Topological **M**ethods in **N**onlinear **A**nalysis
Volume 49, No. 1, 2017, 105–132
DOI: 10.12775/TMNA.2016.064

ON THE CHAOS GAME OF ITERATED FUNCTION SYSTEMS

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Abstract. Every quasi-attractor of an iterated function system (IFS) of continuous functions on a first-countable Hausdorff topological space is renderable by the probabilistic chaos game. By contrast, we prove that the backward minimality is a necessary condition to get the deterministic chaos game. As a consequence, we obtain that an IFS of homeomorphisms of the circle is renderable by the deterministic chaos game if and only if it is forward and backward minimal. This result provides examples of attractors (a forward but no backward minimal IFS on the circle) that are not renderable by the deterministic chaos game. We also prove that every well-fibred quasi-attractor is renderable by the deterministic chaos game as well as quasi-attractors of both, symmetric and non-expansive IFSs.

1. Introduction

Within fractal geometry, iterated function systems (IFSs) provide a method for both generating and characterizing fractal images. An *iterated function system* (IFS) can also be thought of as a finite collection of functions which can be applied successively in any order. Attractors of this kind of systems are self-similar compact sets which draw any iteration of any point in an open neighbourhood of itself.

2010 *Mathematics Subject Classification.* Primary: 37C05, 37C20; Secondary: 37E10.
Key words and phrases. Iterated function system; well-fibred attractors; deterministic and probabilistic chaos game; forward and backward minimality.
The first author was partially supported by the Ministerio de Ciencia e Innovación project MTM2014-56953-P (Spain).

There are two methods for generating an attractor: a *deterministic* algorithm, in which all the transformations are applied simultaneously, and a *random* algorithm, in which the transformations are applied one at a time in random order following the probability. A *chaos game*, popularized by Barnsley [3], is a simple algorithm implementing the random method. There are two different ways to run the chaos game that consists in taking a starting point and then choosing randomly a transformation on each iteration accordingly to the assigned probabilities. The latter starts by choosing a random order iteration and then applying this orbital branch anywhere in the basin of attraction. The first way of implementation is called the *probabilistic chaos game* [9], [7]. The second implementation is called the *deterministic chaos game* (also called the *disjunctive chaos game*) [27], [5], [12].

According to [7], every attractor of an IFS of continuous maps on a first-countable Hausdorff topological space is renderable by the probabilistic chaos game. By contract, we will see that this is not the case of the deterministic chaos game. Namely, we will provide necessary and sufficient conditions to get the deterministic chaos game. As an application we will obtain that an IFS of homeomorphisms of the circle is renderable by the deterministic chaos game if and only if it is forward and backward minimal which provides examples of attractors that are not renderable by the deterministic chaos game.

1.1. Iterated function systems. Let X be a Hausdorff topological space. We consider a finite set $\mathscr{F} = \{f_1, \ldots, f_k\}$ of continuous functions from X to itself. Associated with this set $\mathscr{F}$ we define the *semigroup* $\Gamma = \Gamma_{\mathscr{F}}$ generated by these functions, the *Hutchinson operator* $F = F_{\mathscr{F}}$ on the hyperspace $\mathscr{H}(X)$ of non-empty compact subsets of X

$$F\colon \mathscr{H}(X) \to \mathscr{H}(X), \qquad F(A) = \bigcup_{i=1}^{k} f_i(A)$$

and the *skew-product* $\Phi = \Phi_{\mathscr{F}}$ on the product space of $\Omega = \{1, \ldots, k\}^{\mathbb{N}}$ and X

$$\Phi\colon \Omega \times X \to \Omega \times X, \qquad \Phi(\omega, x) = (\sigma(\omega), f_{\omega_1}(x)),$$

where $\omega = \omega_1\omega_2\ldots \in \Omega$ and $\sigma\colon \Omega \to \Omega$ is the lateral shift map. The action of the semigroup Γ on X is called the *iterated function system* generated by $f_1, \ldots, f_k$ (or, by the family $\mathscr{F}$ for short). Finally, given $\omega = \omega_1\omega_2\ldots \in \Omega$ and $x \in X$,

$$f^n_\omega \stackrel{\text{def}}{=} f_{\omega_n} \circ \ldots \circ f_{\omega_1} \text{ for every } n \in \mathbb{N}, \quad \text{and} \quad O^+_\omega(x) = \{f^n_\omega(x) : n \in \mathbb{N}\}$$

are called, respectively, the *orbital branch* corresponding to ω (or the IFS-iteration driven by the sequence ω) and the ω-*fiberwise orbit* of x. We introduce now a number of different notions of invariant and minimal sets and next give the definition of an attractor. In what follows A denotes a closed subset of X.

1.2. Invariant and minimal sets. We say that A is a *forward invariant* set if $f(A) \subset A$ for all $f \in \Gamma$. We also say that A is a *self-similar* set if

$$A = f_1(A) \cup \ldots \cup f_k(A).$$

Notice that a minimal set regarding to the inclusion of forward invariant non-empty (closed) sets is always a self-similar set. We simply call it a *forward invariant minimal set.* By extension, we say that an IFS is *forward minimal* if the unique forward invariant non-empty closed set is the whole space. It is not difficult to see that forward minimality is equivalent to density of any Γ-orbit. That is, A is a forward invariant minimal set if and only if A coincides with the closure of Γ-orbit

$$\Gamma(x) \stackrel{\text{def}}{=} \{g(x) : g \in \Gamma\} \quad \text{for all } x \in A.$$

Similarly, we will say that A is a *forward minimal set* if A is contained in the closure of $\Gamma(x)$ for all $x \in A$. Thus, forward minimal self-similar sets are forward invariant minimal sets and viceversa.

DEFINITION 1.1. We say that A is a *quasi-attractor* of the IFS generated by $\mathscr{F}$ if it is a forward minimal self-similar compact set, i.e., if

$$A \in \mathscr{H}(X), \quad F(A) = A \quad \text{and} \quad A = \overline{\Gamma(x)} \quad \text{for all } x \in A.$$

Finally, notice that, as a straightforward application of Zorn's lemma, every IFS on a compact space has a quasi-attractor.

1.3. Attractors. We introduce the notion of attractor following [7]–[10]. To accomplish this, we need to define first the pointwise basin of attraction.

Given a compact set K of X, the Ls-*limit set* (also called the ω-limit set or topological upper limit set) of K for F is the set

$$\operatorname{Ls} F^n(K) \stackrel{\text{def}}{=} \bigcap_{m\in\mathbb{N}} \overline{\bigcup_{n\geq m} F^n(K)}.$$

Observe that $\operatorname{Ls} F^n(S)$ is always closed. However it can be non-compact. Now, let A be a compact set. The *pointwise basin of* Ls-*attraction* of A for F is defined to be the set

$$\mathcal{B}_p^*(A) \stackrel{\text{def}}{=} \{x \in X : \operatorname{Ls} F^n(\{x\}) = A\}.$$

Similarly, the *pointwise basin of Vietoris-attraction* for F is the set

$$\mathcal{B}_p(A) \stackrel{\text{def}}{=} \Big\{x \in X : \lim_{n\to\infty} F^n(\{x\}) = A\Big\}.$$

The convergence here is with respect to the Vietoris topology, or equivalently, in the metric space case, with respect to the Hausdorff metric [33, pp. 66–69]. It is not difficult to show that $\mathcal{B}_p(A) \subset \mathcal{B}_p^*(A)$.

A compact set A is a *pointwise attractor* if there is an open set U of X such that $A \subset U \subset \mathcal{B}_p(A)$. A slightly stronger notion of an attractor is the following. A is said to be a *strict attractor* if there is an open neighbourhood U of A such that

$$\lim_{n\to\infty} F^n(K) = A \quad \text{in the Vietoris topology for all compact sets } K \subset U.$$

We denote by $\mathcal{B}(A)$ the *basin* of the strict attractor. That is, the union of all open neighbourhoods U of A such that the above convergence holds.

We remark that it is usual to include in the definition of attractor that $F(A) = A$ (cf. [10, Definition 2.2]). Under our mild assumptions on X, it is unknown the continuity of the Hutchinson operator (see [6]) and thus it is not, a priori, clear if A is a self-similar (F-invariant) set for the IFS. Nevertheless, the following result proves that any attractor is a quasi-attractor and, in the case of a strict attractor, attracts any compact set in the basin of attraction which, a priori, is also not clear from the definition.

Theorem 1.2. *Consider the* IFS *generated by* $\mathscr{F}$ *and a compact subset* A.

(a) *A is a quasi-attractor if and only if $A \subset \mathcal{B}_p^*(A)$. Moreover, in this case,*

$$A = \operatorname{Ls} F^n(K) \quad \textit{for all non-empty compact sets } K \subset A.$$

(b) *If A is a pointwise attractor, it is a quasi-attractor and $\mathcal{B}_p(A) = \mathcal{B}_p^*(A)$.*

(c) *If A is a strict attractor, it is a pointwise attractor, $\mathcal{B}(A) = \mathcal{B}_p(A)$, and for every non-empty compact set $K \subset \mathcal{B}(A)$*

$$\lim_{n\to\infty} F^n(K) = A \quad \textit{in the Vietoris topology.}$$

Another notion of an "attractor" of an IFS is the concept of a *semi-attractor* introduced by Lasota and Myjak in [24]. Semi-attractors, sometimes called semi-fractals, are the smallest (unique) forward invariant sets defined by means of the Kuratowski topological limits. We refer to [25], [26] for a precise definition. Thus, these sets are also forward invariant self-similar sets (in particular closed sets) but in contract with strict/pointwise attractors or quasi-attractors, semi-attractors can be non-compact.

Examples of pointwise attractors that are not strict attractors can be found in [7]. Also, one can easily construct quasi-attractors of IFSs that are neither attractors nor semi-attractors. A simple example is provided by the IFS generated by a minimal map f (for instance a rotation of the circle with irrational rotation number). The whole space A is the unique non-empty forward invariant closed set but it is not the limit in the Hausdorff metric of $F^n(\{x\}) = \{f^n(x)\}$ for any $x \in A$. However, it always holds that $\mathcal{B}_p^*(A) = A$.

1.4. Chaos game. Now, we focus our study to the chaos game of quasi-attractors of the IFS generated by $\mathscr{F}$ on X. In particular, this covers the cases of pointwise attractors, strict attractors, compact semi-attractors and minimal IFSs on a compact topological space. First, we will give a rigorous definition of the chaos game.

Following [9], we consider any probability $\mathbb{P}$ on Ω with the following property: there exists $0 < p \leq 1/k$ so that ω_n is selected randomly from $\{1, \dots, k\}$ in such a way that the probability of $\omega_n = i$ is greater than or equal to p, regardless the preceding outcomes, for all $i \in \{1, \dots, k\}$ and $n \in \mathbb{N}$. More formally, in terms of the conditional probability,

$$\mathbb{P}(\omega_n = i \mid \omega_{n-1}, \dots, \omega_1) \geq p. \tag{1.1}$$

Bernoulli measures on Ω are typical examples of these kinds of probabilities.

Definition 1.3. Let A be a quasi-attractor of the IFS generated by $\mathscr{F}$. We say that A is renderable by

(a) the *probabilistic chaos game* if for any $x \in \mathcal{B}^*_p(A)$ there is $\Omega(x) \subset \Omega$ with $\mathbb{P}(\Omega(x)) = 1$ such that
$$A \subset \overline{O^+_\omega(x)} \quad \text{for all } \omega \in \Omega(x);$$

(b) the *deterministic chaos game* if there is $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$ such that
$$A \subset \overline{O^+_\omega(x)} \quad \text{for all } \omega \in \Omega_0 \text{ and } x \in \mathcal{B}^*_p(A).$$

If the IFS is forward minimal (consequently A is the whole space and $\mathcal{B}^*_p(A) = A$) we simply say that the IFS is renderable by the probabilistic/deterministic chaos game.

The sequences in Ω which have a dense orbit under the shift map $\sigma\colon \Omega \to \Omega$ are called *disjunctive*. That is, the sequence $\omega = \omega_1\omega_2 \ldots \in \Omega$ which contains all finite words $\alpha = \alpha_1 \ldots \alpha_n \in \{1, \dots, k\}^n$ of length n, for all $n \geq 1$. Notice that the set consisting of all disjunctive sequences has $\mathbb{P}$-probability one and its complement is a σ-porous set with respect to the Baire metric in Ω [5]. The following result shows that the existence of a sequence ω such that every point in the basin of attraction has dense ω-fiberwise orbit on the self-similar set is enough to guarantee that for any disjunctive sequence we can also draw the quasi-attractor. This brings to light that actually the deterministic chaos game does not depend on the probability $\mathbb{P}$. This fact deradomizes the algorithm of the chaos game since disjunctive sequences in Ω are a priori well determined sequences. For this reason, the algorithm is called the *deterministic* chaos game (or *disjunctive* chaos game).

Theorem 1.4. *Consider the* IFS *generated by* $\mathscr{F}$ *and let* A *be a compact set of* X *and* $x \in \mathcal{B}^*_p(A)$. *Then,*

(a) A *and* $\overline{\Gamma(x)}$ *are forward invariant compact sets of* X *and* $A \subset \overline{\Gamma(x)}$. *In particular, for every* $n \in \mathbb{N}$ *and* $\omega \in \Omega$

$$\overline{\{f^m_\omega(x) : m \geq n\}} \quad \textit{is a compact set.}$$

(b) $A \subset \overline{O^+_\omega(x)}$ *if and only if*

$$\lim_{n\to\infty} \overline{\{f^m_\omega(x) : m \geq n\}} = A \quad \textit{in the Vietoris topology.}$$

(c) *If* A *is a quasi-attractor, the following are equivalent:*
  (c1) A *is renderable by the deterministic chaos game;*
  (c2) *there is* $\omega \in \Omega$ *such that* $A \subset \overline{O^+_\omega(z)}$ *for all* $z \in \mathcal{B}^*_p(A)$;
  (c3) $A \subset \overline{O^+_\omega(z)}$ *for all* $z \in \mathcal{B}^*_p(A)$ *and disjunctive sequences* $\omega \in \Omega$.

1.4.1. Probabilistic chaos game. Initially, the method was developed for contracting IFSs [3]. Later, it was generalized to attractors of IFSs of continuous functions on proper metric spaces [9]. For minimal IFSs in the case of independent identically distributed random product of continuous maps of a compact metric space the method follows from the Breiman's law of large numbers [17]. Recently in [7], Barnsley, Leśniak and Rypka proved probabilistic chaos games yield pointwise attractors of continuous IFSs on a first-countable Hausdorff topological space (in fact they only need to assume that the attractor is first-countable). Moreover, their proof also works with minor modifications for the general case of quasi-attractors (see Appendix A).

1.4.2. Deterministic chaos game. In the case of attractors of contractive IFSs a very simple justification of the deterministic chaos game can be given along the lines of [18, proof of Theorem 5.1.3]. In [11], the deterministic algorithm was also established for attractors of weakly hyperbolic IFSs, i.e., for point-fibred attractors (see the definition below), which are an extension of the attractors of contractive IFSs. Later, in [5] the deterministic chaos was obtained for a more general class of attractors, the so-called strongly-fibred attractors.

An attractor A is said to be *strongly-fibred* if for every open set $U \subset X$ such that $U \cap A \neq \emptyset$, there exists $\omega \in \Omega$ so that

$$A_\omega \stackrel{\text{def}}{=} \bigcap_{n=1}^{\infty} f_{\omega_1} \circ \ldots \circ f_{\omega_n}(A) \subset U.$$

Similarly, A is said to be *point-fibred* if A_ω is a singleton for all $\omega \in \Omega$. The same definitions can be given for quasi-attractors. We are going to introduce a similar category that we will call *well-fibred* following the proposal of Kieninger's classification of IFS attractors [22, p. 97], [8].

Definition 1.5. We say that a quasi-attractor A of the IFS is *well-fibred* if for every compact set K in A so that $K \neq A$ and for any open cover $\mathcal{U}$ of A, there exist $g \in \Gamma$ and $U \in \mathcal{U}$ such that $g(K) \subset U$. Equivalently, if there are $\omega \in \Omega$ and $U \in \mathcal{U}$ so that

$$K_\omega \stackrel{\text{def}}{=} \bigcap_{n=1}^{\infty} f_{\omega_1} \circ \ldots \circ f_{\omega_n}(K) \subset U.$$

It is not difficult to see that, in the metric space case, a quasi-attractor A is well-fibred if and only if for every compact set K in A so that $K \neq A$, there is a sequence $(g_n)_n \subset \Gamma$ such that the diameter $\operatorname{diam} g_n(K)$ converges to zero as $n \to \infty$. On the other hand, it is easy to show that strongly-fibred implies well-fibred. In fact, we will prove that if $f_i(A)$ is not equal to A for some generator f_i then both notions, strongly-fibred and well-fibred, are equivalent. After this observation, we can say that the following result generalizes [5].

Theorem 1.6. *Every well-fibred quasi-attractor A of an* IFS *of continuous maps of a Hausdorff topological space is renderable by the deterministic chaos game. Moreover, if A is either strongly-fibred or the generators of the* IFS *restricted to A are homeomorphisms, then*

$$\Omega \times A = \overline{\{\Phi^n(\omega, x) : n \in \mathbb{N}\}} \quad \textit{for all disjunctive } \omega \in \Omega \textit{ and } x \in A.$$

As a consequence, we will prove that every forward and backward minimal IFS of homeomorphisms $\mathscr{F}$ of a metric space so that the associated semigroup has a map with exactly two fixed points, one attracting and one repelling, is renderable by the deterministic chaos game (see Corollary 3.18). Backward minimality here means that the IFS generated by $\mathscr{F}^{-1} = \{f^{-1} : f \in \mathscr{F}\}$ is forward minimal.

New examples of attractors renderable by the deterministic chaos game which are not necessarily well-fibred were given in [12], [27]. Namely, in [12] the deterministic chaos was established for any forward and backward minimal IFS of homeomorphisms of the circle and for every IFS of a compact metric space that contains a minimal map. In [27] the deterministic algorithm was shown to work also for attractors of IFSs comprising maps which do not increase distances. In fact, basically with the same proof (see Appendix A), the result of Leśniak also holds for quasi-attractors of *non-expansive* IFSs, i.e., iterated function systems generated by a finite family $\mathscr{F}$ of maps of a metric space X so that

$$d(f(x), f(y)) \leq d(x, y) \quad \text{for all } f \in \mathscr{F}.$$

This class of systems includes equicontinuous IFSs (see [28, Lemma 3.2] and [30, Proposition 8]) and weakly hyperbolic IFSs (see [4, Theorem 1] and [2, Corollary 6.4]). However, a priori, there are no relations between quasi-attractors of non-expansive IFSs and strongly-fibred or well-fibred attractors.

In brief, it is known that the deterministic chaos algorithm holds in the following cases:

(1) well-fibred quasi-attractors of IFSs on Hausdorff topological spaces,
(2) quasi-attractors of non-expansive IFSs on metric spaces,
(3) forward and backward minimal IFSs of homeomorphisms of the circle,
(4) IFSs on a compact metric space having a minimal map.

The following theorem adds a new class of systems to this list: the quasi-attractors of symmetric IFSs. We say that an IFS generated by a family of homeomorphisms $\mathscr{F}$ of X is *symmetric* if for each $f \in \mathscr{F}$ it holds that $f^{-1} \in \mathscr{F}$.

Theorem 1.7. *Every quasi-attractor of a symmetric* IFS *on a Hausdorff topological space is renderable by the deterministic chaos game.*

We will give examples of symmetric non-minimal IFSs with a quasi-attractor which is not an attractor (Remark 3.5) and attractors of symmetric IFSs which are not included in the above list (Example 3.20). Moreover, we will prove that the phase space of a forward minimal symmetric IFS on a connected space is a strict attractor (Proposition 3.8).

1.4.3. Necessary condition to get a deterministic chaos game. The next result goes in the direction to provide necessary conditions to yield a deterministic chaos game. First we need to introduce the notion of backward minimality. A set A of X is said to be *backward invariant* for the IFS if

$$\emptyset \neq f^{-1}(A) \subset A \quad \text{for all } f \in \Gamma,$$

where $f^{-1}(A)$ denotes the preimage of A by the continuous map f. We say that the IFS is *backward minimal* if the unique backward invariant non-empty closed set is the whole space.

Theorem 1.8. *Every forward minimal* IFS *generated by continuous maps of a compact Hausdorff topological space that is renderable by the deterministic chaos game must also be backward minimal.*

As an application of the above result we can complete the main result in [12] obtaining the following corollary:

Corollary 1.9. *Let $f_1, \ldots, f_k$ be circle homeomorphisms. Then the following statements are equivalent:*

(a) *the* IFS *generated by $f_1, \ldots, f_k$ is renderable by the deterministic chaos game;*
(b) *there exists $\omega \in \Omega$ such that $\overline{O^+_\omega(x)} = S^1$ for all $x \in S^1$;*
(c) *the* IFS *generated by $f_1, \ldots, f_k$ is forward and backward minimal.*

This result allows us to construct a contra-example of the deterministic chaos game for general IFSs. More specifically, any forward minimal but not backward minimal IFS of homeomorphisms of the circle does not is renderable by the deterministic chaos game. Observe that for ordinary dynamical systems on the circle, the minimality of a map T is equivalent to that of T^{-1}. However this fact does not hold for IFSs with more than one generator:

Corollary 1.10. *There exists an* IFS *of homeomorphisms of the circle that is forward minimal but not backward minimal. Moreover, S^1 is a strict attractor of this* IFS *which, consequently, is not renderable by a deterministic chaos game.*

We want to indicate that, as we will see, most of minimal IFSs of homeomorphisms of the circle have S^1 as a strict attractor. Namely, we will prove that S^1 is a strict attractor of a minimal IFS of homeomorphisms of the circle if there is no common invariant measure for the generators (Proposition 3.10).

Organization of the paper. In Section 2 we study the basin of attraction of pointwise/strict attractors and we prove Theorem 1.2 and the two first conclusions of Theorem 1.4. We complete the proof of Theorem 1.4 in Section 3.1 where we study a deterministic chaos game. In Section 3.2 we prove Theorem 1.8 and in Section 3.3 we study a deterministic chaos game on the circle proving Corollaries 1.9 and 1.10. The proofs of Theorems 1.6 and 1.7 are developed in Section 3.4 where we study sufficient conditions for the deterministic chaos game. Finally, for completeness of the paper, we include an appendix where we extend the main results of [7] and [27] for the general case of quasi-attractors.

Standing notation. In the sequel, X denotes a Hausdorff topological space. We assume that we work with an IFS of continuous maps $f_1, \dots, f_k$ on X and we hold the above notations introduced in this section.

2. On the basin of attraction

We will study the basin of attraction of quasi-attractors. This allows us to prove Theorem 1.2 and statements (a) and (b) of Theorem 1.4.

2.1. Topological preliminaries. We start giving a basic topological lemma:

Lemma 2.1. *Let A and B be two compact sets in X.*

(a) *If $A \cap B = \emptyset$ then there exist disjoint open neighbourhoods of A and B.*
(b) *If $\{U_1, \dots, U_s\}$ is a finite open cover of A then there exist compact sets $A_1, \dots, A_s$ in X so that*

$$A = A_1 \cup \ldots \cup A_s \quad \textit{and} \quad A_i \subset U_i \quad \textit{for } i = 1, \dots, s.$$

Proof. The first statement is a well-known equivalent definition of a Hausdorff topological space (see [31, Lemma 26.4 and Exercice 26.5]). Hence, we only need to prove the second statement. First of all, notice that it suffices to prove the result for an open cover of A with two sets. So, let $\{U_1, U_2\}$ be an open cover of A. Since X is Hausdorff, A is a closed subset of X. Let us consider compact subsets $K_1 = A \setminus U_2 \subset U_1$ and $K_2 = A \setminus U_1 \subset U_2$. If $A \subset K_1 \cup K_2$ then we set $A_1 = K_1$, $A_2 = K_2$ and it is done. Otherwise, $A \setminus (K_1 \cup K_2)$ is a non-empty subset of A and it is easy to see that $K_1 \cap K_2 = \emptyset$. Since X is Hausdorff and K_1 and K_2 are compact disjoint subsets of X, by the first statement, there are disjoint open subsets V_i of X so that $K_i \subset V_i$, for $i = 1, 2$. We may assume that $V_i \subset U_i$. Now let us take $A_1 = A \setminus V_2 \subset U_1$ and $A_2 = A \setminus V_1 \subset U_2$. Then A_1 and A_2 are compact subsets of X and $A = A_1 \cup A_2$ which concludes the proof. $\square$

Let A and A_n, $n \geq 1$, be compact subsets of X. Following [15], we define the *upper Kuratowski limit* of $(A_n)_n$ as the set

$$\mathrm{Ls}\, A_n \stackrel{\text{def}}{=} \bigcap_{m \geq 1} \overline{\bigcup_{n \geq m} A_n}.$$

Observe that $\mathrm{Ls}\, A_n$ is a closed set and $\mathrm{Ls}\, A_n \subset B$ that provide $A_n \subset B$, for n sufficiently large, and B is a closed set. On the other hand, we recall that the Vietoris topology in $\mathscr{H}(X)$ is generated by the basic sets of the form

$$O\langle U_1, \ldots, U_m \rangle = \left\{ K \in \mathscr{H}(X) : K \subset \bigcup_{i=1}^{m} U_i,\ K \cap U_i \neq \emptyset \text{ for } k = 1, \ldots, m \right\},$$

where $U_1, \ldots, U_m$ are open sets in X and $m \in \mathbb{N}$. Hence, if $A_n \to A$ in the Vietoris topology then $A_n \in O\langle U \rangle$ for any n large enough and any open set U in X such that $A \subset U$. In particular, $A_n \subset U$ for all n sufficiently large. Moreover, we have the following:

Lemma 2.2. *$A_n \to A$ in the Vietoris topology if and only if for any pair of open sets U and V, such that $A \subset U$ and $A \cap V \neq \emptyset$, there is $n_0 \in \mathbb{N}$ so that*

$$\bigcup_{n \geq n_0} A_n \subset U \quad \textit{and} \quad V \cap A_n \neq \emptyset \quad \textit{for all } n \geq n_0.$$

In particular,

$$A = \mathrm{Ls}\, A_n \stackrel{\text{def}}{=} \bigcap_{m \geq 1} \overline{\bigcup_{n \geq m} A_n}. \tag{2.1}$$

Proof. Assume that $A_n \to A$ in the Vietoris topology. Let U be any open set such that $A \subset U$. By applying the above observation, there is $n_0 \in \mathbb{N}$ such that $A_n \subset U$ for all $n \geq n_0$. Now we will see that for any open set V with $A \cap V \neq \emptyset$, it holds that $A_n \cap V \neq \emptyset$ for all n sufficiently large. By the compactness of A, we extract open sets $U_1, \ldots, U_s$ in X such that

$$A \cap U_i \neq \emptyset \quad \text{and} \quad A \subset V \cup U_1 \cup \ldots \cup U_s.$$

Hence $O\langle V, U_1, \dots, U_s\rangle$ is an open neighbourhood of A in $\mathscr{H}(X)$. Since A_n converges to A then $A_n \in O\langle V, U_1, \dots, U_s\rangle$ for all n large enough and in particular $A_n \cap V \neq \emptyset$ for all n large.

We will prove the converse. Let $O\langle U_1, \dots, U_m\rangle$ be a basic open neighbourhood of A. Thus, $U_1, \dots, U_m$ are open sets in X and

$$A \subset U_1 \cup \ldots \cup U_m \stackrel{\text{def}}{=} U \quad \text{and} \quad A \cap U_i \neq \emptyset \quad \text{for all } i = 1, \dots, m.$$

By assumption, there is n_0 such that $A_n \subset U$ for all $n \geq n_0$. Moreover, since $A \cap U_i \neq \emptyset$, we also get n_i such that $A_n \cap U_i \neq \emptyset$ for all $n \geq n_i$ and $i = 1, \dots, m$. Therefore

$$A_n \in O\langle U_1, \dots, U_m\rangle \quad \text{for all } n \geq N = \max\{n_i : i = 0, \dots, m\}.$$

This implies that $A_n \to A$ in the Vietoris topology.

Finally we will prove (2.1). We have that $A \subset \operatorname{Ls} A_n$ since for every open neighbourhood V of any point in A there is $n_0 \in \mathbb{N}$ such that

$$A_n \cap V \neq \emptyset \quad \text{for all } n \geq n_0.$$

The reverse inclusion is equivalent to proving that for every compact set K such that $K \cap A = \emptyset$, there exists $n_0 \in \mathbb{N}$ so that $A_n \cap K = \emptyset$ for all $n \geq n_0$. But this is again a consequence of Lemma 2.1. Indeed, since K and A are compact sets, we can find disjoint open sets U and V such that $A \subset U$ and $K \subset V$. By the above characterization of the Vietoris convergence, there is $n_0 \in \mathbb{N}$ such that $A_n \subset U$ for all $n \geq n_0$. In particular $A_n \cap K = \emptyset$ for all $n \geq n_0$. $\square$

2.2. Proof of Theorem 1.2 and statements (a) and (b) of Theorem 1.4. We start proving statement (a) of Theorem 1.4.

Proposition 2.3. *Let A be a compact subset of X. If $x \in \mathcal{B}_p^*(A)$ then both, A and $\overline{\Gamma(x)}$, are forward invariant compact sets such that*

$$A = \operatorname{Ls} F^n(\{x\}) \stackrel{\text{def}}{=} \bigcap_{m \geq 1} \overline{\bigcup_{n \geq m} F^n(\{x\})} \quad \text{and} \quad \overline{\Gamma(x)} = \bigcup_{n \geq 1} F^n(\{x\}) \cup A.$$

In particular, $\overline{\{f_\omega^m(x) : m \geq n\}}$ is a compact set for all $n \in \mathbb{N}$ and $\omega \in \Omega$.

Proof. Set $K \stackrel{\text{def}}{=} \overline{\Gamma(x)}$. Since $x \in \mathcal{B}_p^*(A)$, by definition the above characterization of A, and consequently of K, follows.

Now we will show that K is compact. Let $\{U_\alpha : \alpha \in I\}$ be an open cover of K. Since $A \subset K$, by the compactness of A there exists a finite subset J_1 of I such that

$$A \subset \bigcup_{\alpha \in J_1} U_\alpha \stackrel{\text{def}}{=} U.$$

Again, by the above characterization of the set A and since

$$\overline{\bigcup_{n \geq m} F^n(\{x\})} \quad \text{for } m \geq 1$$

is a nested sequence, there is $n_0 \in \mathbb{N}$ such that the union of $F^n(\{x\})$ for $n \geq n_0$ is contained in U. On the other hand, the set $F(\{x\}) \cup \ldots \cup F^{n_0-1}(\{x\})$ is a finite union of compact sets and thus, it is compact. Hence, there is a finite subset J_2 of I such that

$$F(\{x\}) \cup \ldots \cup F^{n_0-1}(\{x\}) \subset \bigcup_{\alpha \in J_2} U_\alpha.$$

Putting all together and setting $J = J_1 \cup J_2$ we get

$$K = \overline{\Gamma(x)} = A \cup F(\{x\}) \cup \ldots \cup F^{n_0-1}(\{x\}) \cup \bigcup_{n \geq n_0} F^n(\{x\}) \subset \bigcup_{\alpha \in J} U_\alpha$$

implying that K is compact.

Moreover, clearly $F(K) \subset K$. Thus we have obtained that K is a compact Hausdorff topological space so that $F(K) \subset K$ and $A \subset K$. Hence, we can restrict the map F to the set of non-empty compact subsets of K.

According to [22, Proposition 1.5.3 (iv)], see also [6], the Hutchinson operator $F \colon \mathscr{K}(K) \to \mathscr{K}(K)$ is continuous and from the above characterization of the set A, it is easy to conclude that A is also a forward invariant compact set. This completes the proof of the proposition. □

Now, we characterize the quasi-attractors (statements (a) and (b) of Theorem 1.2).

Proposition 2.4. *Let A be a compact subset of X. Then*:

(a) $\mathcal{B}_p(A) \subset \mathcal{B}_p^*(A)$.

(b) *A is a quasi-attractor if and only if $A \subset \mathcal{B}_p^*(A)$. Moreover, in this case,*

$$\operatorname{Ls} F^n(K) = A \quad \textit{for every non-empty compact set } K \subset A$$
$$\textit{and } \overline{\Gamma(x)} \subset \mathcal{B}_p^*(A) \qquad \textit{for all } x \in \mathcal{B}_p^*(A).$$

(c) *If A is a pointwise attractor, it is a quasi-attractor and $\mathcal{B}_p(A) = \mathcal{B}_p^*(A)$.*

Proof. The first statement follows from the characterization (2.1) of the limit of $F^n(\{x\})$ in the Vietoris topology given in Lemma 2.2.

Assume that A is a quasi-attractor and let $x \in A$. We want to prove that $A = \operatorname{Ls} F^n(\{x\})$. Since $F^n(\{x\}) \subset A$ for all $n \geq 1$ and A is a closed set then $\operatorname{Ls} F^n(\{x\}) \subset A$. As in Proposition 2.3, $\operatorname{Ls} F^n(\{x\})$ is a forward invariant closed set and thus, by the minimality of A, $\operatorname{Ls} F^n(\{x\}) = A$ and $x \in \mathcal{B}_p^*(A)$. Moreover, the same argument also proves that $\operatorname{Ls} F^n(K) = A$ for every non-empty compact set $K \subset A$. In fact, since Proposition 2.3 implies that

$$\overline{\Gamma(y)} = \Gamma(y) \cup A \quad \text{for all } y \in \mathcal{B}_p^*(A),$$

for all $z \in \overline{\Gamma(y)}$, it holds that if $z \in A$ then $\operatorname{Ls} F^n(\{z\}) = A$ and if $z \in \Gamma(y)$ then $\operatorname{Ls} F^n(\{z\}) \subset \operatorname{Ls} F^n(\{y\}) = A$, and from the above arguments $\operatorname{Ls} F^n(\{z\}) = A$. Therefore the closure of $\Gamma(y)$ is contained in $\mathcal{B}_p^*(A)$ for all $y \in \mathcal{B}_p^*(A)$.

Suppose now that $A \subset \mathcal{B}_p^*(A)$. Hence $A = \operatorname{Ls} F^n(\{x\})$ for all $x \in A$. In particular, A is a forward minimal set and by Proposition 2.3 it is also a forward invariant set. Thus A is a forward invariant minimal set, that is, a quasi-attractor.

Finally, we will proof the last statement. By the first statement, it suffices to show that if A is a strict attractor then $\mathcal{B}_p^*(A) \subset \mathcal{B}_p(A)$. To accomplish this, let us consider $x \in \mathcal{B}_p^*(A)$ and open sets U, V such that $A \subset U$ and $A \cap V \neq \emptyset$. As A is a pointwise attractor, there is an open neighbourhood W of A so that $F(\{z\}) \to A$ in the Vietoris topology for all $z \in W$. Without loss of generality, we can assume that $U \subset W$. Since $A = \operatorname{Ls} F^n(\{x\})$, there exists $n_1 \in \mathbb{N}$ such that $F^n(\{x\}) \subset U$ for all $n \geq n_1$. Then, for every $z \in F^{n_1}(\{x\})$ we have that $F^n(\{z\})$ converges to A in the Vietoris topology and thus, by Lemma 2.2, there is $n_2 = n_2(z) \in \mathbb{N}$ so that

$$F^n(\{z\}) \cap V \neq \emptyset \quad \text{for all } n \geq n_2.$$

Taking $n_0 = \max\{n_2(z) : z \in F^{n_1}(\{x\})\}$ we get that $F^n(\{x\}) \cap V \neq \emptyset$ for all $n \geq n_0$ and therefore, Lemma 2.2 implies that $x \in \mathcal{B}_p(A)$. $\square$

We complete the proof of Theorem 1.2 by proving statement (c).

Proposition 2.5. *If A is a strict attractor, then:*

(a) $F^n(K) \to A$ *in the Vietoris topology for all compact sets* $K \subset \mathcal{B}(A)$.

(b) A *is a pointwise attractor and* $\mathcal{B}(A) = \mathcal{B}_p(A)$.

Proof. The first statement is a consequence of Lemma 2.1. Indeed, given any compact set K in $\mathcal{B}(A)$, by compactness we can find open neighbourhoods $U_1, \ldots, U_s$ of A such that $K \subset U_1 \cup \ldots \cup U_s$ and $F^n(S) \to A$ for any compact set S in U_i, for all $i = 1, \ldots, s$. By Lemma 2.1, there are compact sets $K_i \subset U_i$, $i = 1, \ldots, s$, such that $K = K_1 \cup \ldots \cup K_s$. Then,

$$F^n(K) = F^n(K_1) \cup \ldots \cup F^n(K_s)$$

and thus $F^n(K)$ converges to A in the Vietoris topology.

Let us pass to the second statement. Due to the first statement, $\mathcal{B}(A) \subset \mathcal{B}_p(A)$. Thus, since $\mathcal{B}(A)$ is an open set containing A we get that A is a pointwise attractor. To conclude, we will show that $\mathcal{B}_p(A) \subset \mathcal{B}(A)$. Given $x \in \mathcal{B}_p(A)$, we want to prove that x belongs to $\mathcal{B}(A)$.

Claim 2.6. *If there exists a neighbourhood V of x such that $F^n(K) \to A$ in the Vietoris topology for all non-empty compact sets $K \subset V$ then $x \in \mathcal{B}(A)$.*

Proof. Since A is an attractor there exists a neighbourhood U_0 of A such that $F^n(S) \to A$ for all compact sets S in U_0. Take $U = U_0 \cup V$. Clearly, U is a neighbourhood of A and $x \in U$. On the other hand, by Lemma 2.1, any compact set K in U can be written as the union of two compact sets K_0 and K_1 contained in U_0 and V, respectively. Now, since $F^n(K) = F^n(K_0) \cup F^n(K_1)$

it follows that $F^n(K)$ converges to A for all non-empty compact sets K in the neighbourhood U of A. This implies that $x \in \mathcal{B}(A)$. $\square$

Now, we will get a neighbourhood V of x in the assumptions of the above claim. Since $\mathcal{B}_p(A)$ is an open neighbourhood of A and $x \in \mathcal{B}(A)$, there is $m \in \mathbb{N}$ such that $F^m(\{x\}) \subset \mathcal{B}(A)$. Equivalently,

$$f_{\omega_m} \circ \ldots \circ f_{\omega_1}(x) \in \mathcal{B}(A) \quad \text{for all } \omega_i \in \{1, \ldots, k\}, \text{ for } i = 1, \ldots, m.$$

By the continuity of the generators $f_1, \ldots, f_k$ of the IFS, we get an open set V such that $x \in V$ and

$$f_{\omega_m} \circ \ldots \circ f_{\omega_1}(V) \subset \mathcal{B}(A) \quad \text{for all } \omega_i \in \{1, \ldots, k\}, \text{ for } i = 1, \ldots, m.$$

In particular, for every compact set K in V it holds that $F^m(K) \subset \mathcal{B}(A)$ and thus, by the first conclusion, $F^n(K)$ converges to A. Finally, Claim 2.6 implies that $x \in \mathcal{B}(A)$ as we wanted to show. $\square$

To end this section we will prove statement (b) of Theorem 1.4.

Proposition 2.7. *Consider $\omega \in \Omega$ and $x \in \mathcal{B}_p^*(A)$. Then the following statements are equivalent*:

(a) $A \subset \overline{O_\omega^+(x)}$.
(b) $\lim_{n\to\infty} \overline{\{f_\omega^m(x) : m \geq n\}} = A$ *in the Vietoris topology.*
(c) $A = \bigcap_{n\geq 1} \overline{\{f_\omega^m(x) : m \geq n\}}$.

Proof. According to Proposition 2.3, $A_n = \overline{\{f_\omega^m(x) : m \geq n\}}$ is a compact set for all $n \in \mathbb{N}$. Moreover, $A_{n+1} \subset A_n$ and hence, by Lemma 2.2, if $A_n \to A$ in the Vietoris topology,

$$A = \bigcap_{n\geq 1} A_n \subset A_1 = \overline{O_\omega^+(x)}.$$

This proves that (b) implies (a).

Reciprocally, let U and V be open sets such that $A \subset U$ and $V \cap A \neq \emptyset$. Since $x \in \mathcal{B}_p^*(A)$ then $A = \operatorname{Ls} F^n(\{x\})$ and thus there exists $n_0 \in \mathbb{N}$ such that

$$\bigcup_{n\geq n_0} F^n(\{x\}) \subset U.$$

In particular, the union of A_n for $n \geq n_0$ is contained in U. Moreover, since $A \subset \overline{O_\omega^+(x)}$ we have $A_n \cap V \neq \emptyset$ for all n large enough. Lemma 2.2 implies that $A_n \to A$ in the Vietoris topology, completing the proof that (a) implies (b).

Finally, by Lemma 2.2, we have that (b) implies (c) and one can easily see that (c) implies (a), concluding the proof of the proposition. $\square$

3. Deterministic chaos game

3.1. Equivalence. We will conclude the proof of Theorem 1.4 proving statement (c).

Proposition 3.1. *Let A be a quasi-attractor. Then the following statements are equivalent*:

(a) *There exists $\omega \in \Omega$ such that $A \subset \overline{O^+_\omega(x)}$ for all $x \in \mathcal{B}^*_p(A)$.*
(b) *$A \subset \overline{O^+_\omega(x)}$ for all disjunctive sequences $\omega \in \Omega$ and $x \in \mathcal{B}^*_p(A)$.*
(c) *There is $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$ such that*

$$A \subset \overline{O^+_\omega(x)} \quad \textit{for all } \omega \in \Omega_0 \textit{ and } x \in \mathcal{B}^*_p(A).$$

Proof. It suffices to show that (a) implies (b). Let x be a point in $\mathcal{B}^*_p(A)$. According to Proposition 2.3,

$$K \stackrel{\text{def}}{=} \overline{\Gamma(x)} \subset \mathcal{B}^*_p(A)$$

and it is a forward invariant compact set.

The following claim will be useful to prove the density of disjunctive fiberwise orbits, i.e., of fiberwise orbits driven by disjunctive sequences:

Claim 3.2. *Let Z be a forward invariant set such that $A \subset Z$. If for any non-empty open set $I \subset X$ with $A \cap I \neq \emptyset$ there is $f_{i_s} \circ \ldots \circ f_{i_1} \in \Gamma$ such that*

$$\textit{for each } z \in Z \textit{ there is } t \in \{1, \ldots, s\} \textit{ so that } f_{i_t} \circ \ldots \circ f_{i_1}(z) \in I,$$

then $A \subset \overline{O^+_\omega(x)}$ for all disjunctive sequences $\omega \in \Omega$ and $x \in Z$.

Proof. Consider any open set I such that $A \cap I \neq \emptyset$, $x \in Z$ and a disjunctive sequence $\omega \in \Omega$. Using the fact that ω is a disjunctive sequence and that Z is a forward invariant set we can choose $m \geq 1$ such that

$$[\sigma^m(\omega)]_j = i_j \quad \text{for } j = 1, \ldots, s \text{ and } z = f^m_\omega(x) \in Z.$$

Hence, by assumption, there exists such $t = t(z)$ that $f^{m+t(z)}_\omega(x) \in I$ which proves the density on A of the ω-fiberwise orbit of x. $\square$

Notice that $F(K) \subset K$ and hence we can take $Z = K$ in the above claim. Let I be an open set so that $I \cap A \neq \emptyset$. By assumption, since $Z \subset \mathcal{B}_p(A)$, there exists a sequence $\omega \in \Omega$ such that for each point $z \in Z$ the ω-fiberwise orbit of z is dense in A. In particular, there is $n = n(z) \in \mathbb{N}$ such that

$$\{f^m_\omega(z) : m \leq n\} \cap I \neq \emptyset.$$

By continuity of generators $f_1, \ldots, f_k$ of the IFS, there exists an open neighbourhood V_z of z such that

$$\{f^m_\omega(y) : m \leq n\} \cap I \neq \emptyset \quad \text{for all } y \in V_z.$$

Then, the compactness of Z implies that we can extract open sets $V_1, \ldots, V_r$ and positive integers $n_1, \ldots, n_r$ such that $Z \subset V_1 \cup \ldots \cup V_r$ and

$$\{f^m_\omega(z) : m \leq n_i\} \cap I \neq \emptyset \quad \text{for all } z \in V_i \text{ and } i = 1, \ldots, r.$$

Hence the assumptions of Claim 3.2 hold for $f_{\omega_s} \circ \ldots \circ f_{\omega_1} \in \Gamma$ where $s = \max\{n_i : i = 1, \ldots, r\}$. Therefore, since the initial point $x \in \mathcal{B}^*_p(A)$ belongs to Z, we conclude that any disjunctive fiberwise orbit of x is dense in A, what completes the proof. □

3.2. Necessary condition.

Proof of Theorem 1.8. Clearly if there is a minimal orbital branch, i.e., $\omega = \omega_1\omega_2 \ldots \in \Omega$ such that $O^+_\omega(x)$ is dense for all x, then the IFS is forward minimal.

We will assume that it is not backward minimal. Then, there exists a non-empty closed set $K \subset X$ such that $\emptyset \neq f^{-1}(K) \subset K \neq X$ for all $f \in \Gamma$. Consider

$$K^-_n = \bigcap_{i=1}^{n} f^{-1}_{\omega_1} \circ \ldots \circ f^{-1}_{\omega_i}(K) = f^{-1}_{\omega_1} \circ \ldots \circ f^{-1}_{\omega_n}(K) \quad \text{and} \quad K^-_\omega = \bigcap_{n=1}^{\infty} K^-_n.$$

It is a nested sequence of closed sets. By assumption of theorem, the space X, where the IFS is defined, is a compact Hausdorff topological space. As a consequence, K^-_ω is not empty and then for every $x \in K^-_\omega$ we have that $O^+_\omega(x) \subset K$. Since K is not equal to X it follows that there exists a point $x \in X$ so that the ω-fiberwise orbit of x is not dense. But this is a contradiction and we are done. □

As we notified in the introduction, an IFS is forward minimal if and only if every point has a dense Γ-orbit. To complete the section we want to point out the following straightforward equivalent definition of backward minimality.

Lemma 3.3. *Consider an* IFS *of surjective continuous maps of a topological space X. Then the* IFS *is backward minimal if and only if*

$$X = \overline{\Gamma^{-1}(x)} \quad \textit{for all } x \in X,$$

where $\Gamma^{-1}(x) \stackrel{\text{def}}{=} \{y \in X : \textit{there exists } g \in \Gamma \textit{ such that } g(y) = x\}$.

3.3. Minimal IFSs of homeomorphisms of the circle. As a consequence of Theorem 1.8, we will obtain that the deterministic chaos game is totally characterized for forward minimal IFSs of homeomorphisms of the circle. Moreover, this characterization allows us to construct attractors of IFSs that are not renderable by a deterministic chaos game (counterexamples).

3.3.1. Characterization. In [12, Theorem A] it was proved that every forward and backward minimal IFS of preserving-orientation homeomorphisms of the circle is renderable by the deterministic chaos game. However, the assumption of preserving-orientation can be removed from this statement as we explain below.

The main tool in the proof of [12, Theorem A] was Antonov's Theorem [1] (see [12, Theorem 2.1]), stated for preserving-orientation homeomorphisms of the circle. A key lemma for the proof was [12, Lemma 2.2]. The proof of this lemma, by means of Antonov's result, is the only point where the preserving-orientation assumption was used. This lemma can be improved removing the preserving-orientation assumption by two different ways. The first is by observing in the original proof of Antonov that in fact this assumption is not necessary (as one can easily see from the argument described in [21, proof of Theorem 2]). Another way is to use the recent generalization of Antonov's result [29, Theorem D] instead of the key lemma.

Proof of Corollary 1.9. From the above, every forward and backward minimal IFS of homeomorphisms of the circle is renderable by the deterministic chaos game. That is, (c) implies (a). On the other hand, the fact that (a) implies (c) follows from Theorem 1.8. Finally, to complete the proof of the corollary it suffices to note that according to Theorem 1.4, (a) and (b) are equivalent. □

3.3.2. Counterexample. We will prove now Corollary 1.10. As we mentioned in the introduction, for ordinary dynamical systems the minimality of a map T is equivalent to that of T^{-1}. Nevertheless this is not the case for dynamical systems with several maps, as Kleptsyn and Nalskii pointed at [23, p. 271]. However, they omitted to include examples of forward but not backward minimal IFSs. Hence, to provide a complete proof of Corollary 1.10 we will show that indeed such IFSs of homeomorphisms of S^1 can be constructed.

(A) Forward but not backward minimal IFSs on the circle. Consider a group G of homeomorphisms of the circle. Then, there can occur only one of the following three options [32], [20]:

(1) there is a finite G-orbit,
(2) every G-orbit is dense on the circle, or
(3) there is a unique G-invariant minimal Cantor set.

By a G-orbit we understand the action of G at a point $x \in S^1$. That is the set of points $G(x) = \{g(x) : g \in G\}$. If $G(x)$ has finitely many different elements then it is called a *finite orbit*, while if its closure is S^1 it is called a *dense orbit*. The Cantor set K in (3) is usually called the *exceptional minimal set.* This set is G-invariant and minimal, that is,

$$g(K) = K \quad \text{for all } g \in G \quad \text{and} \quad K = \overline{G(x)} \quad \text{for all } x \in K.$$

Notice that these properties are the same as to say that K is minimal regarding to the inclusion of G-invariant closed sets. The following proposition is stated in [32, Exercise 2.1.5]. For completeness, we include the proof.

Proposition 3.4. *There exists a finitely generated group G of homeomorphisms of S^1 admitting an exceptional minimal set K such that the G-orbit of every point of $S^1 \setminus K$ is dense in S^1.*

Proof. Let f be a homeomorphism of the circle with a minimal exceptional set K and such that there is only one class of gaps, which means that for all gaps I, J, there exists n in $\mathbb{Z}$ such that $f^n(I) = J$. For instance, the classic Denjoy map. Let I_0 be a gap of K. Let $u\colon I_0 \to \mathbb{R}$ be a homeomorphism, $\widetilde{f}_1$ and $\widetilde{f}_2$ be respectively the translations $x \mapsto x+1$ and $x \mapsto x+\sqrt{2}$ on $\mathbb{R}$, and let us define two homeomorphisms f_1 and f_2 of S^1 by $f_i = u^{-1}\widetilde{f}_i u$ on I_0, $f_i = \mathrm{id}$ on $S^1 \setminus I_0$. We claim that the group G generated by f, f_1 and f_2 satisfies the required properties. Obviously, K is also the minimal exceptional set of G since $f_i|_K = \mathrm{id}$. On the other hand, the subgroup H generated by f_1 and f_2 leaves the gap I_0 invariant and acts minimally on it since the group generated by $\widetilde{f}_1$ and $\widetilde{f}_2$ acts minimally on $\mathbb{R}$. Hence, let x be in $S^1 \setminus K$ and I be an interval of the circle. Since there is only one class of gaps one can find m and n in $\mathbb{Z}$ such that $f^m(x) \in I_0$ and $f^n(I) \cap I_0 \neq \emptyset$. Next, by minimality of the action of H on I_0, one can find h in H such that $h(f^m(x)) \in f^n(I)$. Thus, the element $g = f^{-n} h f^m$ of G sends x into I. Since I is arbitrary by G, the orbit of x by G is dense. $\square$

We say that a subset $\mathscr{F}$ of a group G is a *symmetric generating system* of G if G is generated by $\mathscr{F}$ as a semigroup. Moreover, we require that if $f \in \mathscr{F}$ then also $f^{-1} \in \mathcal{F}$. Hence, we can see the action of the group as a symmetric IFS generated by $\mathscr{F}$.

Remark 3.5. Let $\mathscr{F}$ be the finite symmetric generating system of the group G given in Proposition 3.4. From the above observation, it follows that the exceptional minimal set K is the unique quasi-attractor of the symmetric IFS generated by $\mathscr{F}$ and $\mathcal{B}^*_p(K) = K$. This provides an example of a quasi-attractor of a non-minimal IFS which cannot be a pointwise attractor.

We will use the following:

Lemma 3.6. *Let G and K be as in Proposition 3.4. Then the closed subsets of S^1 which are invariant by G are $\emptyset, K$ and S^1.*

Proof. Let B be a closed subset of S^1 invariant by G. If $B \neq \emptyset$, then $K \subset B$ by minimality of K, and if $B \neq K$, it means that B contains a point x in $S^1 \setminus K$, and by invariance, B contains the orbit of x by G which is dense, hence $B = S^1$. $\square$

On the other hand, any two Cantor sets are homeomorphic. In fact, if K_I and K_J are two Cantor sets in intervals I and J, respectively, there exists a homeomorphism $g\colon I \to J$ so that $g(K_I) = K_J$ (see for instance [16]). Hence given any Cantor set K in S^1, one can find a homeomorphisms h of S^1 so that $h(K)$ is strictly contained in K (or $h(K)$ strictly contains K).

Proposition 3.7. *Let G and K be as in Proposition 3.4 and $f_1, \dots, f_n$ be a symmetric system of generators of G. Consider any homeomorphism h of S^1 such that $h(K)$ strictly contains K. Then the IFS generated by $f_1, \dots, f_n, h$ is forward minimal but not backward minimal.*

Proof. Let $K_1 \stackrel{\text{def}}{=} h(K)$. By assumption, $K \subsetneq K_1$. We claim that the IFS generated by $f_1, \dots, f_n, h$ is forward minimal but not backward minimal.

(a) The IFS is *not backward minimal*: since K is invariant by the group G,

$$f_i^{-1}(K) = K \quad \text{ for } i = 1, \dots, n.$$

We also have $h^{-1}(K) \subset h^{-1}(K_1) = K$. Thus, K is forward invariant by $f_1^{-1}, \dots, f_n^{-1}, h^{-1}$ and so the IFS is not backward minimal.

(b) The IFS is *forward minimal*: let $B \subset S^1$ be a forward invariant by $f_1, \dots, f_n, h$ closed set. In particular, B is invariant by G, hence $B \in \{\emptyset, K, S^1\}$ by Lemma 3.6. Moreover, $B \neq K$ since K is not invariant by h (otherwise $K_1 = h(K) = h(B) \subset B = K$ but K_1 strictly contains K). So, $B \in \{\emptyset, S^1\}$, which means that the IFS is forward minimal. □

(B) Strict attractors. To complete the proof of Corollary 1.10 we need to show that S^1 is a strict attractor of the IFS generated by $f_1, \dots, f_n, h$ in Proposition 3.7. We infer this from the next result.

We say that the IFS generated by a family $\mathscr{F}$ of continuous maps of X is *quasi-symmetric* if there is $f \in \mathscr{F}$ so that its inverse map $f^{-1} \in \mathscr{F}$.

Proposition 3.8. *Consider a minimal quasi-symmetric IFS on a compact connected Hausdorff space X. Then X is a strict attractor of this IFS.*

We shall need the following lemma (c.f. [29, Lemma 4.15]). Again, for completeness, we include the proof.

Lemma 3.9. *Consider the minimal IFS generated by a family $\mathscr{F}$ of continuous maps of a connected Hausdorff topological space X. Then the IFS generated by $\mathscr{F}^2 = \{f \circ g : f, g \in \mathscr{F}\}$ is also minimal.*

Proof. Throughout the proof, we extend the Hutchinson operator $F = F_{\mathscr{F}}$ to the hyperspace of non-empty closed sets. We want to prove that if B is a non-empty closed subset of X so that $F^2(B) \subset B$ then $B = X$. Notice that

$$B' \stackrel{\text{def}}{=} B \cup F(B) \quad \text{and} \quad B'' \stackrel{\text{def}}{=} B \cap F(B)$$

are both forward invariant sets, i.e., $F(B') \subset B'$ and $F(B'') \subset B''$. By the minimality of the IFS generated by $\mathscr{F}$ it follows that $B' = X$. Hence, since X is a connected space and both B and $F(B)$ are closed we get that $B'' \neq \emptyset$. Thus, again by the minimality we have that $B'' = X$ and therefore $B = X$. $\square$

Proof of Proposition 3.8. Let K be a compact set of X. We want to show that $F^n(K) \to X$ in the Vietoris topology. The first observation is that, since the IFS is quasi-symmetric, then $K \subset F^2(K)$. So,

$$(F^2)^n(K) \subset (F^2)^{n+1}(K) \quad \text{for all } n \geq 1.$$

By Lemma 3.9, the IFS generated by $\mathscr{F}^2$ is also minimal, and thus for every open set V of X there is $n_0 \in \mathbb{N}$ such that $(F^2)^{n_0}(K) \cap V \neq \emptyset$. So, by the monotonicity of this sequence, $(F^2)^n(K) \cap V \neq \emptyset$ for all $n \geq n_0$. Thus, according to Lemma 2.2, we have that $(F^2)^n(K) \to X$ in the Vietoris topology. Due to the continuity of the Hutchinson operator F and since X is a self-similar set, we also have that $F^{2n+1}(K) \to X$. Thus, we conclude that $F^n(K) \to X$. $\square$

Let us remark that in the case of homeomorphisms of the circle we have a stronger result:

Proposition 3.10. *Let $f_1, \ldots, f_k$ be homeomorphisms of S^1 without a common invariant probability measure, and such that the* IFS *generated by them is minimal. Then S^1 is a strict attractor for this* IFS.

Proof. Let x in S^1, and let μ_n be the law of $f^n_\omega(x) = f_{\omega_n} \circ \ldots \circ f_{\omega_1}(x)$, where $\omega_1, \ldots, \omega_n$ are chosen independently and uniformly on $\{1, \ldots, k\}$. By [29, Corollary 2.6], the sequence $(\mu_n)_{n\in\mathbb{N}}$ converges weakly as $n \to \infty$ to the unique stationary probability measure μ of the system, i.e., to the self-similar measure

$$\mu = \frac{1}{k}\,((f_1)_*\mu + \ldots + (f_k)_*\mu).$$

Moreover, this measure μ has total support because its topological support is invariant by $f_1, \ldots, f_k$. Consequently, for any interval I of the circle, $\mu(I) > 0$ and so $\mu_n(I) > 0$ for n large enough. Since we clearly have that $\operatorname{supp}(\mu_n) \subset F^n(\{x\})$, we deduce that $F^n(\{x\}) \cap I \neq \emptyset$ for all n sufficiently large, and hence we conclude by Lemma 2.2 that S^1 is an attractor. $\square$

3.4. Sufficient conditions. In what follows, A denotes a quasi-attractor.

3.4.1. Well-fibred attractors. We start studying the relation between strongly-fibred and well-fibred quasi-attractors.

Proposition 3.11. *If A is strongly-fibred then it is well-fibred. Moreover, if in addition A is a strict attractor then for every compact set K in $\mathcal{B}(A)$ and every open set U so that $A \cap U \neq \emptyset$ there exists $g \subset \Gamma$ such that $g(K) \subset U$.*

Proof. Consider a compact set K in A and let U be any open set such that $A \cap U \neq \emptyset$. Since A is strongly-fibred, we get $\omega \in \Omega$ such that

$$A_\omega = \bigcap_{n=1}^{\infty} f_{\omega_1} \circ \ldots \circ f_{\omega_n}(A) \subset U.$$

Notice that since $f_i(A) \subset A$ for $i = 1, \ldots, k$ then $f_{\omega_1} \circ \ldots \circ f_{\omega_n}(A)$ is a nested sequence of compact sets and thus, for n large enough, $f_{\omega_1} \circ \ldots \circ f_{\omega_n}(A) \subset U$. In particular, taking $h = f_{\omega_1} \circ \ldots \circ f_{\omega_n} \in \Gamma$ we have that $h(K) \subset U$. This proves that A is well-fibred.

We will assume now that A is a strict attractor and consider K in $\mathcal{B}(A)$. As above we have that $h(A) \subset U$. We claim that there exists a neighbourhood V of A such that $h(V) \subset U$. Indeed, it suffices to note that h is a continuous map and hence $h^{-1}(U)$ is an open set containing the compact set A. Since A is a strict attractor, $F^n(K) \to A$ in the Vietoris topology and in particular, there is $f \in \Gamma$ such that $f(K) \subset V$. Thus, taking $g = h \circ f \in \Gamma$, it follows that $g(K) \subset h(V) \subset U$. $\square$

Remark 3.12. If A is strongly fibred we have proved that one can contract any compact set in A. In particular we can contract A and this implies that there exists some generator f_i such that $f_i(A) \neq A$.

Now, we give an example of an IFS defined on S^1 whose unique strict attractor is the whole space (that is the IFS is minimal) and it is well-fibred but not strongly-fibred. This example shows that these two properties are not equivalent. See also Corollary 3.18 at the end of this subsection.

Example 3.13. Consider the IFS generated by two diffeomorphisms g_1, g_2, where g_1 is the rotation with irrational rotation number and g_2 is an orientation preserving diffeomorphism with a unique fixed point p such that $Dg_2(p) = 1$ and the α-limit and ω-limit sets of each point $q \in S^1$ are equal to $\{p\}$. Clearly, the IFS acts minimally on S^1 and has no common invariant measure thus $A = S^1$ is an attractor. Since g_1 and g_2 map S^1 onto itself, it follows that for each $\omega \in \Omega$, the fiber $A_\omega = S^1$. This implies that S^1 is not strongly-fibred, but it is still well-fibred. Indeed, let K be any compact set so that $K \neq S^1$. Then, there is an open arc J of S^1 which is not dense in S^1 such that $K \subset J$. If J contains the fixed point p, there is an integer n such that $g_1^n(J)$ does not contain p. So, without loss of generality, we may assume that $p \notin J$. Now, it is easy to see that $g_2^k(J)$ tends to p as $k \to \infty$. This implies that $A = S^1$ is well-fibred. $\square$

The above example is based on the fact that A satisfies that $f_i(A) = A$ for all $i = 1, \ldots, k$. The above and the next propositions show that if $f_i(A)$ is not equal to A for some generator f_i then both properties are equivalent.

Proposition 3.14. *If A is well-fibred and $f_i(A) \neq A$ for some $i \in \{1, \ldots, k\}$ then A is strongly-fibred.*

Proof. First of all note that it suffices to prove that for any open set U with $U \cap A \neq \emptyset$, there is $h \in \Gamma$ so that $h(A) \subset U$. To this end, notice that since A is a quasi-attractor then the action of Γ restricted to A is minimal. Then, there exist $h_1, \ldots, h_m \in \Gamma$ so that $A \subset h_1^{-1}(U) \cup \ldots \cup h_m^{-1}(U)$. On the other hand, by assumption, there is $i \in \{1, \ldots, k\}$ such that $f_i(A) \neq A$. Hence $f_i(A)$ is a compact set strictly contained in A and since A is well-fibred there exist $g \in \Gamma$ and $j \in \{1, \ldots, m\}$ such that $g(f_i(A)) \subset h_j^{-1}(U)$. Thus, taking $h = h_j \circ g \circ f_i \in \Gamma$, it follows that $h(A) \subset U$, concluding the proof. $\square$

In order to proof Theorem 1.6, we need a lemma (compare with Claim 3.2). Here we understand $f_{i_t} \circ \ldots \circ f_{i_1}$ for $t = 0$ as the identity map.

Lemma 3.15. *If for any non-empty open set $I \subset X$ with $A \cap I \neq \emptyset$, there exist a neighbourhood Z of A and $f_{i_s} \circ \ldots \circ f_{i_1} \in \Gamma$ such that*

$$\text{for each } z \in Z \text{ there is } t \in \{0, \ldots, s\} \text{ so that } f_{i_t} \circ \ldots \circ f_{i_1}(z) \in I$$

then $A \subset \overline{O^+_\omega(x)}$ for all disjunctive sequences $\omega \in \Omega$ and $x \in \mathcal{B}^*_p(A)$.

Proof. Consider $x \in \mathcal{B}^*_p(A)$, as disjunctive sequence $\omega \in \Omega$ and any open set I such that $A \cap I \neq \emptyset$. As Z is a neighbourhood of A and $\operatorname{Ls} F^n(\{x\}) = A$, we can choose $m \geq 1$ such that

$$[\sigma^m(\omega)]_j = i_j \quad \text{for } j = 1, \ldots, s \qquad \text{and} \qquad z = f^m_\omega(x) \in Z.$$

Hence, by assumption, there exists $t = t(z)$ such that $f^{m+t}_\omega(x) \in I$ which proves the density on A of the ω-fiberwise orbit of x. $\square$

The following result proves the first part in Theorem 1.6.

Proposition 3.16. *If A is a well-fibred quasi-attractor then it is renderable by the deterministic chaos game.*

Proof. In order to apply Lemma 3.15, we consider any non-empty open set I with $I_A \overset{\text{def}}{=} A \cap I \neq \emptyset$. Hence, $K = A \setminus I_A$ is a compact set so that $K \neq A$. Since A is a quasi-attractor, the action of Γ restricted to A is minimal and thus, there exist $h_1, \ldots, h_m \in \Gamma$ such that $A \subset h_1^{-1}(I) \cup \ldots \cup h_m^{-1}(I)$. On the other hand, since A is well-fibred, there exist $i \in \{1, \ldots, m\}$ and $g \in \Gamma$ such that $g(K) \subset h_i^{-1}(I)$. Take $h = h_i \circ g$. By continuity of the generators we can find an open set U with $K \subset U$ such that $h(U) \subset I$. Take, $Z = U \cup I$ and $f_{i_s} \circ \ldots \circ f_{i_1} = h \in \Gamma$. Clearly, Z is open with $A = K \cup I_A \subset U \cup I = Z$ and for every $z \in Z$, there is $t \in \{0, s\}$ such that $f_{i_t} \circ \ldots \circ f_{i_1}(z) \in I$. Lemma 3.15 implies that A is renderable by the deterministic chaos game. $\square$

Now, we conclude the proof of Theorem 1.6.

Proposition 3.17. *Consider a well-fibred forward minimal* IFS *generated by continuous maps of a compact Hausdorff topological space A. Assume the* IFS *is either strongly-fibred or invertible (its generators are homeomorphisms). Then*

$$\Omega \times A = \overline{\{\Phi^n(\omega,x) : n \in \mathbb{N}\}} \quad \textit{for all disjunctive sequences } \omega \textit{ and } x \in A.$$

Proof. Let $\omega \in \Omega$ be a disjunctive sequence and consider $x \in A$. We want to show that (ω,x) has a dense orbit in $\Omega \times A$ under the skew-product Φ. In order to prove this, let $C^+_\alpha \times I$ be a basic open set of $\Omega \times A$. That is, C^+_α is a cylinder in Ω around a finite word $\alpha = \alpha_1 \dots \alpha_\ell$ and I is an open set in A. In fact, we can assume that I is not equal to the whole space. It suffices to prove that there exists an iteration by Φ of (ω,x) that belongs to $C^+_\alpha \times I$. To do this, similarly as in the previous proposition, we use the forward minimality of Γ on A to find maps $h_1,\dots,h_m \in \Gamma$ such that $A = h_1^{-1}(I) \cup \dots \cup h_m^{-1}(I)$. Set $f = f_{\alpha_\ell} \circ \dots \circ f_{\alpha_1} \in \Gamma$.

Assume first that the IFS is strongly-fibred. Then there exists a generator f_i such that $K = f_i(A) \neq A$. By Proposition 3.11, the IFS is well-fibred and thus we can find $g \in \Gamma$ such that $g(K) \subset h_\ell^{-1}(I)$ for some $\ell \in \{1,\dots,m\}$. Hence, $h(K) \subset I$ where $h = h_\ell \circ g$. Let $f \circ h \circ f_i = f_{i_s} \circ \dots \circ f_{i_1}$. Since $\omega \in \Omega$ is a disjunctive sequence, we can choose $m \geq 1$ such that

$$[\sigma^m(\omega)]_j = i_j \quad \text{for } j = 1,\dots,s. \tag{3.1}$$

Set $z = f^m_\omega(x)$. Then, $h \circ f_i(z) \in I$. Moreover,

$$\Phi^{m+t}(\omega,x) = \Phi^t(\sigma^m(\omega),z) = (\sigma^{m+t}(\omega), h \circ f_i(z)) \in C^+_\alpha \times I,$$

where $t = 1 + |h|$ with $|h|$ the length of h with respect to $\mathscr{F} = \{f_1,\dots,f_k\}$.

Now, assume that the well-fibred forward minimal IFS is also invertible. Hence f is a homeomorphism of A and thus $\emptyset \neq f(I) \neq A$ is an open set. Let $K = A \setminus f(I)$. Notice that K is a non-empty compact set different from A and by means of the "contractibility" of the IFS we get $g \in \Gamma$ so that $h(K) \subset I$ where $h = h_\ell \circ g$ for some $\ell \in \{1,\dots,m\}$. Let $f \circ h \circ f = f_{i_s} \circ \dots \circ f_{i_1}$. Similar as above, since ω is a disjunctive sequence we choose $m \geq 1$ satisfying (3.1) and denote $z = f^m_\omega(x)$. If $z \in I$, $\Phi^m(\omega,x) = (\sigma^m(\omega),z) \in C^+_\alpha \times I$. Otherwise, $f(z) \in K$ and then $h \circ f(z) \in I$ and thus

$$\Phi^{m+t}(\omega,x) = \Phi^t(\sigma^m(\omega),z) = (\sigma^{m+t}(\omega), h \circ f(z)) \in C^+_\alpha \times I,$$

where $t = |f| + |h|$ with $|f|$ and $|h|$ the lengths of f and h respectively. $\square$

We end this subsection presenting a broad family of IFSs with a well-fibred quasi-attractor which are not strongly-fibred. Notice that this family contains the IFS from Example 3.13.

Corollary 3.18. *Consider a forward and backward minimal* IFS *of homeomorphisms of a metric space X and assume that there is a map h in the semigroup Γ generated by these maps with exactly two fixed points, one attracting and one repelling. Then X is a well-fibred quasi-attractor and consequently is renderable by the deterministic chaos game.*

Proof. The forward minimality implies that X is a quasi-attractor. Consider now any compact set $K \subset X$ such that $K \neq X$. By the backward minimality there exist $T_1, \dots, T_s \in \Gamma$ such that

$$X = \bigcup_{i=1}^{s} T_i(X \setminus K).$$

Let p and q be, respectively, the attracting and the repelling fixed points of h. Then there is $i \in \{1, \dots, s\}$ so that $q \in T_i(X \setminus K)$. Therefore, $q \notin T_i(K)$ and then the diameter of $h^n \circ T_i(U)$ converges to zero. This shows that the action is well-fibred and completes the proof. □

3.4.2. Quasi-attractors of symmetric IFS*s.* We will proof Theorem 1.7. This theorem extends [19, Theorem 3.3] for compact Hausdorff topological spaces.

Proposition 3.19. *If A is a quasi-attractor of a symmetric* IFS *on a Hausdorff topological space then it is renderable by the deterministic chaos game.*

Proof. We will use Lemma 3.15. To accomplish this, let I be an open set so that $A \cap I \neq \emptyset$. By the minimality of the action of Γ restricted to A, there are $h_1, \dots, h_m \in \Gamma$ so that $A \subset h_1^{-1}(I) \cup \dots \cup h_m^{-1}(I)$. Set Z be the union of these open sets. Since the IFS has a symmetric system of generators $\mathscr{F} = \{f_1, \dots, f_k\}$, we can write

$$f_{i_s} \circ \dots \circ f_{i_1} = h_m^{-1} \circ h_m \circ \dots \circ h_2^{-1} \circ h_2 \circ h_1^{-1} \circ h_1.$$

Hence,

$$Z \subset \bigcup_{j=1}^{t} f_{i_1}^{-1} \circ \dots \circ f_{i_j}^{-1}(I).$$

This implies that for each $z \in Z$, there is $t \in \{1, \dots, s\}$ so that $f_{i_t} \circ \dots \circ f_{i_1}(z) \in I$. Thus, by Lemma 3.15, A is renderable by the deterministic chaos game. □

To end this subsection, we give an example of a quasi-attractor of a symmetric IFS on the torus $\mathbb{T}^2$ which is neither well-fibred nor a quasi-attractor of a non-expansive IFS nor has a minimal map.

Example 3.20. Let $f \colon \mathbb{T}^2 \to \mathbb{T}^2$ be a generalized north-south pole diffeomorphism on the torus $\mathbb{T}^2$. By this we mean that the non-wandering set of f, $\Omega(f)$, consists of one fixed source, q, one fixed sink, p, and saddle type periodic orbits. Let S be the set of all saddle type periodic points of f. For simplicity we assume that S consists of two saddle points so that $\mathcal{W} = W^s(S) \cup W^u(S) \cup \{p, q\}$ consists of four circles: two disjoint circles following the meridian direction and two disjoint circles following the parallel directions. For every $x \in \mathbb{T}^2 \setminus \mathcal{W}$ it holds that $f^n(x) \to p$ and $f^{-n}(x) \to q$ as $n \to \infty$. On the other hand, consider the translation $R_\lambda \colon \mathbb{T}^2 \to \mathbb{T}^2$, $R_\lambda(x, y) = (x+\lambda_1, y+\lambda_2)$, where $\lambda = (\lambda_1, \lambda_2)$ is an irrational vector, i.e. $\lambda_i \in \mathbb{R} \setminus \mathbb{Q}$ for $i = 1, 2$. Since the IFS generated by $f, f^{-1}, R_\lambda, R_\lambda^{-1}$ on $\mathbb{T}^2$ has minimal elements, according to [12, Proposition 1], it is renderable by the deterministic chaos game. Moreover, it is not difficult to see that this IFS is C^1-robustly minimal, i.e. the minimality persists under small C^1-perturbations on the generators (indeed, one can easily construct a "blending region" around the attracting fixed point and then apply [13, Theorem 6.3]). Thus, there is a rational vector α close to λ so that the IFS generated by $f, f^{-1}, R_\alpha, R_\alpha^{-1}$ acts minimally on $\mathbb{T}^2$. By Theorem 1.7, this IFS is renderable by the deterministic chaos game. Clearly, it does not contain any minimal element and it is not a non-expansive IFS. Also, it is not well-fibred (indeed, it suffices to consider a compact neighbourhood of a circle that contains p and the unstable manifold of one saddle).

Appendix A

Here we extend the results due to Bransley, Leśniak and Rypka (see [7], [27]) on the probabilistic and the deterministic chaos game for attractors of IFSs to the general case of quasi-attractors. The proofs basically follow the same ideas of [7], [27] with some minor modifications and improvements.

On Bransley, Leśniak and Rypka probabilistic chaos game for quasi-attractors.

Theorem A.1. *Every first-countable quasi-attractor of an* IFS *of continuous maps of a Hausdorff topological space is renderable by the probabilistic chaos game.*

Proof. Let x_0 be a point of $\mathcal{B}^*_p(A)$ and let U be an open subset of A. We want to prove that the event

$$E = E(x_0, U) \stackrel{\text{def}}{=} \{\omega \in \Omega : O^+_\omega(x_0) \cap U \neq \emptyset\}$$

has probability 1. Since A is a minimal forward invariant set, for any x in A we can find a finite sequence $i_1, \ldots, i_m$ such that $f_{i_m} \circ \ldots \circ f_{i_1}(x)$ belongs to U. Then, by using the compactness of A and the continuity of the generators, we can actually find an integer m_0 and functions $i_1, \ldots, i_{m_0}$ from a neighbourhood V of A into $\{1, \ldots k\}$ such that for every $x \in A$, there is $m \leq m_0$ such that $f_{i_m(x)} \circ \ldots \circ f_{i_1(x)}(x) \in U$. Since $\operatorname{Ls} F^n(\{x_0\}) \subset A$ then $x_n = f^n_\omega(x_0) \in V$ for all n large enough. Then $E_n = \{\omega \in \Omega : \omega_{n+m_0} = i_{m_0}(x_n), \ldots, \omega_{n+1} = i_1(x_n)\}$

is obviously contained in E, and since the random variable x_n depends only on $\omega_1, \ldots, \omega_n$, we obtain from (1.1) that

$$\mathbb{P}(E \mid \omega_1, \ldots, \omega_n) \geq \mathbb{P}(E_n \mid \omega_1, \ldots, \omega_n) \geq p^{m_0} \stackrel{\text{def}}{=} \delta_0.$$

In particular, for every set C in the σ-algebra generated by $\omega_1, \ldots, \omega_n$, we have the inequality $\mathbb{P}(E \cap C) \geq \delta_0 \, \mathbb{P}(C)$, and since n is arbitrary, this inequality actually holds for any Borel set C of Ω. Choosing $C = \Omega \setminus E$, we deduce that $\mathbb{P}(\Omega \setminus E) = 0$ concluding that E has full probability.

Finally, we will prove that with probability 1, $O^+_\omega(x_0)$ is dense in A. First notice that any quasi-attractor is a separable set. Hence, let us choose $(z_i)_{i\in\mathbb{N}}$ a sequence dense in A and for each i, $(U_{i,j})_{j\in\mathbb{N}}$ a basis of neighbourhoods of z_i. Let

$$\Omega(x_0) = \bigcap_{i\in\mathbb{N}} \bigcap_{j\in\mathbb{N}} E(x_0, U_{i,j}).$$

From the above, $\Omega(x_0)$ has full probability. Given any open set U of X so that $U \cap A \neq \emptyset$. Then we can find $U_{i,j}$ so that $U_{i,j} \subset U$. Then, for every $\omega \in \Omega(x_0)$, the set $O^+_\omega(x_0)$ intersects $U_{i,j}$ and in particular U. Thus $O^+_\omega(x_0)$ is dense in A.□

On Leśniak deterministic chaos game for quasi-attractor of non-expansive IFS.

Theorem A.2. *Every quasi-attractor of a non-expansive* IFS *on a metric space is renderable by the deterministic chaos game.*

Proof. We will use Lemma 3.15. To accomplish this, let I be an open set so that $A \cap I \neq \emptyset$. We can suppose that $I = B_{2\varepsilon}(y_0)$ is an open ball of radius $2\varepsilon > 0$ and centered at $y_0 \in A$. By the compactness of A, we can find $y_1, \ldots, y_m \in A$ such that $A \subset B_\varepsilon(y_0) \cup B_\varepsilon(y_1) \cup \ldots \cup B_\varepsilon(y_m) \stackrel{\text{def}}{=} Z$. As the action of Γ on A is minimal, we can find $h_1 \in \Gamma$ such that $h_1(y_1) \in B_\varepsilon(y_0)$. Recursively, suppose h_{i-1} is constructed, we can find $h_i \in \Gamma$ such that $h_i \circ \ldots \circ h_1(y_i) \in B_\varepsilon(y_0)$. On the other hand, for each $z \in Z$, there is $i \in \{0, \ldots, m\}$ such that $d(z, y_i) \leq \varepsilon$. Since the IFS is non-expansive,

$$d(h_i \circ \ldots \circ h_1(z), h_i \circ \ldots \circ h_1(y_i)) \leq d(z, y_i) \leq \varepsilon.$$

Since $d(h_i \circ \ldots \circ h_1(y_i), y_0) \leq \varepsilon$, it follows that $d(h_i \circ \ldots \circ h_1(z), y_0) \leq 2\varepsilon$. That is, $h_i \circ \ldots \circ h_1(z) \in I$ for some $i \in \{0, \ldots, m\}$ where we recall that $h_i \circ \ldots \circ h_1$ for $i = 0$ denotes the identity map. Hence writing

$$f_{i_s} \circ \ldots \circ f_{i_1} = h_m \circ \ldots \circ h_1 \quad \text{where } f_{i_j} \in \mathscr{F}$$

we have obtained that there is $t \in \{0, \ldots, s\}$ so that $f_{i_t} \circ \ldots \circ f_{i_1}(z) \in I$. Thus, by Lemma 3.2, A is renderable by the deterministic chaos game. □

REFERENCES

[1] V.A. ANTONOV, *Modeling cyclic evolution processes: synchronization by means of random signal*, Universitet Vestnik Matematika Mekhanika Astronomiia (1984), 67–76.

[2] T. BANAKH, W. KUBIŚ, N. NOVOSAD, M. NOWAK AND F. STROBIN *Contractive function systems, their attractors and metrization*, Topol. Methods Nonlinear Anal. (2015) (to appear).

[3] M.F. BARNSLEY, *Fractal Everywhere*, Academic Press Professional, 1988.

[4] M. BARNSLEY AND K. IGUDESMAN, *Topological Contractive Systems*, Lobachevskii J. Math. **32** (2011), no. 3, 220–223.

[5] M.F. BARNSLEY AND K. LEŚNIAK, *The chaos game on a general iterated function system from a topological point of view*, Internat. J. Bifur. Chaos, **24** (2014).

[6] ———, *On the continuity of the Hutchinson operator*, Symmetry, **7** (2015), no. 4, 1831–1840.

[7] M.F. BARNSLEY, K. LEŚNIAK AND M. RYPKA, *Chaos game for* IFS*s on topological spaces*, arXiv, 2014.

[8] M.F. BARNSLEY, K. LEŚNIAK AND C. WILSON, *Some Recent Progress Concerning Topology of Fractals*, Recent Progress in General Topology III, (2014), 69–92.

[9] M.F. BARNSLEY AND A. VINCE, *The chaos game on a general iterated function system*, Ergodic Theory Dynam. Sytems. **31** (2011), 1073–1079.

[10] ———, *Developments in fractal geometry*, Bull. Aust. Math. Soc. **3** (2013), 299–348.

[11] P.G. BARRIENTOS, A. FAKHARI AND A. SARIZADEH, *Weakly hyperbolic and fiberwise orbits*, extended abstract to IMC44, 2013.

[12] ———, *Density of fiberwise orbits in minimal iterated function systems on the circle*, Discrete Contin. Dyn. Syst. **34** (2014), no. 9, 3341–3352.

[13] P.G. BARRIENTOS, A. FAKHARI, D. MALICET AND A. SARIZADEH, *Expanding actions: minimality and ergodicity*, Stoch. Dyn. (2016) (to appear).

[14] P.G. BARRIENTOS AND A. RAIBEKAS, *Dynamics of iterated function systems on the circle close to rotations*, Ergodic Theory Dynam. Sysems **35** (2015), no. 5, 1345–1368.

[15] G. BEE, *Topologies on closed and closed convex sets*, Math. Appl., **268** (1993).

[16] R. BOWEN, *A horseshoe with positive measure*, Invent. Math. **29** (1975), no. 3, 203–204.

[17] L. BREIMAN, *The strong law of large numbers for a class of Markov chains*, Ann. Math. Statist. **31** (1960), no. 3, 801–803.

[18] G.A. EDGAR, *Integral, Probability, and Fractal Measures*, Springer–Verlag, 1998.

[19] A. FURMAN, *Random walks on groups and random transformations*, Handbook of Dynamical Systems, Vol. 1A, North-Holland, 2002, pp. 931–1014.

[20] E. GHYS, *Groups acting on the circle*, Enseign. Math. (2) **47** (2001), 329–407.

[21] T. GOLENISHCHEVA-KUTUZOVA, A.S. GORODETSKI, V. KLEPTSYN AND D. VOLK, Translation numbers define generators of $F_k^+ \to \mathrm{Homeo}_+(\mathbb{S}^1)$, Mosc. Math. J. **14** (2014), no. 2, 291–308.

[22] B. KIENINGER, *Iterated Function Systems on Compact Hausdorff Spaces*, Ph.D. Thesis, Augsburg University, Berichte aus der Mathematik, Shaker–Verlag, Aachen, 2002.

[23] V.A. KLEPTSYN AND M.B. NALSKIĬ, *Contraction of orbits in random dynamical systems on the circle*, Funct. Anal. Appl. **38** (2004), 267–282.

[24] A. LASOTA AND J. MYJAK *Semifractals*, Bull. Polish Acad. Sci. Math. **44** (1996), 5–21.

[25] ———, *Fractals, semifractals and Markov operators*, Internal. J. Bifur. Chaos **9** (1999), no. 2, 307–325.

[26] A. LASOTA, J. MYJAK AND T. SZAREK, *semifractals and Markov operators*, Progr. Probab. **57** (2004), 3–22.

[27] K. Leśniak, *Random iteration for nonexpansive iterated function systems: derandomised algorithm*, Internat. J. Appl. Nonlinear Sci. **1** (2014), no. 4, 360–363.

[28] ______, *Random iteration for infinite nonexpansive iterated function systems*, Chaos **25** (2015).

[29] D. Malicet, *Random walks on* Homeo(S^1), arXiv 2015.

[30] R. Miculescu and A. Mihail J. Math. Anal. Appl. **422** (2015), 265–271.

[31] J.R. Munkres, *Topology*, Pretince Hall, Incorporated, 2nd Edition, 2000.

[32] A. Navas, *Groups ofCcircle Diffeomorphisms*, University of Chicago Press, 2011.

[33] S.M. Srivastava, *A Course on Borel Sets*, Graduate Texts in Mathematics, 1998.

Manuscript received July 16, 2015

accepted October 29, 2015

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