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THE FINITE LEGENDRE TRANSFORMATION OF GENERALIZED FUNCTIONS

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ABSTRACT. Some techniques employed successfully to investigate a class of infinite integral transforms are now used to study the finite Legendre transformation in a suitable space of generalized functions. Particularly, the close similarity between this transform and the infinite Mehler-Fock transform is emphasized. Inversion and uniqueness theorems are established, and the operational calculus generated is applied in solving certain boundary-value problems.

1. Introduction. L. Schwartz [**11**] was very probably the first who considered the series expansions of generalized functions to investigate the Fourier and Hermite expansions of a certain class of distributions. Later, Gelfand and Shilov [**5**], Walter [**16**], Chébli [**1**] and Trimèche [**15**] studied other series expansions with regard to more general differential operators. The use of Hilbert-space techniques allowed to Zemanian [**17, 18**] to introduce a wide variety of finite distributional transformations, amongst others, the finite Legendre transformation

$$l\{f(x)\} = F(n) = \int_{-1}^{1} P_n(x) f(x)\, dx, \quad n = 0, 1, 2, \dots, \tag{1.1}$$

where $P_n(x)$ are the well-known Legendre polynomials [**2, 4**]. More recently, these results are extended to some testing function spaces of L^p type and their duals by Pathak [**9**].

By using a quite different technique, Dube [**3**] and Pathak and Singh [**10**] have also extended the finite Hankel transformation to certain spaces of distributions. Precisely the aim of this paper is to investigate the finite Legendre transformation of generalized functions following this procedure. In this point we remember that an analogous to the Riemann-Lebesgue lemma plays an important role in the proof

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of inversion formulas for the finite Hankel transforms. Inasmuch as a similar result for the Fourier-Legendre series expansions is not available, in what we know, we have to modify substantially the procedure usually employed in the literature with regard to the finite transforms. Our approach is inspired by the works of Zemanian about Laplace and Meijer transforms [**18**] and Tiwari and Pandey [**14**] dealing with the infinite Mehler-Fock transform. In this way, inversion and uniqueness theorems are established in an appropriate space of generalized functions, making it unnecessary to restrict results to the Schwartz distributional space $\mathcal{D}'(-1,1)$. It is worthwhile to remark that there is an extraordinary resemblance between finite Legendre and infinite Mehler-Fock transformations. Finally we apply the operational calculus generated in solving a distributional differential equation.

The classical inversion formula of the transform (1.1) is given by the corresponding convergence theorem for the Fourier-Legendre series, [**2**, p. 234].

Theorem 1. *Let $f(x)$ be a piecewise continuous function on the interval $(-1,1)$ and A_n denote the coefficients*

$$A_n = \frac{2n+1}{2}\int_{-1}^{1} P_n(x)f(x)\,dx, \quad n = 0,1,2,\dots . \tag{1.2}$$

Then at each point x in that interval where $f(x)$ is continuous and has derivatives from the right and left, the Legendre series

$$\sum_{n=0}^{\infty} A_n P_n(x) \tag{1.3}$$

converges to $f(x)$.

In the sequel, $K_N(t,x)$ stands for the kernel

$$K_N(t,x) = \sum_{m=0}^{N}\left(m+\frac{1}{2}\right)P_m(t)P_m(x) \tag{1.4}$$

which, by virtue of the Christoffel-Darboux formula [**2**, p. 235], can be rewritten

$$K_N(t,x) = \frac{N+1}{2}\,\frac{P_{N+1}(t)P_N(x)-P_N(t)P_{N+1}(x)}{t-x}, \quad (t \neq x). \tag{1.5}$$

When $t = x$ we assign to $K_N(t,x)$ the value

$$K_N(x,x) = \frac{N+1}{2}(P'_{N+1}(x)P_N(x) - P'_N(x)P_{N+1}(x)). \tag{1.6}$$

The sum of the first $m+1$ terms of the Legendre series (1.3) is given by

$$S_m(x) = \sum_{n=0}^{m} A_n P_n(x) = \int_{-1}^{1} K_m(t,x) f(t)\, dt. \tag{1.7}$$

We recall other properties of Legendre polynomials we shall need in this paper [**2, 4, 13**],

$$|P_n(x)| \le 1, \quad -1 < x < 1, \quad n = 0, 1, 2, \dots \tag{1.8}$$

$$P_n(1) = 1, \qquad P_n(-1) = (-1)^n \tag{1.9}$$

$$|P_n(x)| \le \sqrt{\frac{\pi}{2n(1-x^2)}}, \quad -1 < x < 1, \quad n = 1, 2, \dots \tag{1.10}$$

$$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x), \quad n = 1, 2, \dots, \tag{1.11}$$

$\mathcal{D}(-1,1)$ denotes the space of infinitely differentiable functions whose supports are contained in $(-1,1)$ and its dual $\mathcal{D}'(-1,1)$ is the space of Schwartz distributions. Finally, $\mathcal{E}(-1,1)$ represents the space of all infinitely differentiable functions on $(-1,1)$ and its dual $\mathcal{E}'(-1,1)$ is the space of distributions with compact supports.

2. The testing function space $\mathcal{L}(-1,1)$ and its dual. $\mathcal{L}(-1,1)$ is the space of all complex-valued infinitely differentiable functions $\varphi(x)$ defined on the open interval $(-1,1)$ such that

$$\gamma_k(\varphi) = \sup_{-1<x<1} |R_x^k \varphi(x)| < \infty, \tag{2.1}$$

for every nonnegative integer k, where R_x denotes the differential operator

$$R_x = D(x^2-1)D = (x^2-1)D^2 + 2xD, \qquad D = \frac{d}{dx}. \tag{2.2}$$

The linear space $\mathcal{L}(-1,1)$, with the topology generated by the collection of semi-norms $\{\gamma_k\}$, turns out to be a Fréchet space [**18**, p. 12]. $\mathcal{L}'(-1,1)$ symbolizes the dual space of $\mathcal{L}(-1,1)$ and is equipped with the usual weak topology [**18**, p. 21].

We now list some properties of these spaces:

(i) The inclusion relations $\mathcal{D}(-1,1) \subset \mathcal{L}(-1,1) \subset \mathcal{E}(-1,1)$ hold algebraically and topologically, $\mathcal{L}(-1,1)$ being dense in $\mathcal{E}(-1,1)$. Therefore, $\mathcal{E}'(-1,1)$ is a subspace of $\mathcal{L}'(-1,1)$.

(ii) If $f(x)$ is a function defined and absolutely integrable on $(-1,1)$, then $f(x)$ gives rise to a regular generalized function on $\mathcal{L}'(-1,1)$ through

$$\langle f, \varphi \rangle = \int_{-1}^{1} f(x)\varphi(x)\, dx, \quad \varphi(x) \in \mathcal{L}(-1,1). \tag{2.3}$$

Indeed, f is clearly linear. Moreover, since

$$|\langle f, \varphi \rangle| \le \gamma_0(\varphi) \int_{-1}^{1} |f(x)|\, dx,$$

f is a continuous functional on $\mathcal{L}(-1,1)$. Particularly, every function $f(x) \in \mathcal{L}(-1,1)$ is absolutely integrable because

$$\int_{-1}^{1} |f(x)|\, dx \le 2\gamma_0(f) < \infty.$$

Hence, by identifying $f(x) \in \mathcal{L}(-1,1)$ with the regular distribution $f(x)$ generated in $\mathcal{L}'(-1,1)$ by means of (2.3), we can justify the inclusion $\mathcal{L}(-1,1) \subset \mathcal{L}'(-1,1)$.

(iii) The differential operator R_x, defined by (2.2), is a continuous linear mapping of $\mathcal{L}(-1,1)$ into itself. This is an immediate consequence of

$$\gamma_k(R_x\varphi(x)\} \le \gamma_{k+1}(\varphi), \quad \varphi \in \mathcal{L}(-1,1).$$

Consequently, the generalized operator R_x defined on $\mathcal{L}'(-1,1)$ as the adjoint of the operator (2.2), that is to say,

$$\begin{gathered} \langle R_x f(x), \varphi(x) \rangle = \langle f(x), R_x \varphi(x) \rangle, \\ f \in \mathcal{L}'(-1,1), \quad \varphi \in \mathcal{L}(-1,1), \end{gathered} \tag{2.4}$$

is also a continuous linear mapping of $\mathcal{L}'(-1,1)$ into itself.

(iv) Since $P_n(x)$ satisfies the differential equation $(x^2-1)y''+2xy'-n(n+1)y=0$ [**2, 4**], one has

$$R_x P_n(x) = n(n+1)P_n(x). \tag{2.5}$$

Next we characterize the members of the space $\mathcal{L}(-1,1)$.

Proposition 2.1. *An infinitely differentiable function $\varphi(x)$ on $(-1,1)$ is a member of $\mathcal{L}(-1,1)$ if and only if, for all nonnegative integers k, $D^k\varphi(x) = O(1)$ when $x \to 1-0$ and $x \to -1+0$.*

Proof. It is easily seen that

$$R_x^k \varphi(x) = \sum_{j=1}^{2k} p_j(x) D^j \varphi(x), \tag{2.6}$$

where $p_j(x)$ are polynomials of jth degree. This implies immediately that a function $\varphi(x) \in C^\infty(-1,1)$ satisfying the limit conditions $D^k\varphi(x) = O(1)$ as $x \to \pm 1 \mp 0$ belongs to $\mathcal{L}(-1,1)$. To prove the reciprocal, we refer to an analogous property verified by Glaeske and Hess [**6**, Proposition 2.3 and Lemma 2.1] in relation with the infinite Mehler-Fock transformation. $\square$

The boundedness of all derivatives of a function $\varphi(x) \in \mathcal{L}(-1,1)$ for $x \to 1-0$ and $x \to -1+0$ implies the existence of their corresponding limits, in accordance with Glaeske and Hess [**6**]. This permits us to obtain a characterization of the members of the space $\mathcal{L}(-1,1)$ similarly to that reached by Zemanian [**18**] for the Hankel transformable testing functions.

Proposition 2.2. *$\varphi(x)$ is a member of $\mathcal{L}(-1,1)$ if and only if $\varphi(x)$ is an infinitely differentiable complex-valued function on $-1 < x < 1$ which possesses Taylor expansions near the endpoints $x = 1$ and $x = -1$.*

3. The generalized finite Legendre transformation. The generalized finite Legendre transformation is defined on $\mathcal{L}'(-1,1)$ by means of

$$(l'f)(n) = F(n) = \langle f(x), P_n(x)\rangle, \quad n = 0, 1, 2, \dots, \tag{3.1}$$

for every $f(x) \in \mathcal{L}'(-1,1)$. Inasmuch as

$$\gamma_k\{P_n(x)\} = \sup_{-1<x<1} |n^k(n+1)^k P_n(x)| \le n^k(n+1)^k,$$

one has that $P_n(x) \in \mathcal{L}(-1,1)$ and, therefore, Definition (3.1) makes sense.

Lemma 3.1. *Let $f(x) \in \mathcal{L}'(-1,1)$. Then, for any positive integer N and for an arbitrary $\varphi(x) \in \mathcal{L}(-1,1)$, we have*

$$\int_{-1}^{1} \langle f(t), K_N(t,x)\rangle \varphi(x)\, dx = \left\langle f(t), \int_{-1}^{1} K_N(t,x)\varphi(x)\, dx \right\rangle, \tag{3.2}$$

where the kernel $K_N(t,x)$ is given by (1.4).

Proof. Let us fix x in $(-1,1)$. By invoking (1.4) and (2.5), we yield

$$R_t^k K_N(t,x) = \frac{1}{2}\sum_{n=0}^{N} (2n+1)[n(n+1)]^k P_n(t)P_n(x).$$

Thus $K_N(t,x)$ is a smooth function of t and $\gamma_k\{K_N(t,x)\} < \infty$ for all $k = 0, 1, 2, \dots$. Hence, $K_N(t,x) \in \mathcal{L}(-1,1)$ as a function of t. Similarly, we can establish that

$$\int_{-1}^{1} K_N(t,x)\varphi(x)\, dx \in \mathcal{L}(-1,1).$$

So both sides of equality (3.2) make sense. To verify the equality, note that

$$\begin{aligned}
\int_{-1}^{1} \langle f(t), K_N(t,x)\rangle \varphi(x)\,dx \\
&= \int_{-1}^{1} \left\langle f(t), \sum_{m=0}^{N} \left(m+\frac{1}{2}\right) P_m(t)P_m(x)\right\rangle \varphi(x)\,dx \\
&= \int_{-1}^{1} \sum_{m=0}^{N} \left(m+\frac{1}{2}\right) P_m(x)\langle f(t), P_m(t)\rangle \varphi(x)\,dx \\
&= \sum_{m=0}^{N} \left(m+\frac{1}{2}\right) \langle f(t), P_m(t)\rangle \int_{-1}^{1} P_m(x)\varphi(x)\,dx \\
&= \sum_{m=0}^{N} \left(m+\frac{1}{2}\right) \left\langle f(t), P_m(t) \int_{-1}^{1} P_m(x)\varphi(x)\,dx \right\rangle \\
&= \left\langle f(t), \sum_{m=0}^{N} \left(m+\frac{1}{2}\right) P_m(t) \int_{-1}^{1} P_m(x)\varphi(x)\,dx \right\rangle \\
&= \left\langle f(t), \int_{-1}^{1} \left\{ \sum_{m=0}^{N} \left(m+\frac{1}{2}\right) P_m(t)P_m(x)\right\} \varphi(x)\,dx \right\rangle \\
&= \left\langle f(t), \int_{-1}^{1} K_N(t,x)\varphi(x)\,dx \right\rangle
\end{aligned}$$

because of the linearity of $f(t)$. ◻

Theorem 3.1 (Inversion theorem). *Let $f(x)$ be an arbitrary generalized function in the space $\mathcal{L}'(-1,1)$, and let $F(m)$ be its finite Legendre transform as given by* (3.1). *Then*

$$\lim_{N\to\infty} \sum_{m=0}^{N} \left(m+\frac{1}{2}\right) F(m)P_m(x) = f(x)$$

in the sense of the convergence in $\mathcal{L}'(-1,1)$.

Proof. We emphasize here that the inversion proof of Dube [**3**] and Pathak and Singh [**10**] with regard to the finite Hankel transformations is based on the existence of one analogous to the Riemann-Lebesgue lemma. But we have to modify this procedure because the Fourier-Legendre series expansion lacks a similar result, in what we know. Our approach is inspired by the works of Zemanian about the Laplace and Meijer transforms [**18**] and Tiwari and Pandey [**14**] for the infinite Mehler-Fock transformation. In this way we achieve verification of the inversion formula in its own space $\mathcal{L}'(-1,1)$ instead of $\mathcal{D}'$, as is usually in the available literature.

Assume that $\varphi(x)$ is any member of $\mathcal{L}(-1,1)$. Note that $\sum_{m=0}^{N}(m+1/2)F(m)P_m(x)$ is an absolutely integrable function on $(-1,1)$ for any fixed N. Therefore, it gives rise to a regular member in $\mathcal{L}'(-1,1)$ by virtue of (2.3), and we may write

$$\left\langle \sum_{m=0}^{N}\left(m+\frac{1}{2}\right)F(m)P_m(x),\varphi(x)\right\rangle = \int_{-1}^{1}\sum_{m=0}^{N}\left(m+\frac{1}{2}\right)F(m)P_m(x)\varphi(x)\,dx. \tag{3.3}$$

By using (3.1), the linearity of $f(x)$ and Lemma 3.1, the righthand side in (3.3) adopts the form

$$\begin{aligned}\int_{-1}^{1}\sum_{m=0}^{N}\left(m+\frac{1}{2}\right)&\langle f(t),P_m(t)\rangle P_m(x)\varphi(x)\,dx\\ &=\int_{-1}^{1}\langle f(t),K_N(t,x)\rangle\varphi(x)\,dx\\ &=\left\langle f(t),\int_{-1}^{1}K_N(t,x)\varphi(x)\,dx\right\rangle.\end{aligned} \tag{3.4}$$

Our next objective is to show that

$$\int_{-1}^{1}K_N(t,x)\varphi(x)\,dx\longrightarrow\varphi(t)\quad\text{as}\quad N\to\infty\quad\text{in }\mathcal{L}(-1,1),$$

that is to say,

$$R_t^k\left\{\int_{-1}^{1}K_N(t,x)\varphi(x)\,dx-\varphi(t)\right\}\longrightarrow 0$$

uniformly in $t \in (-1, 1)$ as $N \to \infty$.

In view of (1.4), it is obvious that

$$R_t\{K_N(t,x)\} = R_x\{K_N(t,x)\}. \tag{3.5}$$

On the other hand, by the smoothness of the integrand, we may carry the differential operator R_t under the integral sign, to obtain

$$R_t\bigg\{\int_{-1}^{1} K_N(t,x)\varphi(x)\,dx\bigg\} = \int_{-1}^{1} R_t\{K_N(t,x)\}\varphi(x)\,dx. \tag{3.6}$$

Then, in view of (3.5) and (2.2), the last integral in (3.6) can be rewritten

$$\begin{aligned}\int_{-1}^{1} R_x\{K_N(t,x)\}\varphi(x)\,dx &= \int_{-1}^{1} D_x\{(x^2-1)D_xK_N(t,x)\}\varphi(x)\,dx\\ &= [(x^2-1)\{D_xK_N(t,x)\}\varphi(x)]_{x\to-1+0}^{x\to1-0}\\ &\quad - \int_{-1}^{1}(x^2-1)\{D_xK_N(t,x)\}D_x\varphi(x)\,dx\\ &= -[K_N(t,x)(x^2-1)D_x\varphi(x)]_{x\to-1+0}^{x\to1-0}\\ &\quad + \int_{-1}^{1} K_N(t,x)\{D_x(x^2-1)D_x\varphi(x)\}\,dx\\ &= \int_{-1}^{1} K_N(t,x)R_x\varphi(x)\,dx,\end{aligned}$$

once we have integrated twice by parts and taken into account that the limit terms are equal to zero because of Proposition 2.1. By reiterating this process k times, we arrive quickly at

$$R_t^k \int_{-1}^{1} K_N(t,x)\varphi(x)\,dx = \int_{-1}^{1} K_N(t,x)R_x^k\varphi(x)\,dx. \tag{3.7}$$

On the other hand, by setting $f(x) = P_0(x) = 1$ in Theorem 1.1 [**2**, p. 236], we get

$$1 = \int_{-1}^{1} K_N(t,x)\,dx.$$

Definitively, bearing in mind all of the above considerations, we have to prove that

$$(3.8)\quad \int_{-1}^{1} K_N(t,x) R_x^k \varphi(x)\,dx - R_t^k\varphi(t) = \int_{-1}^{1} K_N(t,x)[R_x^k\varphi(x) - R_t^k\varphi(t)]\,dx \longrightarrow 0,$$

uniformly in $t \in (-1,1)$, as $N \to \infty$.

For the moment, let us take for granted that it is possible to break the integral in (3.8) into

$$(3.9)\quad \int_{-1}^{1} K_N(t,x)[\varphi_k(x) - \varphi_k(t)]\,dx = \int_{-1}^{-1+\delta} + \int_{-1+\delta}^{1-\delta} + \int_{1-\delta}^{1} = I_{1,N}(t) + I_{2,N}(t) + I_{3,N}(t)$$

where $I_{1,N}$, $I_{2,N}$ and $I_{3,N}$ denote, respectively, integrals on the intervals $(-1,-1+\delta)$, $(-1+\delta, 1-\delta)$ and $(1-\delta, 1)$, δ being an indeterminate number such that $0 < \delta < 1$. We stress that $\varphi_k(x) = R_x^k\varphi(x)$ belongs to $\mathcal{L}(-1,1)$, in line with property (iii) of the second section.

In the sequel $\Phi_k(x,t)$ represents $[\varphi_k(x) - \varphi_k(t)](x-t)^{-1}$. Notice that $\Phi_k(x,t)$ is a continuous function for all $t \neq x$. Moreover, $\Phi_k(x,t)$ tends to $\to D\varphi_k(x)$ as $t \to x$. If we assign this value to $\Phi_k(x,x)$, it becomes a continuous function everywhere. Furthermore, we assume analogously that

$$\frac{\partial}{\partial x}\Phi_k(x,t) = \frac{\varphi_k'(x)(x-t) - \varphi_k(x) + \varphi_k(t)}{(x-t)^2}$$

has been defined by continuity at $t = x$. By this reason and Proposition 2.1, $\Phi_k(x,t)$ and $D_x\Phi_k(x,t)$ are bounded on the square domain $-1 \le t \le 1$, $-1 \le x \le 1$, say, by the positive constant C.

Firstly we study simultaneously the terms $I_{1,N}(t)$ and $I_{3,N}(t)$. Bearing in mind (1.5) and using the said function $\Phi_k(x,t)$, we can write, for $i = 1, 3$,

$$(3.10)\quad I_{i,N}(t) = \int_{x_{i,1}}^{x_{i,2}} K_N(t,x)[\varphi_k(x) - \varphi_k(t)]\,dx = I_{i,N}^{(1)}(t) - I_{i,N}^{(2)}(t)$$

where

$$I^{(1)}_{i,N}(t) = \frac{N+1}{2} P_{N+1}(t) \int_{x_{i,1}}^{x_{i,2}} P_N(x)\Phi_k(x,t)\, dx \tag{3.11}$$

and

$$I^{(2)}_{i,N}(t) = \frac{N+1}{2} P_N(t) \int_{x_{i,1}}^{x_{i,2}} P_{N+1}(x)\Phi_k(x,t)\, dx. \tag{3.12}$$

Remember that $x_{1,1} = -1$, $x_{1,2} = -1+\delta$, $x_{3,1} = 1-\delta$ and $x_{3,2} = 1$.

By integrating by parts, we get in view of (1.11),

$$\begin{aligned} I^{(1)}_{i,N}(t) &= \frac{N+1}{4N+2} P_{N+1}(t)[(P_{N+1}(x) - P_{N-1}(x))\Phi_k(x,t)]^{x=x_{i,2}}_{x=x_{i,1}} \\ &\quad - \frac{N+1}{4N+2} P_{N+1}(t) \int_{x_{i,1}}^{x_{i,2}} (P_{N+1}(x) - P_{N-1}(x)) \frac{\partial \Phi_k(x,t)}{\partial x}\, dx \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} I^{(2)}_{i,N}(t) &= \frac{N+1}{4N+6} P_N(t)[(P_{N+2}(x) - P_N(x))\Phi_k(x,t)]^{x=x_{i,2}}_{x=x_{i,1}} \\ &\quad - \frac{N+1}{4N+6} P_N(t) \int_{x_{i,1}}^{x_{i,2}} (P_{N+2}(x) - P_N(x)) \frac{\partial \Phi_k(x,t)}{\partial x}\, dx. \end{aligned} \tag{3.14}$$

We next discuss the four terms with integrals, $i = 1, 3$, that appear in the righthand sides of (3.13) and (3.14). We now exploit the fact that $P_N(t), P_{N+1}(t), P_{N+1}(x) - P_{N-1}(x)$ and $P_{N+2}(x) - P_N(x)$ are uniformly bounded in $(-1, 1)$ according to (1.8); as well as that $x_{i,2} - x_{i,1} = \delta$, $i = 1, 3$, and $\Phi_k(x,t)$ and $D_x\Phi_k(x,t)$ are bounded by C. Thus, given an $\varepsilon > 0$, we can make all these terms less than or equal to $C\delta$ and, consequently,

$$C\delta < \frac{\varepsilon}{9}, \tag{3.15}$$

if we choose δ sufficiently small.

Henceforth, in this proof, we fix $\delta < \min\{\varepsilon/(9C), 1\}$.

On the other hand, one of the terms within the braces in (3.13) and (3.14) is equal to zero, for $i = 1, 3$, because the functions $P_{N+1}(x) - P_{N-1}(x)$ and $P_{N+2}(x) - P_N(x)$ vanish in both the end points $x = x_{1,1} = -1$ and $x = x_{3,2} = 1$. As the next step, we investigate the remaining terms, which verify by invoking (1.10) that

$$\begin{aligned} &\left| \frac{N+1}{4N+2} P_{N+1}(t)[P_{N+1}(\pm x_{3,1}) - P_{N-1}(\pm x_{3,1})]\Phi_k(\pm x_{3,1}, t) \right| \\ &\qquad \le C\left\{ \sqrt{\frac{\pi}{2(N+1)(1-x_{3,1}^2)}} + \sqrt{\frac{\pi}{2(N-1)(1-x_{3,1}^2)}} \right\} \\ &\qquad < C\sqrt{\frac{2\pi}{(N-1)\delta}} < \frac{\varepsilon}{9}, \end{aligned} \tag{3.16}$$

$N > \nu_1$ for sufficiently large $\nu_1 > 0$, uniformly in $t \in (-1, 1)$, and

$$\begin{aligned} &\left| \frac{N+1}{4N+6} P_N(t)[P_{N+2}(\pm x_{3,1}) - P_N(\pm x_{3,1})]\Phi_k(\pm x_{3,1}, t) \right| \\ &\qquad \le C\sqrt{\frac{2\pi}{(N-2)\delta}} < \frac{\varepsilon}{9}, \end{aligned} \tag{3.17}$$

$N > \nu_2$ for adequately large $\nu_2 > 0$, uniformly in $t \in (-1, 1)$ as well.

All that remains to be investigated is the term $I_{2,N}(t)$. For this purpose, making use of the function $\Phi_k(x, t)$, we set

$$\begin{aligned} I_{2,N}(t) &= \int_{-1+\delta}^{1-\delta} K_N(t, x)[\varphi_k(x) - \varphi_k(t)]\, dx \\ &= I_{2,N}^{(1)}(t) - I_{2,N}^{(2)}(t) \end{aligned} \tag{3.18}$$

where

$$I_{2,N}^{(1)}(t) = \frac{N+1}{2} P_{N+1}(t) \int_{-1+\delta}^{1-\delta} P_N(x)\Phi_k(x, t)\, dx$$

and

$$I_{2,N}^{(2)}(t) = \frac{N+1}{2} P_N(t) \int_{-1+\delta}^{1-\delta} P_{N+1}(x)\Phi_k(x, t)\, dx.$$

An integration by parts and the use of (1.11) yield

$$I_{2,N}^{(1)}(t) = \frac{N+1}{4N+2} P_{N+1}(t) \bigg\{ [P_{N+1}(x_{3,1}) - P_{N-1}(x_{3,1})]\Phi_k(x_{3,1}, t) - [P_{N+1}(x_{1,2}) - P_{N-1}(x_{1,2})]\Phi_k(x_{1,2}, t) - \int_{-1+\delta}^{1-\delta} [P_{N+1}(x) - P_{N-1}(x)] \frac{\partial \Phi_k(x,t)}{\partial x}\, dx \bigg\}.$$

By invoking (1.8), (1.10) and the boundedness of functions $\Phi_k(x,t)$ and $\partial\Phi_k(x,t)/\partial x$, we infer from the last expression that

$$|I_{2,N}^{(1)}(t)| < \frac{8(N+1)C}{4N+2} \sqrt{\frac{\pi}{2(N-1)\delta}} = O(N^{-1/2}), \tag{3.19}$$

for a sufficiently large number N, uniformly in $t \in (-1,1)$. Similarly,

$$|I_{2,N}^{(2)}(t)| = O(N^{-1/2}), \quad \text{uniformly in} \quad t \in (-1,1). \tag{3.20}$$

These results (3.19) and (3.20) and expression (3.18) allow us to make

$$|I_{2,N}(t)| < \frac{\varepsilon}{9}, \tag{3.21}$$

$N > \nu_3$ for a suitable $\nu_3 > 0$.

Finally, by virtue of our conclusions (3.15), (3.16), (3.17) and (3.21), we are led from (3.9) to the desired result if we choose $N > \nu = \max\{\nu_1, \nu_2, \nu_3\}$. □

Theorem 3.2 (The uniqueness theorem). *Let f and g be arbitrary members of $\mathcal{L}'(-1,1)$. If $F(m)$ and $G(m)$ denote their respective Legendre transforms and $F(m) = G(m)$ for each $m = 0, 1, 2, \ldots,$ then $f = g$ in the sense of the equality in $\mathcal{L}'(-1,1)$.*

Proof. Theorem 3.1 and the fact that $F(m) = G(m)$ by hypothesis imply that

$$\langle f(x) - g(x), \varphi(x)\rangle = \lim_{N\to\infty} \bigg\langle \sum_{m=0}^{N} \left(m + \frac{1}{2}\right)[F(m) - G(m)]P_m(x), \varphi(x) \bigg\rangle = 0,$$

for all $\varphi \in \mathcal{L}(-1,1)$.

Finally, we shall derive the more important operation transform formula which, together with the inversion theorem, shows the usefulness of the Legendre transformation in solving certain partial differential equations. Let $f(x) \in \mathcal{L}'(-1,1)$. From definitions (3.1) and (2.4) we get, in view of the operational rule (2.5),

$$\begin{aligned}[l'\{R_x^k f(x)\}](n) &= \langle R_x^k f(x), P_n(x)\rangle \\ &= \langle f(x), R_x^k P_n(x)\rangle \\ &= \langle f(x), [n(n+1)]^k P_n(x)\rangle \\ &= [n(n+1)]^k \langle f(x), P_n(x)\rangle \\ &= [n(n+1)]^k \{l' f(x)\}(n).\end{aligned}$$

Thus, we have obtained the formula

$$[l'\{R_x^k f(x)\}](n) = [n(n+1)]^k \{l' f(x)\}(n) \tag{3.22}$$

for each nonnegative integer k. $\square$

Remark 3.1. The next numerical example illustrates the inversion Theorem 3.1. Consider the Dirac delta function $\delta(x-a)$, $-1 < a < 1$. Inasmuch as $\delta(x-a) \in \mathcal{E}'(-1,1)$, it follows that $\delta(x-a) \in \mathcal{L}'(-1,1)$ by virtue of property (i) of the second section. On the one hand, according to Definition (3.1), the finite Legendre transform of $\delta(x-a)$ is

$$[l'\{\delta(x-a)\}](n) = \langle \delta(x-a), P_n(x)\rangle = P_n(a).$$

On the other hand, by virtue of Theorem 1.1, one has for all $\varphi(x) \in \mathcal{L}(-1,1)$,

$$\begin{aligned}&\left\langle \sum_{m=0}^{N} \left(m+\frac{1}{2}\right) P_m(a) P_m(x), \varphi(x)\right\rangle \\ &\qquad = \int_{-1}^{1} \sum_{m=0}^{N} \left(m+\frac{1}{2}\right) P_m(a) P_m(x) \varphi(x)\, dx \\ &\qquad = \int_{-1}^{1} K_N(a,x)\varphi(x)\, dx \longrightarrow \varphi(a) \\ &\qquad = \langle \delta(x-a), \varphi(x)\rangle \quad \text{as } N \to \infty.\end{aligned}$$

Therefore,

$$\delta(x-a) = \lim_{N\to\infty} \sum_{m=0}^{N} \left(m+\frac{1}{2}\right) P_m(a) P_m(x).$$

4. Applications. The finite Legendre transformation is useful for solving various boundary-value problems when a spherical coordinate system is chosen. Thus, we consider the problem of finding the steady temperatures $v(r,\theta)$ in a hollow sphere $a < r < b$ such that $v(r,\theta)$ assumes prescribed values $\varphi(\theta)$ in the internal surface whereas that external is maintained at temperature zero. This classical problem can be posed as follows

$$\begin{gathered} r^2 \frac{\partial^2 v}{\partial r^2} + 2r\frac{\partial v}{\partial r} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v}{\partial\theta}\right) = 0, \\ a < r < b, \quad 0 < \theta < \pi, \\ u(a,\theta) = \varphi(\theta), \qquad u(b,\theta) = 0, \\ 0 \le \theta \le \pi. \end{gathered} \tag{4.1}$$

By setting $x = \cos\theta$ and $u(r,x) = v(r,\theta)$, the Dirichlet problem (4.1) reduces now to seek a function $u(r,x)$ satisfying the partial differential equation

$$r^2 \frac{\partial^2 u}{\partial r^2} + 2r\frac{\partial u}{\partial r} + (1-x^2)\frac{\partial^2 u}{\partial x^2} - 2x\frac{\partial u}{\partial x} = 0, \tag{4.2}$$

and the boundary conditions:

(a) As $r \to a$, $u(r,x)$ converges to $f(x) \in \mathcal{L}'(-1,1)$.

(b) As $r \to b$, $u(r,x)$ converges to zero in $\mathcal{L}'(-1,1)$.

(c) For r fixed, $a < r < b$, $u(r,x)$ remains finite as $x \to \pm 1$.

By applying to (4.2) the finite Legendre transformation, denoting $U(r,n) = l'\{u(r,x)\}(n)$ and taking into account the operational rule (3.22), we convert the above problem into the ordinary differential equation

$$r^2 \frac{\partial^2 U(r,n)}{\partial r^2} + 2r\frac{\partial U(r,n)}{\partial r} - n(n+1)U(r,n) = 0,$$

whose general solution is

$$U(r,n) = Ar^n + Br^{-n-1}, \quad n = 0, 1, 2, \dots .$$

The adequate values of A and B will be suggested by the boundary conditions (a) and (b). Thus, we yield the solution

$$U(r,n) = \frac{r^{2n+1} - b^{2n+1}}{a^{2n+1} - b^{2n+1}} F(n) \left(\frac{a}{r}\right)^{n+1}, \tag{4.3}$$

where $F(n) = l'\{f(x)\}(n)$. By inverting (4.3) according to Theorem 3.1, we derive the desired solution

$$\begin{aligned} u(r,x) &= l'^{-1}\{U(r,n)\}(x) \\ &= \sum_{n=0}^{\infty} \frac{2n+1}{2} \frac{r^{2n+1} - b^{2n+1}}{a^{2n+1} - b^{2n+1}} \left(\frac{a}{r}\right)^{n+1} F(n) P_n(x). \end{aligned} \tag{4.4}$$

To justify that (4.4) is truly a solution, we need two previous results:

(i) Let $\varphi(x) \in \mathcal{L}(-1,1)$. Then $\Phi(n) = l\{\varphi(x)\}(n)$, given by (1.1), is of rapid descent as $n \to \infty$.

(ii) For every $f(x) \in \mathcal{L}'(-1,1)$, we have that $F(n) = l'\{f(x)\}(n)$, defined by (3.1), is of slow growth as $n \to \infty$.

To verify (i), suppose $n \neq 0$ and let an arbitrary nonnegative integer stand for k. Then, we can write

$$\begin{aligned} n^k(n+1)^k \Phi(n) &= \int_{-1}^{1} \{R^k P_n(x)\} \varphi(x)\, dx \\ &= \frac{1}{n(n+1)} \int_{-1}^{1} \{R^{k+1} P_n(x)\} \varphi(x)\, dx. \end{aligned} \tag{4.5}$$

Upon integrating the last expression in (4.5) $2(k+1)$ times by parts, we yield in view of Proposition 2.1,

$$n^k(n+1)^k \Phi(n) = \frac{1}{n(n+1)} \int_{-1}^{1} P_n(x) \{R^{k+1} \varphi(x)\}\, dx,$$

which implies that

$$|n^k \Phi(n)| \leq \frac{2\gamma_{k+1}(\varphi)}{n(n+1)^{k+1}} \longrightarrow 0, \quad \text{as} \quad n \to \infty.$$

To prove (ii), it suffices to remember that there exist positive constants C and C_1 and a nonnegative integer s such that [**18**, Theorem 1.8-1]:

$$|F(n)| = |\langle f(x), P_n(x)\rangle| \leq C \max_{0\leq k\leq s} \gamma_k\{P_n(x)\} \leq C_1 n^{2s}.$$

Making use of (ii), we can easily show that the series (4.4) and the series obtained by applying D_r, D_r^2, D_x and D_x^2 separately under the summation sign converge uniformly on $a < r \leq b$ and $-1 \leq x \leq 1$. Thus, on the one hand, some tedious calculations allow us to get

$$r^2 \frac{\partial^2 u}{\partial r^2} + 2r \frac{\partial u}{\partial r} = \sum_{n=0}^{\infty} \frac{2n+1}{2} \frac{r^{2n+1} - b^{2n+1}}{a^{2n+1} - b^{2n+1}} \cdot n(n+1) \left(\frac{a}{r}\right)^{n+1} F(n) P_n(x). \tag{4.6}$$

On the other hand, from (2.5) we immediately infer

$$\begin{aligned} (1-x^2)\frac{\partial^2 u}{\partial x^2} - 2x\frac{\partial u}{\partial x} &= -R_x u(r,x) \\ &= -\sum_{n=0}^{\infty} \frac{2n+1}{2} \frac{r^{2n+1} - b^{2n+1}}{a^{2n+1} - b^{2n+1}} \\ &\quad \cdot n(n+1) \left(\frac{a}{r}\right)^{n+1} F(n) P_n(x). \end{aligned} \tag{4.7}$$

By combining (4.6) and (4.7), we conclude that $u(r,x)$ satisfies the equation (4.2) in the domain $a < r < b$, $-1 < x < 1$.

As the next step, we verify that our solution fulfills boundary conditions (a) and (b). To do it, notice that the series (4.4) defines a continuous function of x, $-1 \leq x \leq 1$, and therefore gives rise to a regular member in $\mathcal{L}'(-1,1)$ through (2.3). Hence, we can write for every $\varphi(x) \in \mathcal{L}(-1,1)$,

$$\begin{aligned}\langle u(r,x), \varphi(x)\rangle &= \int_{-1}^{1}\left\{\sum_{n=0}^{\infty}\frac{2n+1}{2}\frac{r^{2n+1}-b^{2n+1}}{a^{2n+1}-b^{2n+1}}\right. \\ &\qquad\qquad \left.\cdot\left(\frac{a}{r}\right)^{n+1}F(n)P_n(x)\right\}\varphi(x)\,dx \\ &= \sum_{n=0}^{\infty}\frac{2n+1}{2}\frac{r^{2n+1}-b^{2n+1}}{a^{2n+1}-b^{2n+1}}\left(\frac{a}{r}\right)^{n+1}F(n)\Phi(n),\end{aligned} \tag{4.8}$$

where $\Phi(n) = l\{\varphi(x)\}(n)$. But the last series in (4.8) converges uniformly for all r because $F(n)$ grows slowly while $\Phi(n)$ decreases rapidly as $n \to \infty$.

Now it is easily seen that $\langle u(r,x), \varphi(x)\rangle \to 0$ as $r \to b$. Finally, by taking the limit as $r \to a$, one has

$$\begin{aligned}\langle u(r,x), \varphi(x)\rangle &\longrightarrow \lim_{N\to\infty}\sum_{n=0}^{N}\frac{2n+1}{2}F(n)\Phi(n) \\ &= \lim_{N\to\infty}\sum_{n=0}^{N}\frac{2n+1}{2}F(n)\int_{-1}^{1}P_n(x)\varphi(x)\,dx \\ &= \lim_{N\to\infty}\int_{-1}^{1}\sum_{n=0}^{N}\frac{2n+1}{2}F(n)P_n(x)\varphi(x)\,dx \\ &= \lim_{N\to\infty}\left\langle\sum_{n=0}^{N}\frac{2n+1}{2}F(n)P_n(x), \varphi(x)\right\rangle \\ &= \langle f(x), \varphi(x)\rangle,\end{aligned}$$

by virtue of Theorem 3.1.

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