\varepsilon\|_s \le C_s\|v_\varepsilon\|$ for any $s \ge 2$. Thus, for some $t > N$, $\|f(v_\varepsilon(x))\|_{t/2} \le C$. By regularization's estimates in [25] (also see in [36]), one has for any $B_2(z) \subset \mathbb{R}^N \setminus B(y, R+1)$,

$$\sup_{x\in B_1(z)} v_\varepsilon(x) \le C\big(\|v_\varepsilon\|_{L^2(B_2(z))} + \|g\|_{t/2}\big). \tag{2.33}$$

Hence, by (2.32) and (2.33), we have $\lim\limits_{|x|\to\infty} v_\varepsilon(x) = 0$ independent of small $\varepsilon > 0$. Then, for $y \in O$, there exist $R > 0$ such that

$$\frac{f(v_\varepsilon)}{\alpha_0} \le \frac{C}{\alpha_0}(|v_\varepsilon| + |v_\varepsilon|^p) < \frac{V}{2\alpha_0} \quad \text{for } x \in \mathbb{R}^N \setminus B(y, R+1).$$

Thus, through calculating, one has

$$-\Delta v_\varepsilon + \frac{V}{2\alpha_0}\, v_\varepsilon \le 0.$$

Moreover, take a function $d(x) \in C^2(\mathbb{R}^N \setminus B(y, R))$ such that for $r(x) := |x - y|$,

$$\|d(x) - r(x)\|_{C^2(\mathbb{R}^N\setminus B(y,R))} \le \frac{1}{10}. \tag{2.34}$$

Then we choose $c > 0$ independent of $\varepsilon > 0$ such that

$$\begin{aligned}\Delta \exp\big(-c(d(x)-R-1)\big) &- \frac{V}{2\alpha_0}\exp(-c(d(x)-R-1)) \\ &\le \left(c\Delta d + c^2|\nabla d|^2 - \frac{V}{2\alpha_0}\right)\exp(-c(d(x)-R-1)) < 0.\end{aligned}$$

And we take $C > 0$ such that

$$v_\varepsilon(x) \le C\exp(-c(d(x)-R-1)) \quad \text{on } \partial B(y, R+1).$$

Setting $\varphi := C\exp(-c(d(x)-R-1)) - v_\varepsilon(x)$, we have

$$\begin{cases} -\Delta\varphi + \dfrac{V}{2\alpha_0}\,\varphi \ge 0 & \text{in } \mathbb{R}^N \setminus B(y, R+1), \\ \varphi \ge 0 & \text{on } \partial B(y, R+1), \\ \lim\limits_{|x|\to\infty} \varphi(x) = 0. \end{cases}$$

According to maximum principle, one has $\varphi \ge 0$ in $\mathbb{R}^N \setminus B(y, R+1)$. Consequently, from (2.34), there exist $C, c > 0$ such that

$$v_\varepsilon(x) \le C\exp(-c(|x-y| - R - 1)) \quad \text{for } x \in \mathbb{R}^N \setminus B(y, R+1)$$

independent of small $\varepsilon > 0$. $\square$

For any $u \in \mathcal{R}(2r_2)$, from the proof of (2.15) and Lemma 2.2, we have that, for small $\varepsilon > 0$,

$$\int_{\mathbb{R}^N\setminus B(\Upsilon(u),1/\sqrt{\varepsilon})} |\nabla u|^2 + u^2\, dx \le 6\, r_2^2.$$

Then, in Lemma 2.6, taking $y = \Upsilon(u)$, $R = 1/\sqrt{\varepsilon}$ and $b = 2\sqrt{3}r_2$, there exists unique minimizer

$$\tau(u) := v_\varepsilon(u, \Upsilon(u), 1/\sqrt{\varepsilon}) \quad \text{for } I^R_{y,b}(u).$$

Thus, $\tau(u) \in \mathcal{R}(2r_2)$ and $\Gamma_\varepsilon\big(\tau(u)\big) \le \Gamma_\varepsilon(u)$.

From the next lemma, we can see that the center of mass of $\tau(u)$ does not go far away.

Lemma 2.7. *For small $\varepsilon > 0$, $|\Upsilon\big(\tau(u)\big) - \Upsilon(u)| \le 2R_0$ for any $u \in \mathcal{R}(2r_2)$, where R_0 is given in Lemma 2.2.*

Proof. For any $u \in \mathcal{R}(2r_2)$, there exist $U \in \widehat{S}$, $y \in \mathbb{R}^N$ such that $|\Upsilon(u) - y| \le R_0$ and

$$\int_{B(\Upsilon(u),1/\sqrt{\varepsilon})} |\nabla(u - U(\,\cdot\, - y))|^2 + |u - U(\,\cdot\, - y)|^2\, dx \le \frac{(2r_2)^2}{2}.$$

Moreover, from the definition of $\tau(u)$ and Lemma 2.6, we observe that $\tau(u) \in \mathcal{R}(2r_2)$ and $\tau(u) = u$ on $B(\Upsilon(u), 1/\sqrt{\varepsilon})$. Then $\tau(u)$ satisfies either

$$\int_{\mathbb{R}^N \setminus B(\Upsilon(u),1/\sqrt{\varepsilon})} |\nabla\tau(u)|^2 + \tau^2(u) - 2F(u)\, dx \le \frac{(2r_2)^2}{2},$$

or, denoting $\alpha_\varepsilon := \int_{B(\Upsilon(u),1/\sqrt{\varepsilon})} |\nabla u|^2\, dx$ and $D_\varepsilon := \mathbb{R}^N \setminus B(\Upsilon(u), 1/\sqrt{\varepsilon})$,

$$\begin{aligned}\int_{\alpha_\varepsilon}^{\alpha_\varepsilon + \int_{D_\varepsilon} |\nabla\tau(u)|^2\, dx} M(t)\, dt + \int_{D_\varepsilon} V_\varepsilon \tau^2(u)\, dx \\ - 2\int_{D_\varepsilon} F(\tau(u))\, dx \le \frac{(2r_2)^2}{2} \min\{\underline{V}, \underline{M_0}\}.\end{aligned}$$

By (2.9), one has

$$\int_{D_\varepsilon} |\nabla\tau(u)|^2 + |\tau(u)|^2\, dx \le \frac{(2r_2)^2}{2(1-q)} \le (2r_2)^2.0$$

Hence, for small $\varepsilon > 0$,

$$\begin{aligned}\|\tau(u) - U(\,\cdot\, - y)\|^2 \le& \int_{B(\Upsilon(u),1/\sqrt{\varepsilon})} |\nabla(u - U(\,\cdot\, - y))|^2 + (u - U(\,\cdot\, - y))^2\, dx \\ &+ \int_{\mathbb{R}^N \setminus B(\Upsilon(u),1/\sqrt{\varepsilon})} |\nabla\tau(u)|^2 + \tau^2(u)\, dx + o(1) \\ \le& 2(r_2)^2 + (2r_2)^2 + r_2^2 = 7r_2^2.\end{aligned}$$

For any $z \in \mathbb{R}^N$ with $|z - \Upsilon(u)| \ge 2R_0$, noting that

$$|z - y| \ge |z - \Upsilon(u)| - |\Upsilon(u) - y| \ge R_0,$$

one has that, for any $\widetilde{U} \in \widehat{S}$,

$$\begin{aligned}
&\big\|\tau(u) - \widetilde{U}(\,\cdot\, - z)\big\|_{H^1(|x-z|\le R_0/2)} \\
&\quad \ge \big\|U(\,\cdot\, - y) - \widetilde{U}(\,\cdot\, - z)\big\|_{H^1(|x-z|\le R_0/2)} - \|\tau(u) - U(\,\cdot\, - y)\| \\
&\quad \ge \frac{3}{4} r_* - \frac{1}{16} r_* - \sqrt{7}\, r_2 \ge \frac{1}{2} r_*,
\end{aligned}$$

here we use that $r_* \ge 16\, r_2$. Then, from Lemma 2.2, $\Psi(d(z,u)) = 0$. Thus $\Upsilon(\tau(u)) \in B(\Upsilon(u), 2R_0)$. □

Since there exist $(t,z) \in [0,T] \times \mathcal{L}_\varepsilon$ such that $A(t,z) \notin \mathcal{R}(2r_2)$, we will use $A(t,z)$ to begin with the iteration, thus we extend continuously the center of mass $\Upsilon(u)$ onto $\mathcal{R}(2r_2) \cup \{A(t,z) : (t,z) \in [0,T] \times \mathcal{L}_\varepsilon\}$ such that, for any $A(t,z) \notin \mathcal{R}(2r_2)$,

$$\left|\Upsilon(A(t,z)) - \frac{\gamma(z)}{\varepsilon}\right| \le 3R_0 \tag{2.35}$$

and

$$\Upsilon(A(t,z)) = \frac{\gamma(z)}{\varepsilon} \quad \text{for } (t,z) \in [0,T] \times \mathcal{N}(L_0),$$

where $\mathcal{N}(L_0) \subset \mathcal{L}_\varepsilon$ is a neighbuorhood of L_0. Moreover, observing the exponential decay of $A(t,z)$ and the proof of Lemma 2.7, we can extend the map τ continuously on $\mathcal{R}(2r_2) \cup \{A(t,z) : (t,z) \in [0,T] \times \mathcal{L}_\varepsilon\}$ such that if $A(t,z) \notin \mathcal{R}(2r_2)$ or $\varepsilon\Upsilon(A(t,z)) \notin \mathcal{M}([-\nu_0, 0])$,

$$\tau(A(t,z)) = A(t,z). \tag{2.36}$$

2.6. Translation operator through the deformation flow of $V(x)$. This part states properties of map $\mathcal{T}_\varepsilon(l,u)$. Since combined with the deformation flow of $V(x)$, functional Γ_ε does not increase energy. Further, when center of mass $\Upsilon(u)$ stays away from local maximal points of $V(x)$, $\Gamma_\varepsilon(\mathcal{T}_\varepsilon(l,u))$ decreases strictly as l varies (see Lemma 2.10). Moreover, through choosing proper cut-off functions, $\mathcal{R}(2r_2)$ is invariant under this translation operator (see Lemma 2.9). Also, center of mass does not go fa away after this translation (see Lemma 2.8).

Take large $R_1 > 0$ such that $\mathcal{M}\big([-\nu_0, 0]\big) \subset B(0, R_1)$ and choose three cut-off functions for the translation operator:

- $\psi_\varepsilon \in C_0^\infty(\mathbb{R}^N, [0,1])$ with $\psi_\varepsilon(x) = 1$ for $|x| \le 2R_1/\varepsilon$, $\psi_\varepsilon(x) = 0$ for $|x| \ge 3R_1/\varepsilon$ and $|\nabla\psi_\varepsilon| \le 2\varepsilon$;
- $\kappa_1 \in C_0^\infty(\mathbb{R}^N, [0,1])$ with $\kappa_1(x) = 1$ for $x \in \mathcal{M}([-3\nu_1, -2\nu_1])$ and $\kappa_1(x) = 0$ for $x \in \mathcal{M}([-\nu_1, 0]) \cup \mathcal{M}([-\nu_0, -4\nu_1])$, where we choose small $\nu_1 \in (0, \nu_0)$ such that $4\nu_1 < \nu_0$.
- $\kappa_2 \in C^2(\mathcal{R}(2r_2), [0,1])$ with $\kappa_2(u) = 1$ for $u \in \mathcal{R}(r_2/6)$ and $\kappa_2(u) = 0$ for $u \notin \mathcal{R}(r_2/4)$.

Then we define a function $\zeta_0\colon [0,1]\times\mathbb{R}^N\times H^1(\mathbb{R}^N)\to\mathbb{R}^N$ by

$$\zeta_0(l,x,u)=\zeta(\kappa_1(x)\kappa_2(u)l,x),$$

where from (c) of (V3), we take $l_0\in(0,1]$ such that

$$|V(\zeta(l,x))-V(x)|\le\frac{\nu_1}{10}\quad\text{for any } l\in[0,l_0] \text{ and } x\in\mathcal{M}([-\nu_0,0]). \tag{2.37}$$

We define translation operator $\mathcal{T}_\varepsilon\colon [0,l_0]\times\mathcal{R}(2r_2)\to H^1(\mathbb{R}^N)$ by

$$\mathcal{T}_\varepsilon(l,u)(x):=(1-\psi_\varepsilon(x))u(x)+(\psi_\varepsilon u)\bigg(x-\frac{\zeta_0(l,\varepsilon\Upsilon(u),u)}{\varepsilon}+\Upsilon(u)\bigg).$$

From this definition, we note that if $u\notin\mathcal{R}(r_2/4)$ or $\varepsilon\Upsilon(u)\in\mathcal{M}([-\nu_0,-4\nu_1]\cup[-\nu_1,0])$, $\zeta_0(l,\varepsilon\Upsilon(u),u)=\varepsilon\Upsilon(u)$, $\mathcal{T}_\varepsilon(l,u)=u$ for any $l\in[0,l_0]$. For simplicity, denote

$$d(l,u)=\frac{\zeta_0(l,\varepsilon\Upsilon(u),u)}{\varepsilon}-\Upsilon(u).$$

Lemma 2.8. *For $u\in\mathcal{R}(2r_2)$,*

$$\bigg|\Upsilon(\mathcal{T}_\varepsilon(l,u))-\frac{\zeta_0(l,\varepsilon\Upsilon(u),u)}{\varepsilon}\bigg|\le 2R_0.$$

Proof. For $u\in\mathcal{R}(2r_2)\setminus\mathcal{R}(r_2/4)$, $\zeta_0(l,\varepsilon\Upsilon(u),u)=\varepsilon\Upsilon(u)$ and $\mathcal{T}_\varepsilon(l,u)=u$, the conclusion follows. Moreover, for $u\in\mathcal{R}(\frac{r_2}{4})$, from Lemma 2.3, we know that $u\in\mathcal{S}(r_2/(4(1-q)))$. Then, from Lemma 2.2, there exist $U\in\widehat{S}$ and $y\in\mathbb{R}^N$ such that

$$|\Upsilon(u)-y|<R_0\quad\text{and}\quad\|u-U(\,\cdot\,-y)\|\le\frac{r_2}{4(1-q)}.$$

By (2.7), for small $\varepsilon>0$,

$$\int_{|x|\ge 2R_1/\varepsilon}|\nabla u|^2+u^2\,dx\le\frac{r_2}{4(1-q)}+o(1).$$

Then, for small $\varepsilon>0$,

$$\begin{aligned}\|\mathcal{T}_\varepsilon(l,u)&-U(\,\cdot\,-y-d(l,u))\|\\&\le\|(1-\psi_\varepsilon)u\|+\|(\psi_\varepsilon u)(\,\cdot\,-d(l,u))-U(\,\cdot\,-y-d(l,u))\|\\&\le 2\|(1-\psi_\varepsilon)u\|+\|u-U(\,\cdot\,-y)\|\le\frac{3r_2}{4(1-q)}+o(1)\le 2r_2.\end{aligned}$$

Hence, by Lemma 2.2, $|\Upsilon(\mathcal{T}_\varepsilon(l,u))-y-d(l,u)|\le R_0$. Thus

$$\bigg|\Upsilon(\mathcal{T}_\varepsilon(l,u))-\frac{\zeta_0(l,\varepsilon\Upsilon(u),u)}{\varepsilon}\bigg|\le|\Upsilon(\mathcal{T}_\varepsilon(l,u))-y-d(l,u)|+|y-\Upsilon(u)|\le 2R_0.\ \square$$

Next we prove that $\mathcal{R}(2r_2)$ is invariant under $\mathcal{T}_\varepsilon\big(l,\tau(u)\big)$. For simplicity, denote

$$d(l,\tau)=\frac{\zeta_0(l,\varepsilon\Upsilon(\tau(u)),\tau(u))}{\varepsilon}-\Upsilon(\tau(u)).$$

LEMMA 2.9. *For* $u \in \mathcal{R}(r_2/4)$, $\mathcal{T}_\varepsilon(l, \tau(u)) \in \mathcal{R}(7r_2/8)$ *for any* $l \in [0, l_0]$. *Thus, for any* $u \in \mathcal{R}(2r_2)$ *and* $l \in [0, l_0]$, $\mathcal{T}_\varepsilon(l, \tau(u)) \in \mathcal{R}(2r_2)$.

PROOF. For $u \in \mathcal{R}(r_2/4)$, $\tau(u) \in \mathcal{R}(r_2/4)$. By Llemmas 2.3 and 2.2, there exist $U \in \widehat{S}$ and $y \in \mathbb{R}^N$ such that

$$\|\tau(u) - U(\,\cdot\, - y)\| \le \frac{r_2}{4(1-q)} \quad \text{and} \quad |\Upsilon(\tau(u)) - y| < R_0.$$

Then, we note that $\varepsilon y + \varepsilon d(l, \tau) \in \mathcal{M}([-\nu_0, 0])$ for small $\varepsilon > 0$, and from Lemma 2.6, there exist $C, c > 0$ such that

$$\int_{|x| \ge R_1/\varepsilon} |\nabla((1-\psi_\varepsilon)\tau(u))|^2 + V_\varepsilon((1-\psi_\varepsilon)\tau(u))^2\, dx \le C \exp\left(-\frac{c}{\varepsilon}\right). \tag{2.38}$$

Thus, for small $\varepsilon > 0$,

$$\begin{aligned}
\|\mathcal{T}_\varepsilon(l, \tau(u)) &- U(\,\cdot\, - y - d(l,\tau))\| \\
&\le \|(1-\psi_\varepsilon)\tau(u)\| + \|(\psi_\varepsilon \tau(u))(\,\cdot\, - d(l,\tau)) - U(\,\cdot\, - y - d(l,\tau))\| \\
&\le 2\|(1-\psi_\varepsilon)\tau(u)\| + \|\tau(u) - U(\,\cdot\, - y)\| \le \frac{(3\sqrt{2}-4)r_2}{16(1+q)} + \frac{r_2}{4(1-q)}.
\end{aligned}$$

Then, observing

$$\sqrt{2}(1+q)\left(\frac{(3\sqrt{2}-4)r_2}{16(1+q)} + \frac{r_2}{4(1-q)}\right) \le \frac{7}{8}\, r_2,$$

it follows from Lemma 2.3 that

$$\mathcal{T}_\varepsilon(l, \tau(u)) \in \mathcal{R}\left(\frac{7}{8}\, r_2\right).$$

Consequently, for $u \in \mathcal{R}(2r_2)$, if $\tau(u) \in \mathcal{R}(r_2/4)$, $\mathcal{T}_\varepsilon(l, \tau(u)) \in \mathcal{R}(2r_2)$ for any $l \in [0, l_0]$. If $\tau(u) \notin \mathcal{R}(r_2/4)$, $\kappa_2(\tau(u)) = 0$ and $\mathcal{T}_\varepsilon(l, \tau(u)) = \tau(u) \in \mathcal{R}(2r_2)$ for any $l \in [0, l_0]$. □

LEMMA 2.10. *For small* $\varepsilon > 0$,

(a) *if* $u \in \mathcal{R}(2r_2)$ *and* $\varepsilon\Upsilon(u) \in \mathcal{M}([-\nu_0, 0])$, $\Gamma_\varepsilon(\mathcal{T}_\varepsilon(l, \tau(u)))$ *is non-increasing on* $[0, l_0]$,
(b) *if* $u \in \mathcal{R}(r_2/10)$ *and* $\varepsilon\Upsilon(u) \in \mathcal{M}([-3\nu_1, -2\nu_1])$, *there exists* $\mu_0 > 0$ *such that* $\Gamma_\varepsilon(\mathcal{T}_\varepsilon(l_0, \tau(u))) - \Gamma_\varepsilon(\mathcal{T}_\varepsilon(0, \tau(u))) \le -\mu_0$.

Proof. Set $0 \le l < l+h \le l_0$. Denote $x_\tau := \varepsilon\Upsilon(\tau(u))$ and $\psi_\varepsilon^C := 1-\psi_\varepsilon$. We compute that

$$
\begin{aligned}
&\Gamma_\varepsilon(\mathcal{T}_\varepsilon(l+h,\tau(u))) - \Gamma_\varepsilon(\mathcal{T}_\varepsilon(l,\tau(u)))\\
&= \frac{1}{2}\int_{\alpha_1+\alpha_2}^{\alpha_1+\alpha_2+\alpha_3} M(t)\,dt - \frac{1}{2}\int_{\alpha_1+\alpha_2}^{\alpha_1+\alpha_2+\alpha_3'} M(t)\,dt\\
&\quad + \int_{\mathbb{R}^N} V_\varepsilon \cdot \psi_\varepsilon^C\tau(u)\cdot[(\psi_\varepsilon\tau(u))(\,\cdot\,-d(l+h,\tau)) - (\psi_\varepsilon\tau(u))(\,\cdot\,-d(l,\tau))]\,dx\\
&\quad + \frac{1}{2}\int_{\mathbb{R}^N}[V(\varepsilon\,\cdot+\zeta_0(l+h,x_\tau,\tau(u)))\\
&\qquad\qquad - V(\varepsilon\,\cdot+\zeta_0(l,x_\tau,\tau(u)))](\psi_\varepsilon\tau(u))^2\bigg(\,\cdot+\frac{x_\tau}{\varepsilon}\bigg)\,dx\\
&\quad - \int_{\mathbb{R}^N}[F(\mathcal{T}_\varepsilon(l+h,\tau(u))) - F(\mathcal{T}_\varepsilon(l,\tau(u)))]\,dx := T_1 - T_2 + T_3 + T_4 - T_5,
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha_1 &= \|\nabla(\psi_\varepsilon^C\tau(u))\|_2^2, \qquad \alpha_2 = \|\nabla(\psi_\varepsilon\tau(u))\|_2^2,\\
\alpha_3 &= \int_{\mathbb{R}^N}\nabla(\psi_\varepsilon^C\tau(u))\cdot\nabla(\psi_\varepsilon\tau(u))(\,\cdot\,-d(l+h,\tau))\,dx,\\
\alpha_3' &= \int_{\mathbb{R}^N}\nabla(\psi_\varepsilon^C\tau(u))\cdot\nabla(\psi_\varepsilon\tau(u))(\,\cdot\,-d(l,\tau))\,dx.
\end{aligned}
$$

Recalling that $\zeta_0(l,x_\tau,\tau(u)) = \zeta(\kappa_1(x_\tau)\kappa_2(\tau(u))l,x_\tau)$, if $x_\tau \in \mathcal{M}([-\nu_1,0]) \cup \mathcal{M}([-\nu_0,-4\nu_1])$ or $\tau(u) \in \mathcal{R}(2r_2)\setminus\mathcal{R}(r_2/4)$, $\mathcal{T}_\varepsilon(l,\tau(u)) = \tau(u)$ for any $l \in [0,l_0]$ and the conclusion hold. Thus we consider the case that $\kappa_1(x_\tau)\kappa_2(\tau(u)) > 0$, that is $\tau(u) \in \mathcal{R}(r_2/4)$ and $x_\tau \in \mathcal{M}([-4\nu_1,-\nu_1])$, then we compute that, denoting $H = \kappa_1(x_\tau)\kappa_2(\tau(u))h$,

$$
\frac{1}{H}[\Gamma_\varepsilon(\mathcal{T}_\varepsilon(l+h,\tau(u))) - \Gamma_\varepsilon(\mathcal{T}_\varepsilon(l,\tau(u)))] = \frac{T_1-T_2}{H} + \frac{T_3}{H} + \frac{T_4}{H} - \frac{T_5}{H}.
$$

Since $\varepsilon d(l,\tau)$, $\varepsilon d(l+h,\tau) \in B(0,R_1)$, then it follows from (2.38), elliptic estimate for the solution $\tau(u)$ of (2.30) and the property (iii) of (V3) that for small $\varepsilon > 0$

$$
\frac{|T_1-T_2|}{H} \le \overline{M}_0\,\frac{|\alpha_3-\alpha_3'|}{H} \le o(1) \quad\text{and}\quad \frac{|T_3|}{H} \le o(1)
$$

and

$$
\begin{aligned}
\frac{T_5}{H} &= \frac{1}{H}\bigg|\int_{\mathbb{R}^N\setminus B(0,2R_1/\varepsilon)} F\big((\psi_\varepsilon^C\tau(u))(\,\cdot\,+d(l+h,\tau)) + \psi_\varepsilon\tau(u)\big)\,dx\\
&\qquad - \int_{\mathbb{R}^N\setminus B(0,2R_1/\varepsilon)} F\big((\psi_\varepsilon^C\tau(u))(\,\cdot\,+d(l,\tau)) + \psi_\varepsilon\tau(u)\big)\,dx\bigg|\\
&= \int_{\mathbb{R}^N\setminus B(0,2R_1/\varepsilon)}\big|f\big((\psi_\varepsilon^C\tau(u))(\,\cdot\,+d(l,\tau))\\
&\qquad\qquad + \psi_\varepsilon\tau(u) + \theta g(l,h,\tau)\big)\big|\frac{|g(l,h,\tau)|}{H}\,dx,
\end{aligned}
$$

where $\theta \in (0,1)$ and $g(l,h,\tau) = \big(\psi_\varepsilon^C \tau(u)\big)(\,\cdot\,+d(l+h,\tau)) - \big(\psi_\varepsilon^C \tau(u)\big)(\,\cdot\,+d(l,\tau))$.

Next we consider the term T_4/H. Denoting

$$\widehat{V_\varepsilon} := V(\varepsilon x + \zeta_0(l+h, x_\tau, \tau(u))) - V(\varepsilon x + \zeta_0(l, x_\tau, \tau(u))),$$

one has that

$$\begin{aligned}\frac{T_4}{H} &= \frac{1}{2H}\int_{|x|\geq \varepsilon^{-3/4}} \widehat{V_\varepsilon}(\psi_\varepsilon \tau(u))^2\Big(\,\cdot\, + \frac{x_\tau}{\varepsilon}\Big)\,dx \\ &\qquad + \frac{1}{2H}\int_{|x|\leq \varepsilon^{-3/4}} \widehat{V_\varepsilon}(\psi_\varepsilon \tau(u))^2\Big(\,\cdot\, + \frac{x_\tau}{\varepsilon}\Big)\,dx := T_{V_1} + T_{V_2}.\end{aligned}$$

For $|T_{V_1}|$, denoting $x_{l+h} := x - \zeta_0(l+h, x_\tau, \tau(u))/\varepsilon + x_\tau/\varepsilon$ and $x_l := x - \zeta_0(l, x_\tau, \tau(u))/\varepsilon + x_\tau/\varepsilon$,

$$\begin{aligned}|T_{V_1}| &= \frac{1}{2H}\bigg|\int_{|x-\zeta_0(l+h,x_\tau,\tau(u))/\varepsilon|\geq\varepsilon^{-3/4}} V_\varepsilon(\psi_\varepsilon\tau(u))^2(x_{l+h})\,dx \\ &\qquad - \int_{|x-\zeta_0(l,x_\tau,\tau(u))/\varepsilon|\geq\varepsilon^{-3/4}} V_\varepsilon(\psi_\varepsilon\tau(u))^2(x_l)\,dx\bigg| \\ &\leq \frac{1}{2H}\int_{D_1} V_\varepsilon(\psi_\varepsilon\tau(u))^2(x_{l+h})\,dx + \frac{1}{2H}\int_{D_2} V_\varepsilon(\psi_\varepsilon\tau(u))^2(x_l)\,dx \\ &\qquad + \frac{1}{2H}\int_{|x-\zeta_0(l,x_\tau,\tau(u))/\varepsilon|\geq\varepsilon^{-3/4}} V_\varepsilon\big|(\psi_\varepsilon\tau(u))^2(x_{l+h}) - (\psi_\varepsilon\tau(u))^2(x_l)\big|\,dx \\ &:= A_1 + A_2 + A_3,\end{aligned}$$

where

$$\begin{aligned}D_1 = \bigg\{x\in\mathbb{R}^N : \bigg|x - \frac{\zeta_0(l+h,x_\tau,\tau(u))}{\varepsilon}\bigg| \geq \varepsilon^{-3/4} \\ \text{and} \quad \bigg|x - \frac{\zeta_0(l,x_\tau,\tau(u))}{\varepsilon}\bigg| \leq \varepsilon^{-3/4}\bigg\},\end{aligned}$$

$$\begin{aligned}D_2 = \bigg\{x\in\mathbb{R}^N : \bigg|x - \frac{\zeta_0(l,x_\tau,\tau(u))}{\varepsilon}\bigg| \geq \varepsilon^{-3/4} \\ \text{and} \quad \bigg|x - \frac{\zeta_0(l+h,x_\tau,\tau(u))}{\varepsilon}\bigg| \leq \varepsilon^{-3/4}\bigg\}.\end{aligned}$$

Noting that $h \geq H > 0$, we can take $h > 0$ small enough such that, for any $x \in D_1$,

$$\begin{aligned}\bigg|x - \frac{\zeta_0(l,x_\tau,\tau(u))}{\varepsilon}\bigg| &\geq \bigg|x - \frac{\zeta_0(l+h,x_\tau,\tau(u))}{\varepsilon}\bigg| \\ &\quad - \frac{1}{\varepsilon}\,|\zeta_0(l+h,x_\tau,\tau(u)) - \zeta_0(l,x_\tau,\tau(u))| \geq \varepsilon^{-3/4} - \mu H\varepsilon^{-1/2} \geq \frac{1}{2}\,\varepsilon^{-3/4},\end{aligned}$$

then we have, for any $x \in D_1 \cup D_2$,

$$\bigg|x - \frac{\zeta_0(l,x_\tau,\tau(u))}{\varepsilon}\bigg| \geq \frac{1}{2}\varepsilon^{-3/4}.$$

Moreover, from the definition of D_1 and D_2 and (c) of (V3), there exist $M>0$ dependent of N such that

$$|D_1|+|D_2|\le \frac{M}{\varepsilon^{3N/4}}\,|\zeta_0(l+h,x_\tau,\tau(u))-\zeta_0(l,x_\tau,\tau(u))|\le M\mu H\varepsilon^{-3N/4}.$$

Hence, observing that $\varepsilon d(l+h,x_\tau,\tau(u))$, $\varepsilon d(l,x_\tau,\tau(u))\in\mathcal{M}([-\nu_0,0])$ and the decay property of $\tau(u)$ on $D_1\cup D_2$, we have $|A_1|+|A_2|\le o(1)$ for small $\varepsilon>0$.

For A_3, it follows from the decay property of $\tau(u)$, standard C^2-estimate for $\tau(u)$ of (2.30) (see [25]) and (c) of (V3) that

$$\begin{aligned}\limsup_{h\to0}\,&|A_3|\\ &\le\frac{\mu}{\varepsilon}\int_{|x-\zeta_0(l,x_\tau,\tau(u))/\varepsilon|\ge\varepsilon^{-3/4}}V_\varepsilon|(\psi_\varepsilon\tau(u))(x_l)|\cdot|\nabla(\psi_\varepsilon\tau(u))(x_l)|\,dx\le o(1).\end{aligned}$$

On the other hand, for T_{V_2}, by (iv) in (V3) and $x_\tau\in\mathcal{M}([-4\nu_1,-\nu_1])$, we have that, for small $\varepsilon>0$ and $H_0:=\kappa_1(x_\tau)\kappa_2(\tau(u))l$,

$$\begin{aligned}\limsup_{h\to0}T_{V_2}=\limsup_{h\to0}\frac{1}{2H}\int_{|x|\le\varepsilon^{-3/4}}&[V(\varepsilon x+\zeta(H_0+H,x_\tau))\\ &-V(\varepsilon x+\zeta(H_0,x_\tau))](\psi_\varepsilon\tau(u))^2\Big(x+\frac{x_\tau}{\varepsilon}\Big)\,dx\\ &\le-\frac{3a}{8}\int_{|x|\le\varepsilon^{-3/4}}(\psi_\varepsilon\tau(u))^2\Big(x+\frac{x_\tau}{\varepsilon}\Big)\,dx.\end{aligned}\tag{2.39}$$

Noting that $|x_\tau/\varepsilon-\Upsilon(u)|\le 2R_0$ in Lemma 2.7, one has

$$B_0:=\liminf_{\varepsilon\to0}\inf_{u\in\mathcal{R}(2r_2)}\int_{|x-x_\tau/\varepsilon|\le\varepsilon^{-3/4}}u^2\,dx>0.$$

Hence $T_{V_1}\le-3aB_0/8$. Above all, we have that in the case of $\kappa_1(x_\tau)\kappa_2(\tau(u))>0$ for any $l\in(0,l_0]$ and small $\varepsilon>0$,

$$\limsup_{h\to0}\frac{1}{H}\big[\Gamma_\varepsilon(\mathcal{T}_\varepsilon(l+h,\tau(u)))-\Gamma_\varepsilon(\mathcal{T}_\varepsilon(l,\tau(u)))\big]\le-\frac{aB_0}{4}.$$

This implies that $\Gamma_\varepsilon(\mathcal{T}_\varepsilon(l,\tau(u)))$ is non-increasing on $l\in[0,l_0]$.

For (b) of this lemma, if $u\in\mathcal{R}(r_2/10)$ and $\varepsilon\Upsilon(u)\in\mathcal{M}([-3\nu_1,-2\nu_1])$, from Lemma 2.7, x_τ belongs to a $2\varepsilon R_0$-neighbourhood of $\mathcal{M}([-3\nu_1,-2\nu_1])$, then for small $\varepsilon>0$, $\kappa_1(x_\tau)\kappa_2(\tau(u))>0$. Take $l=0$, $h=l_0$ and $H=\kappa_1(x_\tau)\kappa_2(\tau(u))l_0$, we can derive that there exists $\mu_0>0$ such that (b) of this lemma holds. □

2.7. Gradient flow of the energy functional Γ_ε. The third map introduced in this part stems from gradient flow of functional Γ_ε. Some other necessary properties are proved in what follows.

Next we define the set: for the positive constants r_3 and δ_2 to be determined later,

$$\mathcal{X}_{r_3}^{C_\varepsilon}:=\{u\in\mathcal{R}(2r_3):\varepsilon\Upsilon(u)\in\mathcal{M}([-\nu_0,0])\text{ and }\Gamma_\varepsilon(u)\le C_\varepsilon\}.$$

Then we look for the critical points in the set $\mathcal{X}_{r_3}^{C_\varepsilon} \setminus \mathcal{X}_{r_3}^{E(m_0)-\delta_2}$. Arguing on the contrary, we assume that $\Gamma_\varepsilon(u)$ does not have the critical points in the set $\mathcal{X}_{r_3}^{C_\varepsilon} \setminus \mathcal{X}_{r_3}^{E(m_0)-\delta_2}$.

Now we consider the following ordinary differential equation:

$$\begin{cases} \dfrac{d\eta}{ds} = -\psi_1(\Gamma_\varepsilon(\eta))\,\psi_2(\eta)\,\dfrac{\Gamma_\varepsilon'(\eta)}{\|\Gamma_\varepsilon'(\eta)\|_{H^{-1}}}, \\ \eta(0,u) = u, \end{cases} \tag{2.40}$$

where two cut-off functions $\psi_1(s) \in C_0^\infty(\mathbb{R},[0,1])$ and $\psi_2(u) \in C^2(\mathcal{R}(2r_3),[0,1])$ are defined as following: $\psi_1(s) = 1$ for $|s - E(m_0)| \le \delta_2/2$ and $\psi_1(s) = 0$ for $|l - E(m_0)| \ge \delta_2$; $\psi_2(u) = 1$ for $u \in \mathcal{R}(3r_3/2)$ and $\psi_2(u) = 0$ for $u \notin \mathcal{R}(2r_3)$. Then there exists a unique solution $\eta(s,u)$ for $s \in [0,\infty)$ such that $\eta_\varepsilon(s,u) \in \mathcal{X}_{r_3}^{C_\varepsilon} \setminus \mathcal{X}_{r_3}^{E(m_0)-\delta_2}$ for $u \in \mathcal{X}_{r_3}^{C_\varepsilon} \setminus \mathcal{X}_{r_3}^{E(m_0)-\delta_2}$. Moreover, observing that the following lemma, the center of mass for $\eta(s,u)$ does not go far away when s varies.

Lemma 2.11. *Assume that $u \in \mathcal{X}_{r_3}^{C_\varepsilon} \setminus \mathcal{X}_{r_3}^{E(m_0)-\delta_2}$ and $0 \le s_1 = s_1(\varepsilon) < s_2 = s_2(\varepsilon)$, there is some $c > 0$ independent of ε such that $|\Upsilon(\eta(s_1,u)) - \Upsilon(\eta(s_2,u))| \ge c/\varepsilon$, then $\lim\limits_{\varepsilon\to 0} |s_2(\varepsilon) - s_1(\varepsilon)| = \infty$.*

Proof. For $[s_1,s_2]$, we take interval division into $s_1 = t_0 < t_1 < \ldots < t_k = s_2$ such that

$$|\Upsilon(\eta(t_{i+1},u)) - \Upsilon(\eta(t_i,u))| \ge \frac{c}{k\varepsilon} \quad \text{for } i = 0,\ldots k-1.$$

Moreover, for each $i = 0,\ldots,k$, since $\eta(t_i,u) \in \mathcal{S}(2r_3)$, by Lemma 2.2, there exist $U_i \in \widehat{S}$ and $y_i \in \mathbb{R}^N$ such that

$$\|\eta(t_i,u) - U_i\| \le 2r_3 \quad \text{and} \quad |\Upsilon(\eta(t_i,u)) - y_i| \le R_0.$$

Then, for small $\varepsilon > 0$, we have that

$$|y_{i+1} - y_i| \ge |\Upsilon(\eta(t_{i+1},u)) - \Upsilon(\eta(t_i,u))| - 2R_0 \ge c/k\varepsilon - 2R_0 \ge R_0.$$

Thus, noting that $r_* \ge 16r_3$ in Lemma 2.2, one has

$$\begin{aligned} \|\eta(t_{i+1},u) - \eta(t_i,u)\| &\ge \|U_{i+1}(\,\cdot\, - y_{i+1}) - U_i(\,\cdot\, - y_i)\| \\ &\quad - \|\eta(t_{i+1},u) - U_{i+1}(\,\cdot\, - y_{i+1})\| - \|\eta(t_i,u) - U_i(\,\cdot\, - y_i)\| \\ &\ge \frac{3}{4}r_* - \frac{1}{16}r_* - 4r_3 \ge \frac{7}{16}r_* > 0. \end{aligned}$$

On the other hand, noting that $\|\eta(t_{i+1},u) - \eta(t_i,u)\| \le |t_{i+1} - t_i|$, there exists $c > 0$ such that for each i, we have $|t_{i+1} - t_i| \ge c$ and hence $|s_1 - s_2| \ge kc$. Taking $k = [1/\sqrt{\varepsilon}\,]$, we complete the proof. $\square$

Before we prove the next lemma, recalling the proof of Lemma 2.4 (c), we can also obtain that there exist $t_0 > 0$ and $\delta_2' > 0$ such that for any $t \in [1-t_0, 1+t_0]$,

$$A(t,z) \in \mathcal{R}\left(\frac{r_3}{10}\right) \quad \text{and} \quad \Gamma_\varepsilon(A(1 \pm t_0, z)) \leq E(m_0) - \frac{\delta_2'}{2}. \tag{2.41}$$

Lemma 2.12. *For some $r_3 \in (0, r_2)$ and $\delta_2 \in (0, \delta_1)$, if $u \in \mathcal{X}_{r_3}^{C_\varepsilon}$ with $\eta(l,u) \in \mathcal{R}(7r_3/8) \setminus \mathcal{R}(r_3/10)$, $l \in [0, r_3/30]$, then for small $\varepsilon > 0$, $\Gamma_\varepsilon(\eta(r_3/30, u)) \leq E(m_0) - \delta_2/2$.*

Proof. We divide the proof into the following two cases:

Case 1. For $l \in [r_3/60, r_3/30]$, since $\eta(l,u) \in \mathcal{R}(7r_3/8) \setminus \mathcal{R}(r_3/10)$ and from Lemma 2.3, we also have that $\eta(l,u) \in \mathcal{S}(7r_3/(8(1-q))) \setminus \mathcal{S}(r_3/(10\sqrt{2}(1+q)))$. Setting $s \in [l - r_3/60, l]$, we note that that

$$\|\eta(l,u) - \eta(s,u)\| \leq |l - s| \leq \frac{r_3}{60}. \tag{2.42}$$

Then, noting that $\lim\limits_{r \to 0} q(r) = 0$ in Lemma 2.3, we choose $r_3 \in (0, r_2)$ such that

$$\frac{3\sqrt{2}}{4(1+q)} - \frac{7}{8(1-q)} > \frac{1}{60} \quad \text{and} \quad \frac{1}{10\sqrt{2}(1+q)} - \frac{\sqrt{1}}{60(1-q)} > \frac{1}{60}.$$

Thus, from (2.42) and Lemma 2.3,

$$\eta(s,u) \in \mathcal{S}\left(\frac{3\sqrt{2}r_3}{4(1+q)}\right) \setminus \mathcal{S}\left(\frac{r_3}{60(1-q)}\right) \subset \mathcal{R}\left(\frac{3r_3}{2}\right) \setminus \mathcal{R}\left(\frac{r_3}{60}\right).$$

Consequently, in Lemma 2.5, we take

$$r' = \frac{r_3}{60} \quad \text{and} \quad \delta_2 < \min\left\{\frac{r_3}{60}\,\delta_1\left(r_3, \frac{r_3}{60}\right), \delta_2'\right\}.$$

Moreover, we assume that for any $s \in [l - r_3/60, l]$, $\Gamma_\varepsilon(\eta(s,u)) \geq E(m_0) - \delta_2/2$, then, for small $\varepsilon > 0$,

$$\begin{aligned}
&\Gamma_\varepsilon(\eta(l,u)) - E(m_0) + o(1) \leq \Gamma_\varepsilon(\eta(l,u)) - \Gamma_\varepsilon\left(\eta\left(l - \frac{r_3}{60}, u\right)\right) \\
&= \int_{l-r_3/60}^{l} \Gamma_\varepsilon'(\eta)\,\frac{d\eta}{dt}\,dt = -\int_{l-r_3/60}^{l} \|\Gamma_\varepsilon'(\eta)\|_{H^{-1}}\,dt \leq -\frac{r_3}{60}\,\delta_1\left(r_3, \frac{r_3}{60}\right) < -\delta_2,
\end{aligned}$$

which lead to a contradiction with the assumption.

Case 2. For $l \in [0, r_3/60]$, setting $s \in [l, l + r_3/60]$, then we still have that $\eta(s,u) \in \mathcal{R}(3r_3/2) \setminus \mathcal{R}(r_3/60)$. Similarly to the preceding part, we can also prove that $\Gamma_\varepsilon(\eta(s,u)) \leq E(m_0) - \delta_2/2$. □

3. Existence results of solutions

3.1. Iteration procedure by translation and gradient flow of Γ_ε. We define a map $I\colon \mathcal{X}_{r_3}^{C_\varepsilon} \to \mathcal{X}_{r_3}^{C_\varepsilon}$ by

$$I(u) = \mathcal{T}_\varepsilon(l_0, \cdot\,) \circ \tau \circ \eta(l_1, u),$$

where $l_1 = r_3/30$ and l_0 is given in (2.37). From Lemmas 2.6, 2.10 and 2.12, we observe that Γ_ε is non-increasing under the maps τ, $\mathcal{T}_\varepsilon$ and η. Then one has $\Gamma_\varepsilon(I(u)) \le \Gamma_\varepsilon(u)$ for any $u \in \mathcal{R}(2r_3)$ and $I(u) = u$ if $u \notin \mathcal{R}(2r_3)$. Now we define the iteration I^j by

$$I^j = I^{j-1} \circ I, \quad \text{for } j = 1, 2, \dots$$

Since we assume that there is no critical point in $\mathcal{X}_{r_3}^{C_\varepsilon} \setminus \mathcal{X}_{r_3}^{E(m_0)-\delta_2}$, then there exists $k_\varepsilon > 0$ such that

$$\|\Gamma_\varepsilon'(u)\|_{H^{-1}} \ge k_\varepsilon \quad \text{for } u \in \mathcal{X}_{r_3}^{C_\varepsilon} \setminus \mathcal{X}_{r_3}^{E(m_0)-\delta_2}. \tag{3.1}$$

We consider the iteration map I^k on $\mathcal{X}_{r_3}^{C_\varepsilon}$ in the following lemma:

LEMMA 3.1. *Assume* (3.1) *holds, taking* $j_\varepsilon = [30\delta_2/(k_\varepsilon r_3)] + 1$*, then for fixed small* $\varepsilon > 0$*,*

$$\Gamma_\varepsilon\big(I^{j_\varepsilon} \circ A(t,z)\big) \le E(m_0) - \frac{1}{2}\min\{\delta_2, \mu_0\}, \quad \textit{for any } (t,z) \in [0,T] \times \mathcal{L}_\varepsilon.$$

PROOF. We consider the following sequences:

$$\eta\big(l, I^j \circ A(t,z)\big)] \quad \text{for } j = 0, \dots, j_\varepsilon - 1 \text{ and } l \in [0, r_3/30].$$

First, we assume that the following case holds: For some j and $l \in [0, r_3/30]$,

$$\eta\big(l, I^j \circ A(t,z)\big) \in \mathcal{R}\bigg(\frac{7r_3}{8}\bigg) \setminus \mathcal{R}\bigg(\frac{r_3}{10}\bigg).$$

By Lemma 2.12, one has

$$\Gamma_\varepsilon\big(I^{j_\varepsilon} \circ A(t,z)\big) \le \Gamma_\varepsilon\big(\eta\big(l_1, I^j \circ A(t,z)\big)\big) \le E(m_0) - \frac{\delta_2}{2}.$$

If the preceding case does not happen, then for any $j = 0, \dots, j_\varepsilon - 1$ and $l \in [0, r_3/30]$,

$$\eta\big(l, I^j \circ A(t,z)\big) \notin \mathcal{R}\bigg(\frac{7r_3}{8}\bigg) \setminus \mathcal{R}\bigg(\frac{r_3}{10}\bigg).$$

Noting that (2.41), thus we consider that for any $j = 0, \cdots, j_\varepsilon - 1$ and $l \in [0, r_3/30]$, $\eta\big(l, I^j \circ A(t,z)\big) \in \mathcal{R}(r_3/10)$.

If for some j and $l \in [0, r_3/30]$,

$$\varepsilon\Upsilon\big(\eta\big(l, I^j \circ A(t,z)\big)\big) \in \mathcal{M}\big([-3\nu_1, -2\nu_1]\big),$$

then, by Lemma 2.10, one has

$$\Gamma_\varepsilon\big(I^{j_\varepsilon} \circ A(t,z)\big) \le \Gamma_\varepsilon\big(\eta\big(l_1, I^{j+1} \circ A(t,z)\big)\big) \le E(m_0) - \frac{\mu_0}{2}.$$

Now the rest case is that for any $j = 0, \dots, j_\varepsilon - 1$ and $l \in [0, r_3/30]$, $\eta\big(l, I^j \circ A(t,z)\big) \notin \mathcal{M}\big([-3\nu_1, -2\nu_1]\big)$. Whereas we can claim that $\Gamma_\varepsilon\big(I^{j_\varepsilon} \circ A(t,z)\big) \le E(m_0) - \delta_2/2$. Otherwise, we suppose that for any $j = 0, 1, \cdots, j_\varepsilon - 1$ and $l \in [0, r_3/30]$, $\Gamma_\varepsilon\big(I^{j_\varepsilon} \circ A(t,z)\big) \ge E(m_0) - \delta_2/2$. Then

$$\begin{aligned}\Gamma_\varepsilon&\big(I^{j_\varepsilon} \circ A(t,z)\big)\\ &= \Gamma_\varepsilon(A(t,z)) + \sum_{j=0}^{j_\varepsilon - 1} \big[\Gamma_\varepsilon\big(I^{j+1} \circ A(t,z)\big) - \Gamma_\varepsilon\big(I^j \circ A(t,z)\big)\big]\\ &\le \Gamma_\varepsilon(A(t,z)) + \sum_{j=0}^{j_\varepsilon - 1} \big[\Gamma_\varepsilon\big(\eta\big(l_1, I^j \circ A(t,z)\big)\big) - \Gamma_\varepsilon\big(I^j \circ A(t,z)\big)\big],\end{aligned}$$

where

$$\begin{aligned}\Gamma_\varepsilon&\big(\eta\big(l_1, I^j \circ A(t,z)\big)\big) - \Gamma_\varepsilon\big(I^j \circ A(t,z)\big)\\ &= \int_0^{l_1} \Gamma_\varepsilon'(\eta)\,\frac{d\eta}{ds}\,ds = -\int_0^{l_1} \|\Gamma_\varepsilon'(\eta)\|_{H^{-1}}\,ds \le -\frac{k_\varepsilon r_3}{30} \le -\frac{\delta_2}{j_\varepsilon - 1}.\end{aligned}$$

Thus, for small $\varepsilon > 0$, one has

$$\Gamma_\varepsilon\big(I^{j_\varepsilon}\big(A(t,z)\big)\big) \le \Gamma_\varepsilon(A(t,z)) - \frac{j_\varepsilon}{j_\varepsilon - 1}\,\delta_2 \le E(m_0) - \frac{\delta_2}{2}.$$

Above all, we conclude that

$$\Gamma_\varepsilon\big(I^{j_\varepsilon} \circ A(t,z)\big) \le E(m_0) - \frac{1}{2}\min\{\mu_0, \delta_2\}. \qquad \square$$

3.2. Proof of the existence results. We denote

$$B(t,z) := \tau\big(I^{j_\varepsilon} \circ A(t,z)\big),$$

then it is clear that

$$\Gamma_\varepsilon(B(t,z)) \le \Gamma_\varepsilon\big(I^{j_\varepsilon} \circ A(t,z)\big) \le E(m_0) - \frac{1}{2}\,\min\{\mu_0, \delta_2\}, \tag{3.2}$$

for any $(t,z) \in [0,T] \times \mathcal{L}_\varepsilon$. Moreover, we claim that for $\varepsilon > 0$ small,

$$\begin{aligned}\Gamma_\varepsilon(B(t,z)) \ge \frac{1}{2}\,\widehat{M}\big(\|\nabla B(t,z)\|_2^2\big) &+ \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon\Upsilon(B(t,z)))B^2(t,z)\,dx\\ &- \int_{\mathbb{R}^N} F(B(t,z))\,dx + o(1) := J(B(t,z)) + o(1).\end{aligned} \tag{3.3}$$

Indeed, if $A(t,z) \notin \mathcal{R}(2r_3)$, then $B(t,z) = A(t,z)$ and by (2.35), we have that for small $\varepsilon > 0$,

$$\begin{aligned}\int_{\mathbb{R}^N} V_\varepsilon A^2(t,z)\,dx &= \int_{\mathbb{R}^N} V(\gamma(z))A^2(t,z)\,dx + o(1)\\ &= \int_{\mathbb{R}^N} V(\varepsilon\Upsilon(A(t,z)))A^2(t,z)\,dx + o(1).\end{aligned}$$

Moreover, if $A(t,z) \in \mathcal{R}(2r_3) \subset \mathcal{S}(2r_3)$, by Lemma 2.2 and $A(t,z) = U_0(\cdot - \gamma(z)/\varepsilon) + \varphi_t$ with $\|\varphi_t\| \le 2r_3$, one has $\varepsilon\Upsilon(A(t,z)) = \gamma(z) + o(1)$. Furthermore, it follows from Lemma 2.7, 2.8, 2.11 and the property (a) in (V3) that, for each $j = 1, \ldots, j_\varepsilon$,

$$\begin{aligned}V\big(\varepsilon\Upsilon\big(I^j \circ A\big)\big) &= V\big(\zeta_0\big(l_0, \varepsilon\Upsilon \circ \tau \circ \eta\big(l_1, I^{j-1} \circ A\big)\big)\big) + o(1)\\ &\le V\big(\varepsilon\Upsilon\big(\tau \circ \eta\big(l_1, I^{j-1} \circ A\big)\big)\big) + o(1) = V\big(\varepsilon\Upsilon\big(I^{j-1} \circ A\big)\big) + o(1).\end{aligned}$$

Thus we obtain (3.3).

Next we prove that the following proposition:

PROPOSITION 3.2. *For $\varepsilon > 0$ small, there exists $(t_\varepsilon, z_\varepsilon) \in (0,T) \times \mathcal{L}_\varepsilon \setminus L_0$ such that $J(B(t_\varepsilon, z_\varepsilon)) \ge E(m_0)$.*

PROOF. Denote

$$\begin{aligned}D_\varepsilon(t,z) := \frac{N-2}{2} M\big(\|\nabla B(t,z)\|_2^2\big) &\int_{\mathbb{R}^N} |\nabla B(t,z)|^2\,dx\\ &+ N\int_{\mathbb{R}^N}\bigg[\frac{V(\varepsilon\Upsilon(B(t,z))}{2} B^2(t,z) - F(B(t,z))\bigg]\,dx,\end{aligned}$$

for $(t,z) \in [0,T] \times \mathcal{L}_\varepsilon$. From the definition of $A(t,z)$, we can take $T_0 > 0$ small enough such that $A(T_0, z) \notin \mathcal{R}(2r_3)$ for any $z \in \mathcal{L}_\varepsilon$. Similarly, we also have for $T > 0$ large such that $A_\varepsilon(T,z) \notin \mathcal{R}(2r_3)$ for any $z \in \mathcal{L}_\varepsilon$. Thus, if $t = T_0$ or T, $B(t,z) = A(t,z)$ for any $z \in \mathcal{L}_\varepsilon$. It follows from (M3) and (M5) that

$$D_\varepsilon(T_0, z) > 0 \quad\text{and}\quad D_\varepsilon(T,z) < 0, \quad \text{for any } z \in \mathcal{L}_\varepsilon. \tag{3.4}$$

On the other hand, for $z \in L_0$, from $\varepsilon\Upsilon(A(t,z)) = z \notin \mathcal{M}([-\nu_0, 0])$, (2.36) and the definition of $\mathcal{T}_\varepsilon$ and η, $B(t,z) = A(t,z)$ for any $(t,z) \in (T_0,T) \times L_0$. Hence by (M3) and (M5), there exists unique $t(z) > 0$ such that

$$D_\varepsilon(t(z), z) = 0 \quad\text{and}\quad \frac{\partial D_\varepsilon(t(z),z)}{\partial t} \ne 0 \quad \text{for any } (t,z) \in (T_0,T) \times L_0. \tag{3.5}$$

Now we claim that, for small $\varepsilon > 0$, there exists $(t_\varepsilon, z_\varepsilon) \in (T_0,T) \times \mathcal{L}_\varepsilon \setminus L_0$ such that

$$D_\varepsilon(t_\varepsilon, z_\varepsilon) = 0 \quad\text{and}\quad V(\varepsilon\Upsilon(B(t_\varepsilon, z_\varepsilon))) = m_0.$$

In fact, we use the sequences $D_\varepsilon^{(l)} \in C^{N+1}((T_0,T) \times \mathcal{L}_\varepsilon)$ to approximate $D_\varepsilon(t,z)$ in $C((T_0,T) \times \mathcal{L}_\varepsilon)$ and satisfy $D_\varepsilon^{(l)}(t,z) = D_\varepsilon(t,z)$ for $(t,z) \in (T_0,T) \times L_0$.

For each l, it follows from Sard Theorem that we let $b_i^{(l)}$ be regular value of $D_\varepsilon^{(l)}$ with $b_i^{(l)} \to 0$ as $i \to \infty$. By choosing appropriate subsequences, we take subsequences l_i such that $l_i \to \infty$ and $b_i^{(l_i)} \to 0$ as $i \to \infty$. Since $b_i^{(l_i)}$ is regular value of $D_\varepsilon^{(l_i)}$ for each i, then $\big(D_\varepsilon^{(l_i)}\big)^{-1}\big(b_i^{(l_i)}\big)$ is union of finitely many $k-$dimensional compact connected sub-manifold of $(T_0,T) \times \mathcal{L}_\varepsilon$.

Set $\mathcal{B}^i$ be the connected component which $\big(D_\varepsilon^{(l_i)}\big)^{-1}(b_i^{(l_i)})$ belongs to and intersects with $(T_0,T) \times L_0$. Then, setting $\pi_\varepsilon\colon (T_0,T) \times \mathcal{L}_\varepsilon \to \mathcal{L}_\varepsilon$ be natural projection, it follows from (3.5) that $\pi_\varepsilon\colon \partial\mathcal{B}^i \to \pi_\varepsilon(\partial\mathcal{B}^i) \subset L_0$ is homeomorphic. Moreover, for any $z \in \mathcal{L}_\varepsilon \backslash L_0$, the mod 2 degree $\deg_2(\pi_\varepsilon, \mathcal{B}^i, z)$ is well defined, and $\deg_2(\pi_\varepsilon, \mathcal{B}^i, z) = 1$ for any z close to L_0. Since $\mathcal{L}_\varepsilon$ is connected and $\pi_\varepsilon(\partial\mathcal{B}^i) \subset L_0$, $\deg_2(\pi_\varepsilon, \mathcal{B}^i, z)$ is independent of $z \in \mathcal{L}_\varepsilon \setminus L_0$. Hence $\pi_\varepsilon(\mathcal{B}^i) = \mathcal{L}_\varepsilon$ and $\pi_\varepsilon(\partial\mathcal{B}^i) = L_0$. Then we have $\mathcal{B}^i \in \mathcal{L}(L_0)$. From (2.3), we have

$$\max_{(t,z)\in\mathcal{B}^{(i)}} V(\varepsilon\Upsilon(B(t,z))) \geq m_0.$$

Moreover, for $(t,z) \in (T_0, T) \times L_0$,

$$V(\varepsilon\Upsilon(B(t,z))) = V(z) \leq \max_{x\in L_0} V(x) < m_0.$$

From (3.4), we observe that $D_\varepsilon^{(l_i)}(T_0, z) > 0$ and $D_\varepsilon^{(l_i)}(T, z) < 0$, for i large and $z \in \mathcal{L}_\varepsilon$. Thus there exist $(t_i, z_i) \in (T_0, T) \times \mathcal{L}_\varepsilon \setminus L_0$ such that

$$V(\varepsilon\Upsilon(B(t_i, z_i))) = m_0 \quad \text{and} \quad D_\varepsilon^{(l_i)}(t_i, z_i) = b_i^{(l_i)}.$$

Letting $i \to \infty$, we have that $(t_i, z_i) \to (t_\varepsilon, z_\varepsilon) \in (T_0, T) \times \mathcal{L}_\varepsilon \setminus L_0$,

$$D_\varepsilon(t_\varepsilon, z_\varepsilon) = 0 \quad \text{and} \quad V(\varepsilon\Upsilon(B(t_\varepsilon, z_\varepsilon))) = m_0.$$

By Proposition 2.1 in Appendix, $J(B(t_\varepsilon, z_\varepsilon)) \geq E(m_0)$. □

Combining with (3.3), the conclusion of Proposition 3.2 leads to a contradiction with (3.2). Hence the set $\mathcal{X}_{r_3}^{C_\varepsilon} \setminus \mathcal{X}_{r_3}^{E(m_0)-\delta_2}$ contains the critical points. Next we shall estimate critical points in this set.

4. Multiplicity of solutions

This section is devoted to using the concept of relative category to consider the multiplicity of solutions for (2.1). More concretely, we use relative category to estimate the change of topology between $\mathcal{X}_{r_3}^{E(m_0)+\delta_0}$ and $\mathcal{X}_{r_3}^{E(m_0)-\delta_0}$, where $\delta_0 = \min\{\mu_0, \delta_2\}/2$. Here we are inspired of [15] in which they deal with the case of local minimum in potential V.

In algebraic topology, the notation of a map $f\colon (A, B) \to (A', B')$ means that $f\colon A \to A'$ is continuous, $B \subset A$, $B' \subset A'$ and $f(B) \subset B'$. Then we introduce the following two maps:

$$\Phi_\varepsilon\colon ([1-t_0, 1+t_0] \times \mathcal{M}, \{1 \pm t_0\} \times \mathcal{M}) \to \big(\mathcal{X}_{r_3}^{E(m_0)+\delta_0}, \mathcal{X}_{r_3}^{E(m_0)-\delta_0}\big),$$
$$\Psi_\varepsilon\colon \big(\mathcal{X}_{r_3}^{E(m_0)+\delta_0}, \mathcal{X}_{r_3}^{E(m_0)-\delta_0}\big) \to ([1-t_0, 1+t_0] \times \mathcal{M}([-\nu_0, 0]), [1-t_0, 1+t_0] \setminus \{1\} \times \mathcal{M}([-\nu_0, 0])).$$

They are defined in order to construct a map homotopic to the inclusion between the proper sets, where t_0 is given in (2.41). Define

$$\Phi_\varepsilon(t, y) := U_0\bigg(\frac{x - y/\varepsilon}{t}\bigg).$$

Recalling U_0 is the least energy solution of the equation (2.13), $\Phi_\varepsilon(t,y) \in \mathcal{R}(2r_3)$ for any $t \in [1-t_0, 1+t_0]$. Then, by Lemma 2.2 and (2.7), one has that, for small $\varepsilon > 0$,

$$\varepsilon\Upsilon(\Phi_\varepsilon(t,y)) = y + o(1) \quad \text{and} \quad \Gamma_\varepsilon(\Phi_\varepsilon(t,y)) = L_{m_0}(\Phi_\varepsilon(t,y)) + o(1). \tag{4.1}$$

Thus, from (2.41), we see that Φ_ε is well defined.

Define $\Psi_\varepsilon(u) = \big(\widetilde{\mathcal{P}}_0(u), \varepsilon\Upsilon(u)\big)$, where

$$\widetilde{\mathcal{P}}_0(u) = \begin{cases} 1-t_0 & \text{if } \mathcal{P}_0(u) < 1-t_0, \\ \mathcal{P}_0(u) & \text{if } 1-t_0 \le \mathcal{P}_0(u) \le 1+t_0, \\ 1+t_0 & \text{if } \mathcal{P}_0(u) > 1+t_0. \end{cases}$$

and

$$\mathcal{P}_0(u) = \bigg(\frac{2N}{(N-2)M(\|\nabla u\|_2^2)\|\nabla u\|_2^2} \int_{\mathbb{R}^N} F(u) - \frac{m_0}{2}\, u^2\, dx\bigg)^{1/2}.$$

It is obvious that $\Psi_\varepsilon\big(\mathcal{X}_{r_3}^{E(m_0)+\delta_0}\big) \subset [1-t_0, 1+t_0] \times \mathcal{M}([-\nu_0, 0])$. Then it suffice to prove that for $u \in \mathcal{X}_{r_3}^{E(m_0)-\delta_0}$, one has that $\widetilde{\mathcal{P}}_0 \neq 1$. Indeed, assume that $\widetilde{\mathcal{P}}_0 = 1$, then we have that either $u \in \widehat{S}$ or $u \notin \widehat{S}$.

If $u \in \widehat{S}$, then $L_{m_0}(u) = E(m_0)$ and there exists $y_0 \in \mathbb{R}^N$ such that $u = U_0(\,\cdot\, - y_0) \in \widehat{S}$. Denoting $\widetilde{M} = \sup\limits_{u \in \mathcal{S}(2r_3)} \|u\|_2^2$, we observe that, for small $\varepsilon > 0$,

$$\begin{aligned} E(m_0) - \delta_0 \ge \Gamma_\varepsilon(u) &= L_{V(\varepsilon\Upsilon(u))}(U_0) + o(1) \\ &= L_{m_0}(U_0) + \frac{1}{2}\int_{\mathbb{R}^N} (V(\varepsilon\Upsilon(u)) - m_0) U_0\, dx + o(1) \\ &\ge E(m_0) - \frac{\nu_0}{2}\int_{\mathbb{R}^N} U_0^2\, dx + o(1) \ge E(m_0) - \frac{5}{8}\,\nu_0\,\widetilde{M}. \end{aligned}$$

If $u \notin \widehat{S}$, then it follows from Proposition 2.1 that $L_{m_0}(u) > E(m_0)$ and we have that, for small $\varepsilon > 0$,

$$\begin{aligned} E(m_0) - \delta_0 \ge \Gamma_\varepsilon(u) &= L_{m_0}(u) + \frac{1}{2}\int_{\mathbb{R}^N} (V(\varepsilon\Upsilon(u)) - m_0) u^2\, dx + o(1) \\ &\ge E(m_0) - \frac{\nu_0}{2}\int_{\mathbb{R}^N} u^2\, dx + o(1) \ge E(m_0) - \frac{5}{8}\,\nu_0\,\widetilde{M}. \end{aligned}$$

In both cases, we can choose small $\nu_0 \in (0, 8\delta_0/(5\widetilde{M}))$ to get contradiction. Hence we observe that Ψ_ε is well defined. Next, we show that $\Psi_\varepsilon \circ \Phi_\varepsilon$ is homotopic to the embedding $j(t,y) = (t,y)$, which is essential to estimate the lower bound of the relative category.

Lemma 4.1. *There exists a continuous map*

$$\eta(s,t,y)\colon [0,1] \times [1-t_0, 1+t_0] \times \mathcal{M} \to [1-t_0, 1+t_0] \times \mathcal{M}([-\nu_0, 0])$$

such that $\eta(0,t,y) = \Psi_\varepsilon \circ \Phi_\varepsilon(t,y)$, $\eta(1,t,y) = (t,y) \in [1-t_0, 1+t_0] \times \mathcal{M}$ *and for any* $(s,t,y) \in [0,1] \times \{1 \pm t_0\} \times \mathcal{M}$,

$$\eta(s,t,y) \in [1-t_0, 1+t_0] \setminus \{1\} \times \mathcal{M}([-\nu_0, 0]).$$

Proof. Noting that

$$\Psi_\varepsilon \circ \Phi_\varepsilon(t,y) = \bigg(\widetilde{\mathcal{P}}_0\bigg(U_0\bigg(\frac{x-y/\varepsilon}{t}\bigg)\bigg), \varepsilon\Upsilon\bigg(U_0(\frac{x-y/\varepsilon}{t}\bigg)\bigg)\bigg),$$

define

$$\eta(s,t,y) = \bigg((1-s)\widetilde{\mathcal{P}}_0\bigg(U_0\bigg(\frac{x-y/\varepsilon}{t}\bigg)\bigg) + st, (1-s)\varepsilon\Upsilon\bigg(U_0\bigg(\frac{x-y/\varepsilon}{t}\bigg)\bigg) + sy\bigg).$$

From the first equality (4.1), given small $\varepsilon > 0$, the required properties is satisfied. $\square$

Lemma 4.2. *For any fixed* $\varepsilon > 0$ *and* $\{u_n\} \subset \mathcal{R}(2r_3)$, *assume that* $\Gamma_\varepsilon(u_n)$ *is bounded and* $\Gamma'_\varepsilon(u_n) \to 0$, *then* $\{u_n\}$ *have convergent subsequences.*

Proof. Since $\mathcal{R}(2r_3)$ is bounded in $H^1(\mathbb{R}^N)$, thus, up to subsequence, $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$. In order to prove this lemma, it suffices to prove

$$\lim_{R\to\infty}\lim_{n\to\infty} \|u_n\|_{H^1(\mathbb{R}^N\setminus B_R)} = 0. \tag{4.2}$$

From the proof of (2.15), we have that there is $R_2 > 0$ such that, for any n,

$$\int_{\mathbb{R}^N\setminus B_{R_2}} |\nabla u_n|^2 + u_n^2\,dx \le 5\,r_3^2.$$

Then, denoting $D_i = B_{R_2+i} \setminus B_{R_2+i-1}$, we have

$$\sum_{i=1}^k \|u_n\|^2_{H^1(D_i)} \le \int_{\mathbb{R}^N\setminus B_{R_2}} |\nabla u_n|^2 + u_n^2\,dx \le 5\,r_3^2.$$

For any n, we choose $i_n \in \{1,\ldots,k\}$ such that

$$\|u_n\|^2_{H^1(D_{i_n})} \le \frac{5\,r_3^2}{k}. \tag{4.3}$$

Then, set cut-off function $\chi(x) \in C_0^\infty(\mathbb{R}^N,[0,1])$ such that $\chi(x) = 1$ for $x \in B_{R_2+i_n}$ and $\chi(x) = 0$ for $x \in \mathbb{R}^N \setminus B_{R_2+i_n+1}$ and $|\nabla\chi(x)| \le 2$. We denote $\widetilde{u}_n = (1-\chi(x))u_n$, using (4.3) and (2.18), one has for $C, C' > 0$

$$\begin{aligned}\Gamma'_\varepsilon(u_n)\widetilde{u}_n &= M(\|\nabla u_n\|_2^2)\|\nabla\widetilde{u}_n\|_2^2 + M(\|\nabla u_n\|_2^2)\int_{\mathbb{R}^N} \nabla(\chi u_n)\cdot\nabla((1-\chi)u_n)\,dx \\ &\quad + \int_{\mathbb{R}^N} V_\varepsilon\chi(1-\chi)u_n^2\,dx - \int_{\mathbb{R}^N} f(u_n)\widetilde{u}_n\,dx \ge C\|\widetilde{u}_n\|^2 - \frac{C'}{k}.\end{aligned}$$

Thus

$$\|u_n\|^2_{H^1(\mathbb{R}^N\setminus B_{R_2+k+1})} \le o(1) + \frac{C}{k}.$$

Then (4.2) holds and $\{u_n\}$ have convergent subsequences. $\square$

In the following, we introduce the category $\mathrm{cat}(f)$ and cuplength $\mathrm{cupl}(f)$ of $f\colon (A,B) \to (A',B')$ in algebraic topology which we refer to [4] and [43]. First, the category $\mathrm{cat}(f)$ is the infimum of all integers $k \geq 0$ such that there exists a covering $A = A_0 \cup \ldots \cup A_k$ of A by closed sets $A_i \subset A$ with the following properties:

(1) $B \subset A_0$ and there is a homotopy $h\colon ([0,1] \times A_0, [0,1] \times B) \to (A',B')$ satisfying $h(0,x) = f(x)$ and $h(1,x) \in B'$ for all $x \in A_0$.
(2) For $i = 1, \cdots, k$ the restriction $f\,|_{A_i}\colon A_i \to A'$ is homotopic to a constant map.

If $f = \mathrm{id}_{(A,B)}$ is the identity map, then we denote

$$\mathrm{cat}(A,B) = \mathrm{cat}(\mathrm{id}_{(A,B)}) \quad \text{and} \quad \mathrm{cat}(A) = \mathrm{cat}(A,\emptyset).$$

Now, we introduce the cuplength $\mathrm{cupl}(f)$ of $f\colon (A,B) \to (A',B')$. Let H^* denote the Alexander–Spanier cohomology with coefficients in the field $\mathbb{F}$. We recall that the cup product $\smile$ turns $H^*(A)$ into a ring with unit 1_A, and it turns $H^*(A,B)$ into a module over $H^*(A)$. A continuous map $f\colon (A,B) \to (A',B')$ induces a homomorphism $f^*\colon H^*(A',B') \to H^*(A,B)$ of abelian groups. The number $\mathrm{cupl}(f)$ is defined as the largest integer $k \geq 0$ such that there exist elements $\alpha_1, \cdots, \alpha_k \in \widetilde{H}^*(A')$ and $\beta \in H^*(A',B')$ with

$$f^*(\alpha_1 \smile \cdots \smile \alpha_k \smile \beta) = f^*(\alpha_1) \smile \cdots \smile f^*(\alpha_k) \smile f^*(\beta) \neq 0 \in H^*(A,B).$$

If $\widetilde{H}^*(A') = 0$ and $f^* \neq 0$, we have $\mathrm{cupl}(f) = 0$. If $f^* = 0$, we define $\mathrm{cupl}(f) = -1$. Again, if $f = \mathrm{id}_{(A,B)}$, we denote that

$$\mathrm{cupl}(A,B) = \mathrm{cupl}(\mathrm{id}_{(A,B)}) \quad \text{and} \quad \mathrm{cupl}(A) = \mathrm{cupl}(A,\emptyset).$$

The category and cuplength have the following properties:

LEMMA 4.3.

(a) *If $f\colon (A,B) \to (A',B')$, then* $\mathrm{cat}(f) \geq$ *$cupl(f) + 1$.*
(b) *For two continuous maps $f\colon (A,B) \to (A',B')$, $f'\colon (A',B') \to (A'',B'')$, we have*

$$\mathrm{cupl}(f' \circ f) \leq \min\{\mathrm{cupl}(f), \mathrm{cupl}(f')\}.$$

(c) *If $f, g\colon (A,B) \to (A',B')$ are homotopic, then* $\mathrm{cupl}(f) = \mathrm{cupl}(g)$.

These properties which we state here will be used in the back. For more details and proof, we can refer to [4], [23] and [24].

The next lemma is the important ingredient for our proof due to the continuity property of Alexander–Spanier cohomology (see [42], [43]). And this lemma is proved in [15]. For simplicity, we omit the proof.

Lemma 4.4. *Let* $\mathcal{M} \subset \mathbb{R}^N$ *be a compact set. For its neighbourhood* $\mathcal{M}^d = \Big\{x \in \mathbb{R}^N : \inf_{y\in\mathcal{M}} |x-y| \le d\Big\}$ *and* $I = [1-t_0, 1+t_0]$, $\partial I = \{1 \pm t_0\}$, *the inclusion* $j\colon (I \times \mathcal{M}, \partial I \times \mathcal{M}) \to (I \times \mathcal{M}^d, \partial I \times \mathcal{M}^d)$ *is defined by* $j(s,x) = (s,x)$. *Then*

$$\mathrm{cupl}(j) \ge \mathrm{cupl}(\mathcal{M}) \quad \textit{for small } d > 0.$$

Proof of Theorem 1.1. From (2.40) and Lemmas 2.12 and 4.2, standard argument (see [24]) imply that

$$\sharp\big\{u \in \mathcal{X}_{r_3}^{E(m_0)+\delta_0} \setminus \mathcal{X}_{r_3}^{E(m_0)-\delta_0} : \Gamma_\varepsilon'(u) = 0\big\} \ge \mathrm{cat}\big(\mathcal{X}_{r_3}^{E(m_0)+\delta_0}, \mathcal{X}_{r_3}^{E(m_0)-\delta_0}\big).$$

Then, from Lemma 4.3 (a) and (b), we have

$$\begin{aligned}\mathrm{cat}\big(\mathcal{X}_{r_3}^{E(m_0)+\delta_0}, \mathcal{X}_{r_3}^{E(m_0)-\delta_0}\big)& \\ \ge \mathrm{cupl}\big(\mathcal{X}_{r_3}^{E(m_0)+\delta_0}, \mathcal{X}_{r_3}^{E(m_0)-\delta_0}\big) + 1 &\ge \mathrm{cupl}(\Psi_\varepsilon \circ \Phi_\varepsilon) + 1.\end{aligned}$$

Here, we choose $\nu_0 > 0$ small such that $\mathcal{M}([-\nu_0, 0]) \subset \mathcal{M}^d$. Then, setting the inclusion map

$$\begin{aligned}\sigma_0\colon ([1-t_0, 1+t_0] \times \mathcal{M}([-\nu_0, 0]), [1-t_0, 1+t_0] \setminus \{1\} \times \mathcal{M}([-\nu_0, 0]))& \\ \to \big([1-t_0, 1+t_0] \times \mathcal{M}^d, [1-t_0, 1+t_0] \setminus \{1\} \times \mathcal{M}^d\big)&.\end{aligned}$$

From Lemma 4.1, $\sigma_0 \circ \Psi_\varepsilon \circ \Phi_\varepsilon$ is homotopic to the inclusion map

$$j\colon ([1-t_0, 1+t_0] \times \mathcal{M}, \{1 \pm t_0\} \times \mathcal{M}) \to ([1-t_0, 1+t_0] \times \mathcal{M}^d, \{1 \pm t_0\} \times \mathcal{M}^d).$$

Then it follows from Lemma 4.3 (b) and (c) that

$$\mathrm{cupl}(\Psi_\varepsilon \circ \Phi_\varepsilon) \ge \mathrm{cupl}(\sigma_0 \circ \Psi_\varepsilon \circ \Phi_\varepsilon) = \mathrm{cupl}(j). \tag{4.4}$$

Consequently, by Lemma 4.4 and (4.4), we have

$$\mathrm{cat}(\mathcal{X}_{r_3}^{E(m_0)+\delta_0}, \mathcal{X}_{r_3}^{E(m_0)-\delta_0}) \ge \mathrm{cupl}(\mathcal{M}) + 1.$$

In summary, for any $d > 0$ in (V3), there exists $\varepsilon_d > 0$ such that for $\varepsilon \in (0, \varepsilon_d)$, Γ_ε has at least $\mathrm{cupl}(\mathcal{M}) + 1$ critical points $u_\varepsilon^{(i)} \in \mathcal{X}_{r_3}^{E(m_0)+\delta_0} \setminus \mathcal{X}_{r_3}^{E(m_0)-\delta_0}$ with $\Gamma_\varepsilon(u_\varepsilon^{(i)}) \le C_\varepsilon$, for $i = 1, \ldots, \mathrm{cupl}(\mathcal{M}) + 1$. Then, from

$$\varepsilon\Upsilon(u_\varepsilon^{(i)}) \in \mathcal{M}([-\nu_0, 0]) \subset \mathcal{M}^d,$$

we have, up to subsequence,

$$\lim_{\varepsilon\to 0} \mathrm{dist}(\varepsilon\Upsilon(u_\varepsilon^{(i)}), \mathcal{M}) = 0 \quad \text{and} \quad \limsup_{\varepsilon\to 0} \Gamma_\varepsilon(u_\varepsilon^{(i)}) \le E(m_0).$$

Letting $x_\varepsilon^{(i)}$ be maximum point of $u_\varepsilon^{(i)}$, by Lemma 2.2, one has

$$\lim_{\varepsilon\to 0} \mathrm{dist}(\varepsilon x_\varepsilon^{(i)}, \mathcal{M}) = 0.$$

Moreover, noting that

$$\limsup_{\varepsilon\to 0} \Gamma_\varepsilon(u_\varepsilon^{(i)}) \le E(m_0) \quad \text{and} \quad \Gamma_\varepsilon'(u_\varepsilon^{(i)}) = 0,$$

it follows from the proof of Lemma 2.5, up to subsequence, $u_\varepsilon^{(i)}(\,\cdot\,+x_\varepsilon^{(i)})$ converges to $U \in \widehat{S}$ as $\varepsilon \to 0$. Since $\lim_{\varepsilon\to 0} \operatorname{dist}(\varepsilon x_\varepsilon^{(i)}, \mathcal{M}) = 0$, $U \in S_{m_0}$.

For Theorem 1.1 (c), we refer to the proof of Lemma 2.6 or [21]. Consequently, we complete the proof of the theorem. □

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