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EVOLUTIONARY STABILITY FOR TWO-STAGE HAWK-DOVE GAMES

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ABSTRACT. Although two individuals in a biological species often interact with each other more than once, standard evolutionary game theory does not allow behaviors to depend on past interactions. This paper analyzes two-stage games where pairs of individuals compete twice. It is shown that evolutionary stability follows from a rational backwards induction procedure that first establishes stability at the second stage. All examples are of the hawk-dove type.

1. Introduction. Evolutionary game theory has primarily modelled the frequency evolution of strategy types in a single species where an individual's fitness (i.e., reproductive success) depends on the payoff it obtains in a single contest with a random opponent. In such single-stage games, the ESS (evolutionarily stable strategy) stability concept [**3**] has three intuitive interpretations. Maynard Smith's [**4**] original ESS definition relies on the intuition that a population will be stable if it cannot be successfully invaded by a few individuals using a mutant strategy. For the second interpretation, ESS's indeed become stable equilibria [**6**] for the continuous pure-strategy dynamic (Section 3) that involves no conscious decisions on strategy use by individuals in the population. This paper emphasizes a third interpretation, specifically, an ESS is the strategy a rational individual will choose against a rational opponent (Section 2).

The ESS concept has had limited success in predicting the outcome of evolutionary games where contestants remember previous competitions. For instance, in iterated prisoner's dilemma, cooperative strategies such as Tit-for-Tat may evolve but are typically not ESS's [**3**]. In this paper we analyze evolutionary games where the same pair of individuals compete in two consecutive (i.e., two-stage) hawk-dove games. In Sections 3 and 4, we show dynamic stability follows from rational

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decisions made at each stage of the game. Section 2 first briefly summarizes the standard single-stage hawk-dove game, emphasizing the different interpretations of an ESS mentioned above.

2. The single-stage hawk-dove game. Here hawks (H) and doves (D) compete over a resource of value $2V$. A contest of hawk versus dove results in the hawk winning the resource at no cost. On the other hand, two hawks will fight over the resource, each expecting a payoff V while incurring a cost C of fighting. Lastly, two doves will split the resource evenly. This information is contained in the 2×2 payoff matrix A given by

$$\begin{array}{cc} & \begin{array}{cc} H & \quad D \end{array} \\ \begin{array}{c} H \\ D \end{array} & \begin{pmatrix} V-C & 2V \\ 0 & V \end{pmatrix} \end{array}. \tag{2.1}$$

An ESS is then a (possibly mixed) strategy p^* in the frequency simplex $\Delta^2 = \{(p_1, p_2) \mid p_1 + p_2 = 1, p_i \geq 0\}$ that satisfies

$$\text{(i)} \quad p^* \cdot Ap^* \geq q \cdot Ap^* \quad \text{for all } q \in \Delta^2 \tag{2.2}$$

$$\text{(ii)} \quad p^* \cdot Aq > q \cdot Aq \quad \text{for any } q \neq p^* \quad \text{with equality in (i).} \tag{2.3}$$

Here $q \cdot Ap$ is the expected payoff to an individual using strategy q against one using p. For instance, a hawk versus a dove has payoff $(1,0) \cdot A(0,1) = (1,0) \cdot (2V, V) = 2V$ from (2.1).

This paragraph develops the rational interpretation of the above ESS conditions; that is, an interpretation whereby a rational player in a contest against a rational opponent chooses the ESS strategy. For this development, it is important to realize that evolutionary games used to model interactions in a single species are symmetric and that only game-theoretic symmetric solutions [**5, 7**] in the strategy simplex are considered. This means that both players have the same list of possible strategies and, furthermore, that both players use the same strategy at a solution p^* that represents the average strategy (or state) of the single-species population. The (symmetric) Nash equilibrium condition (2.2) then asserts there is no incentive to unilaterally alter your strategy to q if you and your opponent are currently using the ESS

p^*. However, if (2.2) is indifferent to unilateral change (i.e., if there is equality in (2.2)), the Nash condition does not give a reason to remain at p^*. Suppose, in this case, you contemplate a shift from p^* to q. By symmetry, the population state would also shift to q. At this point, the stability condition (2.3) would intervene to give you an incentive to return to p^*. In other words, the rational decision is to stay at p^*. That is, for single-stage, symmetric, evolutionary games, rational players should adopt the ESS strategy. (Notice that the pronoun "you" is often used in discussing rational players. This is a common feature throughout the paper.)

It is well known [**3**] that all 2×2 payoff matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of the form (2.1) where $b > d$ have a unique ESS; namely, $(1, 0)$ if $a \geq c$ and $(1/(b-d+c-a))(b-d, c-a)$ if $a < c$. Thus, the hawk-dove game (2.1) has two qualitatively different ESS's depending on the relative values of V and C. If $V > C$, a rational player will always play H (i.e., $(1, 0)$ is the unique ESS). On the other hand, if $V < C$, you play H with probability V/C and D the rest of the time (i.e., $(1/C)(V, C - V)$ is the ESS).

The dynamic interpretation of an ESS is also well-known [**3**] for hawk-dove games. Any initial polymorphic population (i.e., one made up of a mixture of individuals who are always hawks or always doves) will evolve under the continuous pure-strategy dynamic (equation (3.2) below) to a population where hawks and doves are at their respective ESS frequencies. In other words, a population of individuals who all use one of the two pure strategies evolves under frequency-dependent selection given in (3.2), that assumes individuals do not make conscious decisions, to a population whose frequencies can be predicted by considering competition between two rational players.

3. A second stage for doves. The simplest evolutionary game model that involves a second stage is the case of two individuals engaging in a repeated contest if they both played the same particular strategy (e.g., both were doves) at the first stage. This game is depicted most concisely by its extensive form game tree (Figure 1). Important features of general symmetric extensive games are given in the following two paragraphs with reference to Figure 1. A complete description is in [**5**].

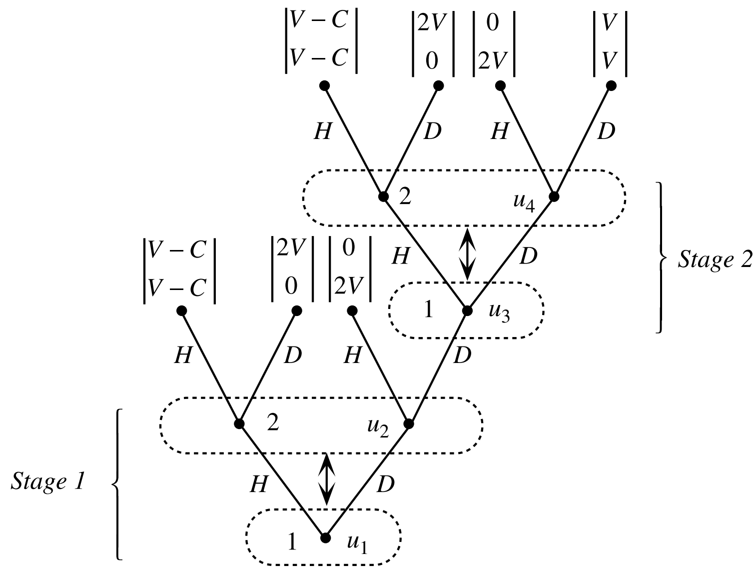

FIGURE 1. The extensive form of a two-stage game where doves compete in a second stage.

The extensive form emphasizes the sequential nature of the game in Figure 1. At stage 1, player 1 makes the first choice, between H and D, at the vertex enclosed in the dashed curve containing u_1 that is called the information set u_1. Player 2 then chooses between H and D at the information set u_2. Since u_2 contains both vertices that branch from u_1, player 2 makes this choice without knowing what player 1 already chose. If both players choose D at stage 1, the game continues to a second stage where another hawk-dove game is played at information sets u_3 and u_4 that represent the respective choices of players 1 and 2 at stage 2.

Evolutionary games also feature a symmetry between the situations of players 1 and 2. This is indicated by double arrows, $\leftrightarrow$, in Figure 1. For instance, information sets u_3 and u_4 are symmetric. At u_4, player 2's choice, between H and D, is based on knowing both it and its opponent chose D at stage 1. Player 1 finds itself in the same situation at u_3. Finally, extensive games indicate payoffs above each endpoint of the game tree. For instance, if both players choose D at stage 1 followed by choices H and D by players 1 and 2, respectively, at stage 2, the game reaches the third endpoint from the right in Figure 1. The payoff in this case, $\begin{vmatrix} 2V \\ 0 \end{vmatrix}$, means player 1 receives payoff $2V$ whereas player 2 receives 0. The payoffs given in Figure 1 correspond to playing the hawk-dove game of Section 2 at stage 1 except when both chose Dove in which case another hawk-dove is played at stage 2 with payoff matrix (2.1).

The remainder of this section analyzes the game from two perspectives; first as a symmetric game between two rational players, then as an evolutionary game where dynamic stability is of utmost importance.

Suppose the game is played by two rational opponents and that $V < C$. The backwards induction procedure of Selten [**5**] finds the solution as follows. If the game reaches the second stage, the subgame that starts at u_3 is the single-stage hawk-dove game of Section 2. Thus, at stage 2, both rational players should adopt the ESS of (2.1); namely, the mixed strategy $p^* = (1/C)(V, C - V)$. In particular, both will receive an expected payoff of $p^* \cdot Ap^* = (V/C)(C - V)$. The complete two-stage game is now reduced to a single-stage game formed from Figure 1 by replacing the subgame that starts at u_3 with an endpoint having payoffs

$$\begin{vmatrix} (V/C) & (C-V) \\ (V/C) & (C-V) \end{vmatrix}.$$

What remains is a single-stage game that has payoff matrix

$$\begin{bmatrix} V-C & 2V \\ 0 & (V/C)(C-V) \end{bmatrix}.$$

Both rational players will then choose the ESS of this 2×2 matrix. Thus, at stage 1, backwards induction yields the mixed strategy $(1/(V^2+C^2))(V^2+VC, C^2-VC)$. That is, a rational individual plays H with probability $(V^2+VC)/(V^2+C^2)$ at stage 1 and, if the game reaches stage 2, plays H with probability V/C there.

This same backwards induction procedure described in [**5**] can be applied to the case $V > C$ where it implies you should always play

H at stage 1. Even though stage 2 is then never reached, the method suggests H should also be played there.

How do these rational outcomes compare with the evolutionary outcome of the pure-strategy dynamic? There are now three pure strategies for this two-stage evolutionary game that will be denoted

$$\begin{aligned} H&\text{—play } H \text{ at stage 1}\\ [D;H]&\text{—play } D \text{ at stage 1 and } H \text{ at stage 2}\\ [D;D]&\text{—play } D \text{ at both stages.}\end{aligned}$$

The 3×3 payoff matrix A, given by

$$\begin{bmatrix} V-C & 2V & 2V \\ 0 & V-C & 2V \\ 0 & 0 & V \end{bmatrix}, \tag{3.1}$$

is called the normal form [**3, 5, 7**] of the game in Figure 1. If p_i, $i = 1, 2, 3$, is the frequency of individuals using pure strategy i in the population at time t, the continuous dynamic [**6**] is

$$\dot{p}_i = p_i(e_i - p) \cdot Ap \tag{3.2}$$

where $p = (p_1, p_2, p_3) \in \Delta^3$ and e_i is the i^{th} unit coordinate vector.

It is not difficult to show that

$$\left(\frac{V^2 + VC}{V^2 + C^2}, \frac{V}{C} \frac{C^2 - VC}{V^2 + C^2}, \frac{C - V}{C} \frac{C^2 - VC}{V^2 + C^2} \right)$$

and $(1, 0, 0)$, respectively, are the unique ESS's in Δ^3 of (3.1) when $V < C$ and $V > C$, respectively. Moreover, these ESS's are the globally stable equilibria of (3.2) for any initial polymorphism in their respective cases. On the other hand, they are precisely the strategies described earlier in this section as the solution reached by rational players. That is, the evolutionary outcome matches the rational outcome in this two-stage game.

The backwards induction procedure does not always lead to an ESS of the normal form game [**7**]. This is true even for the game tree of Figure 1 if the payoffs are chosen appropriately. For instance, if $C = 2V$ in the first stage and the second stage is replaced by values and costs four times those in Figure 1, the payoff matrix (3.1) is altered to

$$\begin{bmatrix} -V & 2V & 2V \\ 0 & -4V & 8V \\ 0 & 0 & 4V \end{bmatrix}. \tag{3.3}$$

In this case, backwards induction implies D should always be played at stage 1 followed by D at stage 2 half the time. This strategy, in normal form $p^* = (0, 1/2, 1/2)$, is not an ESS of (3.3) since it does not satisfy (2.3) for $q = (1/2, 1/2, 0)$. In fact, (3.3) has no ESS. On the other hand, p^* is the globally stable equilibrium of (3.2) for any initial polymorphism. That is, the evolutionary outcome and the rational outcome match in this example as well.

In summary, for two-stage hawk-dove games where the second stage is reached only when both individuals were doves at the first stage, the backwards induction procedure of Selten [**5**] predicts the evolutionary outcome but may or may not yield an ESS of the pure-strategy payoff matrix.

4. The two-stage hawk-dove game. A more substantial test of Selten's procedure is the two-stage game where all individuals engage in a second contest irrespective of their choice at the first stage. Figure 2 depicts such a game in extensive form where payoffs are cumulative and the same values of V and C are used for both stages. An added feature of this game tree is the *asymmetry* that connects the subgames that begin at information sets u_4 and u_5 (indicated by $\leftrightarrow$). The asymmetry refers to the fact that your choice at u_4 is based on the same knowledge as your choice at u_9; namely, that you used H and your opponent D at stage 1. In this asymmetric hawk-dove subgame, an asymptotically stable evolutionary outcome must be a two-species ESS [**3**] given by a pure strategy for each species. Again, the outcome depends on the relative sizes of V and C.

For $V > C$, the only two-species ESS is to play H at u_4 and u_5. From Section 3, it is clear that H must be played at u_3 and u_6 as well. The reduced game is thus a single-stage game with payoff matrix

$$\begin{bmatrix} 2(V-C) & 2V+V-C \\ V-C & V+V-C \end{bmatrix}.$$

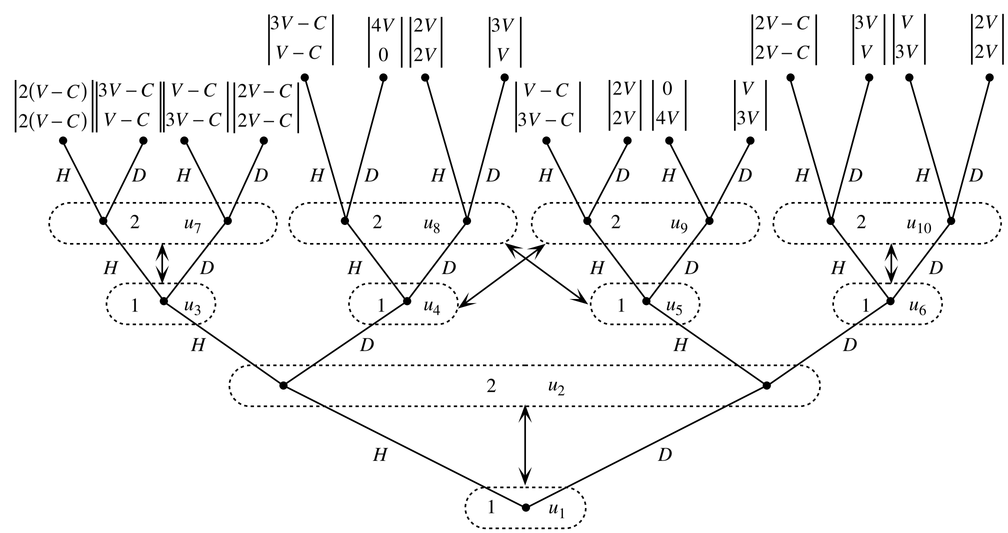

FIGURE 2. The two-stage hawk-dove game in extensive form.

Therefore, backwards induction implies a rational player will use H at stage 1 as well. That is, Selten's method predicts hawk will always be played at both stages.

For $V < C$, there are two ESS's for the asymmetric subgame; namely,

(i) Play H at u_4 and D at u_5

(ii) Play D at u_4 and H at u_5.

At u_3 and u_6, you must play the mixed strategy ESS $(1/C)(V, C-V)$ given in Section 3. The respective reduced single-stage games for these two possibilities have payoff matrices

$$\begin{bmatrix} V - C + \frac{V}{C}(C-V) & 2V + 2V \\ 0 & V + \frac{V}{C}(C-V) \end{bmatrix}$$

and

$$\begin{bmatrix} V - C + \frac{V}{C}(C-V) & 2V \\ 2V & V + \frac{V}{C}(C-V) \end{bmatrix}$$

that imply rational players use the ESS strategies

$$\frac{(V^2 + 2VC, (C-V)^2)}{2V^2 + C^2} \quad \text{and} \quad \frac{(V^2, 2VC + (C-V)^2)}{2V^2 + C^2}, \tag{4.1}$$

respectively, at stage 1.

The exact form of these rational outcomes is less important from the perspective of this paper than the question of whether they are also the evolutionary outcomes. To test this, the relevant evolutionary game now has eight pure strategies: $[H; H, H]$, $[H; H, D]$, $[H; D, H]$, $[H; D, D]$, $[D; H, H]$, $[D; D, H]$, $[D; H, D]$, $[D; D, D]$ where $[a; b, c]$ means play a at stage 1, b at stage 2 if your opponent plays H, c at stage 2 if your opponent plays D. The 8×8 payoff matrix is

$$\begin{bmatrix} 2(V-C) & 2(V-C) & 3V-C & 3V-C & 3V-C & 4V & 3V-C & 4V \\ 2(V-C) & 2(V-C) & 3V-C & 3V-C & 2V & 3V & 2V & 3V \\ V-C & V-C & 2V-C & 2V-C & 3V-C & 4V & 3V-C & 4V \\ V-C & V-C & 2V-C & 2V-C & 2V & 3V & 2V & 3V \\ V-C & 2V & V-C & 2V & 2V-C & 2V-C & 3V & 3V \\ 0 & V & 0 & V & 2V-C & 2V-C & 3V & 3V \\ V-C & 2V & V-C & 2V & V & V & 2V & 2V \\ 0 & V & 0 & V & V & V & 2V & 2V \end{bmatrix}. \tag{4.2}$$

If $V > C$, a careful analysis of the eight-dimensional dynamic (3.2) by means of dominated strategies [**1**] shows that any initial polymorphism evolves to an equilibrium strategy of the form $(p_1^*, 1-p_1^*, 0, 0, 0, 0, 0, 0)$. A population in this state will always play H at both stages—the same result predicted by backwards induction. That is, although the evolutionary outcome is not an ESS (it is not even an ES Set as was erroneously asserted for a similar situation in [**3**]), it remains the same as the rational outcome.

For $V < C$, (4.2) has exactly two ESS's that can be found as in [**2, 3**]. To simplify these mathematically without qualitatively changing the discussion, let us assume $C = 2V$. Then the two ESS's are

$$p^* = \frac{1}{12}(5, 0, 5, 0, 0, 1, 0, 1)$$
$$q^* = \frac{1}{12}(0, 1, 0, 1, 5, 0, 5, 0)$$

and these are both locally asymptotically stable under the dynamic (3.2). It is straightforward to verify that p^* and q^* correspond to the two rational outcomes developed earlier in this section. For instance, p^* always plays H at u_4 and D at u_5 since it only involves the four pure strategies $[H; H, H], [H; D, H][D; D, H]$ and $[D; D, D]$. Furthermore, p^* uses H at u_1 with probability $(1/12)(5+5) = 5/6$ which matches that of the first mixed strategy of (4.1), $(1/6V^2)(5V^2, V^2)$, when $C = 2V$. Computer simulations of the dynamic (2.3) reported in [**2**] suggest essentially all initial polymorphisms evolve to either p^* or q^*. In this sense, the rational and evolutionary outcomes of the two-stage hawk-dove game coincide.

5. Discussion. The backwards induction procedure suggested by Selten [**5**] does not, in general, determine the ESS structure of two-stage evolutionary games. On the other hand, it does determine the evolutionary outcome for all the examples of hawk-dove games analyzed in this paper. Since the primary goal of evolutionary game theory is to predict evolution outcomes based on biological intuition, the procedure is indeed a success for these games. Moreover, its rational foundation should provide a means to analyze more complicate multi-stage games.

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